Arithmetic surfaces and higher adeles

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One-dimensional case: number fields

Let $K$ be a number field such that $[K : \mathbb{Q}] = 1$. Let $E$ be the ring of integers in $K$, i.e. $E \supset \mathbb{Z}$ is the integral closure of $\mathbb{Z}$ in $\mathbb{Q}$. The ring of adeles of $K$ is $A_K = \prod_v K_v = \{ f \in \prod_v K_v | f \in \hat{E}_\sigma \text{ for almost all } \sigma \}$, where $v$ runs over all places of $K$, $K_v$ is the completion of the field $K$ with respect to $v$, $'$ means the restricted product with respect to the subrings $E_\sigma$, $\sigma$ runs over the set of all non-Archimedean places of $K$ (which is the set of all maximal ideals of $E$), $\hat{E}_\sigma$ is the corresponding completion of $E$ at $\sigma$ (which is the ring of integers in $K_\sigma$).
One-dimensional case: number fields

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The ring of adeles of $K$ is

$$\mathbb{A}_K = \prod' K_v = \{ f \in \prod K_v \mid f \in \hat{E}_\sigma \text{ for almost all } \sigma \},$$

where $v$ runs over all places of $K$, $K_v$ is the completion of the field $K$ with respect to $v$, $\prod'$ means the restricted product with respect to the subrings $E_\sigma$, $\sigma$ runs over the set of all non-Archimedean places of $K$ (which is the set of all maximal ideal of $E$), $\hat{E}_\sigma$ is the corresponding completion of $E$ at $\sigma$ (which is the ring of integers in $K_\sigma$).
Quotient group of the ring of adeles

The field $K_v$ has a locally compact topology. Hence the ring $\mathbb{A}_K$ has a topology of the restricted product. Thus, $\mathbb{A}_K$ is a locally compact topological ring.
Quotient group of the ring of adeles

The field $K_v$ has a locally compact topology. Hence the ring $\mathbb{A}_K$ has a topology of the restricted product. Thus, $\mathbb{A}_K$ is a locally compact topological ring. The field $K$ is diagonally embedded in the ring $\mathbb{A}_K$ (through the natural embedding $K \hookrightarrow K_v$). The strong approximation theorem implies the well-known result.

Theorem

There is an exact sequence of groups

$$0 \longrightarrow \prod_{\sigma} \hat{E}_\sigma \longrightarrow \mathbb{A}_K/K \longrightarrow \left( \prod_w K_w \right)/E \longrightarrow 0,$$

where $\sigma$ runs over the set of all non-Archimedean places of $K$, $w$ runs over the (finite) set of all Archimedean places of $K$. 
Example

The group $\prod \hat{E}_\sigma$ is compact.

The group $\left( \prod K_w \right) / E \simeq \mathbb{T}^d$, where $\mathbb{T} = \mathbb{R} / \mathbb{Z}$ is also compact. Thus, the group $\mathbb{A}_K / K$ is compact and is an extension of two compact groups through the above exact sequence.
Example

The group $\prod_{\sigma} \hat{E}_{\sigma}$ is compact.

The group $\left( \prod_{w} K_w \right) / E \simeq T^1$, where $T = \mathbb{R} / \mathbb{Z}$ is also compact. Thus, the group $\mathbb{A}_K / K$ is compact and is an extension of two compact groups through the above exact sequence.

Example

For $K = \mathbb{Q}$ the above sequence is

$$0 \longrightarrow \hat{\mathbb{Z}} \longrightarrow \mathbb{A}_\mathbb{Q} / \mathbb{Q} \longrightarrow \mathbb{R} / \mathbb{Z} \longrightarrow 0,$$

where $\hat{\mathbb{Z}} \simeq \prod_{p} \mathbb{Z}_p$. 
One-dimensional geometric case

In the geometric case there is an analogous exact sequence for the ring of adeles $\mathbb{A}_K$, where $K = \mathbb{F}_q(C)$, and $C$ is a smooth projective curve over a finite field $\mathbb{F}_q$.

Fix a finite set of points $s_1, \ldots, s_n$ (the analog of above set of Archimedean places). Let $A = H^0(C \setminus \bigcup s_i), \mathcal{O})$. Then

$$0 \longrightarrow \prod_{u \in C, u \neq s_i} \mathcal{O}_u \longrightarrow \mathbb{A}_K/K \longrightarrow (\prod_{i} K_{s_i})/A \longrightarrow 0.$$ 

In other notation, $\mathbb{A}_K = \mathbb{A}_{C,01}$, $K = \mathbb{A}_{C,0}$. 

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Arithmetic surfaces and higher adeles

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Adeles on two-dimensional schemes

In higher dimensions rings of adeles were introduced: by A. N. Parshin (1976) for smooth algebraic surfaces, and by A. A. Beilinson (1980) for arbitrary Noetherian schemes.

Rings of higher adeles (and higher-dimensional local fields) are important for the arithmetic properties of schemes: higher-dimensional generalization of the class field theory and possible generalization of the Tate-Iwasawa method (on meromorphic continuation and functional equation for L-functions) from one-dimensional case to the case of higher dimensions.

We restrict ourselves to the case of dimension 2.
Algebraic and arithmetic surfaces

We consider two parallel (and analogous cases):

- Y is an irreducible normal algebraic surface over a finite field $\mathbb{F}_q$,
- X is an arithmetic surface, i.e. a two-dimensional normal integral scheme with projective and surjective morphism to Spec $\mathbb{Z}$.

Then for the rings of adeles we have:

$$A_Y \subset \prod_{x \in D} K_{x,D} \quad A_X \subset \prod_{x \in D} K_{x,D},$$

where $x \in D$ is a pair with a closed point $x \in X$ and an irreducible curve $D \subset Y$ (or an integral closed one-dimensional subscheme $D \subset X$).
Local components

The ring

$$K_{x,D} = \prod_{1 \leq i \leq l} K_i,$$

where $K_i$ is the completion of the field $\text{Frac} \, \hat{O}_x$ with respect to $D_i \subset \text{Spec} \, \hat{O}_x$, where $D \mid_{\text{Spec} \, \hat{O}_x} = \bigcup_{1 \leq i \leq l} D_i$ and $D_i$ is irreducible.

In other words, $K_i$ is associated with every branch of $D$ at the formal neighbourhood of $x \in X$ (or $x \in Y$).
Two-dimensional local fields

The field $K_i$ is a two-dimensional local field:

- $K_i \cong \mathbb{F}_q((u))((t))$ in the geometric case,
- $K_i \supset \mathbb{Q}_p((t))$ or $K_i \supset \mathbb{Q}_p\{\{t\}\}$ in the arithmetic case,

where $f = \sum_{-\infty}^{+\infty} a_i t^i \in \mathbb{Q}_p\{\{t\}\}$ (with $a_i \in \mathbb{Q}_p$) iff

- there is $c_f \in \mathbb{Z}$ such that $\nu_p(a_i) > c_f$ for all $i$,
- and $\lim_{i \to -\infty} \nu_p(a_i) = +\infty$. 
Algebraic surfaces

Let us give an exact definition of $A_Y$.

For any irreducible curve $C$ on $Y$ let $t_C = 0$ be an equation of $C$ on some affine open subset of $Y$. We note that

$$A_C((t_C)) \subset \prod_{x \in C} K_{x,C}.$$ 

Then $g \in A_Y \subset \prod_{x \in D} K_{x,D}$ iff

- $g \in \prod_{D \subset X} A_D((t_D))$,
- and for almost all $D \subset X$ the restriction of $g$ to $D$-component belongs to $A_D[[t_D]]$.

This definition does not depend on the choice of $t_D$. 

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Subgroups of the ring of adeles

Similarly to $A_Y$ one defines the ring $A_X$ (with more technical details).

Let $A_{012} = A_Y$ (or $A_{012} = A_X$).

For any $D \subset Y$ (or $D \subset X$) let $K_D$ be the completion of the field of rational functions $\mathbb{F}_q(Y)$ (or $\mathbb{Q}(X)$) with respect to the discrete valuation given by $D$. We have $K_D \subset \prod_{x \in D} K_{x,D}$. 
Subgroups of the ring of adeles

Similarly to $\mathbb{A}_Y$ one defines the ring $\mathbb{A}_X$ (with more technical details).

Let $\mathbb{A}_{012} = \mathbb{A}_Y$ (or $\mathbb{A}_{012} = \mathbb{A}_X$).

For any $D \subset Y$ (or $D \subset X$) let $K_D$ be the completion of the field of rational functions $\mathbb{F}_q(Y)$ (or $\mathbb{Q}(X)$) with respect to the discrete valuation given by $D$. We have $K_D \subset \prod_{x \in D} K_{x, D}$.

For any $x \in Y$ (or $x \in X$) let $K_x = \hat{O}_x \cdot L$, where $L = \mathbb{F}_q(Y)$ (or $L = \mathbb{Q}(X)$). We have $K_x \subset \prod_{D \ni x} K_{x, D}$.

We define inside $\prod_{x \in D} K_{x, D}$:

$$\mathbb{A}_{01} = \mathbb{A}_{012} \cap \prod_{D} K_D,$$
$$\mathbb{A}_{02} = \mathbb{A}_{012} \cap \prod_{x} K_x.$$
Adelic quotient group for algebraic surface

We fix an ample divisor $C$ on $Y$. Then the complement $U = Y \setminus \text{supp} C$ is affine.

**Theorem**

There is an exact sequence of compact groups for $\mathbb{A}_{012} = \mathbb{A}_Y$:

$$0 \rightarrow G_1 \rightarrow \mathbb{A}_{012}/(\mathbb{A}_{01} + \mathbb{A}_{02}) \rightarrow G_2 \rightarrow 0,$$

where

$$G_1 \simeq \left( \prod_{D \subseteq Y, D \not\subseteq C} \left( \left( \prod_{x \in D} \mathcal{O}_{x, \mathcal{D}} \right) / \hat{\mathcal{O}}_D \right) \right) / \left( \prod_{x \in U} \hat{\mathcal{O}}_x \right),$$

and $\prod'$ means an adelic (on algebraic surface) product, the ring $\mathcal{O}_{x, D}$ is the product of discrete valuation rings from the ring $K_{x,D}$ (for example, if $K_{x,D} = \mathbb{F}_{q^r}((u))((t))$, then $\mathcal{O}_{K_{x,D}} = \mathbb{F}_{q^r}((u))[[t]]$), the ring $\hat{\mathcal{O}}_D$ is the discrete valuation ring in $K_D$. 
Groups $G_1$ and $G_2$

The group $G_1$ is compact, since the group

$$\hat{\mathcal{O}}_x \simeq \mathbb{F}_q[[u, t]]$$

is compact, and for a fixed $D \subset Y$ the group

$$\left( \prod_{x \in D} ' \mathcal{O}_{x, D} \right) / \hat{\mathcal{O}}_D \simeq (\mathbb{A}_D / \mathbb{F}_q(D)) [[t_D]]$$

is also compact.
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is compact, and for a fixed $D \subset Y$ the group

$$\left(\prod_{x \in D}' \mathcal{O}_{x,D}\right)/\hat{\mathcal{O}}_D \simeq \left(\mathbb{A}_D/\mathbb{F}_q(D)\right)[[t_D]]$$

is also compact.

The group $G_2$ has also an explicit presentation. We note only that in the simplest case

$$Y = \mathbb{P}^1_{\mathbb{F}_q} \times_{\mathbb{F}_q} \mathbb{P}^1_{\mathbb{F}_q}, \quad C = \mathbb{P}^1_{\mathbb{F}_q} \times y + y \times \mathbb{P}^1_{\mathbb{F}_q}$$

($y$ is a fixed $\mathbb{F}_q$-rational point on $\mathbb{P}^1_{\mathbb{F}_q}$) we have

$$G_2 \simeq \mathbb{F}_q((u))((t))/\left(\mathbb{F}_q[u^{-1}]((t)) + \mathbb{F}_q((u))[t^{-1}]\right).$$
Arithmetic surfaces

An arithmetic surface \( X \) (with morphism \( X \to \text{Spec} \mathbb{Z} \)) is “not compact” (since \( \text{Spec} \mathbb{Z} \) is “not compact”).
To have the good analogy with projective algebraic surface \( Y \) we have “to compactify” \( X \).
For this goal we have to take into account the fibre over the \( \infty \)-point of \( \text{Spec} \mathbb{Z} \), which corresponds to the Archimedean valuation of \( \mathbb{Q} \).
Arithmetic surfaces

An arithmetic surface $X$ (with morphism $X \to \text{Spec} \mathbb{Z}$) is “not compact” (since $\text{Spec} \mathbb{Z}$ is “not compact”).

To have the good analogy with projective algebraic surface $Y$ we have “to compactify” $X$.

For this goal we have to take into account the fibre over the $\infty$-point of $\text{Spec} \mathbb{Z}$, which corresponds to the Archimedean valuation of $\mathbb{Q}$.

Let a curve $X_\mathbb{Q} = X \times_{\text{Spec} \mathbb{Z}} \text{Spec} \mathbb{Q}$ be the fibre over the generic point of $\text{Spec} \mathbb{Z}$.

We define a ring of arithmetic adeles

$$A_X^{\text{ar}} = A_X \oplus (A_{X_\mathbb{Q}} \hat{\otimes}_\mathbb{Q} \mathbb{R}),$$

where the ring

$$A_{X_\mathbb{Q}} \hat{\otimes}_\mathbb{Q} \mathbb{R} = \lim_{\longrightarrow} \lim_{\longleftarrow} \prod_{D_1 \in \text{Div}(X_\mathbb{Q}) \atop D_2 \leq D_1} \left( \hat{\mathcal{O}}_u(D_1)/\hat{\mathcal{O}}_u(D_2) \right) \otimes_\mathbb{Q} \mathbb{R}.$$
Subgroups of the group of arithmetic adeles

We define subgroups of the group $\mathbb{A}^\text{ar}_{012} = \mathbb{A}^\text{ar}_X$ as

\[
\mathbb{A}^\text{ar}_{01} = \mathbb{A}_{01}, \quad \mathbb{A}^\text{ar}_{02} = \mathbb{A}_{X,02} \oplus (\mathbb{Q}(X) \otimes_\mathbb{Q} \mathbb{R}),
\]

where the group in the first formula is naturally embedded into the both summands for $\mathbb{A}^\text{ar}_{012}$,
the first summand in the second formula is naturally embedded into the first summand (scheme part) for $\mathbb{A}^\text{ar}_{012}$,
the second summand in the second formula is naturally embedded into the second summand (Archimedean part) for $\mathbb{A}^\text{ar}_{012}$. 
Quotient group

We fix a “horizontal divisor” \( C \) on \( X \). Then \( U = X \setminus C \) is affine, and “\( C \) plus Archimedean fibre” is an analog of an ample divisor on a projective algebraic surface.

Theorem
There is an exact sequence of compact groups:

\[
0 \to \tilde{G}_1 \to \mathbb{A}^{ar}_{012}/(\mathbb{A}^{ar}_{01} + \mathbb{A}^{ar}_{02}) \to \tilde{G}_2 \to 0,
\]

where \( \tilde{G}_1 \) is written analogously to the group \( G_1 \) for an algebraic surface (but we have to take into account the fields \( \mathbb{R}((t)) \) and \( \mathbb{C}((t)) \) which come from the Archimedean part of \( \mathbb{A}^{ar}_{012} \)).
The group $\tilde{G}_2$

Let us give the description of the group $\tilde{G}_2$

Example

In the simplest case when $X = \mathbb{P}^1_{\mathbb{Z}}$ and $C$ is a hyperplane section, we have that

$$\tilde{G}_2 \simeq \mathbb{R}((t))/\left(\mathbb{Z}((t)) + \mathbb{R}[t^{-1}]\right).$$
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Proposition

For an arithmetic surface $X$ we have $\tilde{G}_2 \simeq \lim_{\leftarrow m \geq 0} \Theta_m$, where the group

$$\Theta_m \simeq H^1(X, \mathcal{O}_X(-mC)) \otimes_{\mathbb{Z}} \mathbb{T} \simeq \mathbb{T}^{\text{rank}(H^1(X, \mathcal{O}_X(-mC)))}$$

is a finite direct product of copies of $\mathbb{T}$ (we recall that $\mathbb{T} \simeq \mathbb{R}/\mathbb{Z}$).