Arithmetic surfaces and higher adeles

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One-dimensional case: number fields

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The ring of adeles of K is

$$\mathbb{A}_K = \prod_v{}' K_v = \{ f \in \prod_v K_v \mid f \in \hat{E}_\sigma \quad \text{for almost all} \quad \sigma \} \, ,$$

where v runs over all places of K,
K_v is the completion of the field K with respect to v,
$\prod'$ means the restricted product with respect to the subrings E_σ,
σ runs over the set of all non-Archimedean places of K (which is the set of all maximal ideal of E),
$\hat{E}_\sigma$ is the corresponding completion of E at σ (which is the ring of integers in K_σ).

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The field K is diagonally embedded in the ring $\mathbb{A}_K$ (through the natural embedding $K \hookrightarrow K_v$). The strong approximation theorem implies the well-known result.

Theorem

There is an exact sequence of groups

$$0 \longrightarrow \prod_{\sigma} \hat{E}_\sigma \longrightarrow \mathbb{A}_K/K \longrightarrow \left(\prod_{w} K_w \right) /E \longrightarrow 0\,,$$

where σ runs over the set of all non-Archimedean places of K, w runs over the (finite) set of all Archimedean places of K.

Example

The group $\prod_{\sigma} \hat{\mathrm{E}}_{\sigma}$ is compact.

The group $\left(\prod_{\mathrm{w}} \mathrm{K}_{\mathrm{w}}\right)/\mathrm{E} \simeq \mathbb{T}^{\mathrm{l}}$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is also compact.

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Example

For $\mathrm{K} = \mathbb{Q}$ the above sequence is

$$0 \longrightarrow \widehat{\mathbb{Z}} \longrightarrow \mathbb{A}_{\mathbb{Q}}/\mathbb{Q} \longrightarrow \mathbb{R}/\mathbb{Z} \longrightarrow 0\,,$$

where $\widehat{\mathbb{Z}} \simeq \prod_{\mathrm{p}} \mathbb{Z}_{\mathrm{p}}$.

One-dimensional geometric case

In the geometric case there is an analogous exact sequence for the ring of adeles $\mathbb{A}_K$, where $K = \mathbb{F}_q(C)$, and C is a smooth projective curve over a finite field $\mathbb{F}_q$.
Fix a finite set of points $s_1, \ldots, s_n$ (the analog of above set of Archimedean places). Let $A = H^0(C \setminus \bigcup s_i), \mathcal{O})$. Then

$$0 \longrightarrow \prod_{u \in C, u \neq s_i} \hat{\mathcal{O}}_u \longrightarrow \mathbb{A}_K / K \longrightarrow (\prod_i K_{s_i})/A \longrightarrow 0 \,.$$

In other notation, $\mathbb{A}_K = \mathbb{A}_{C,01}$, $K = \mathbb{A}_{C,0}$.

Adeles on two-dimensional schemes

In higher dimensions rings of adeles were introduced: by A. N. Parshin (1976) for smooth algebraic surfaces, and by A. A. Beilinson (1980) for arbitrary Noetherian schemes.

Rings of higher adeles (and higher-dimensional local fields) are important for the arithmetic properties of schemes: higher-dimensional generalization of the class field theory and possible generalization of the Tate-Iwasawa method (on meromorphic continuation and functional equation for L-functions) from one-dimensional case to the case of higher dimensions.

We restrict ourselves to the case of dimension 2.

Algebraic and arithmetic surfaces

We consider two parallel (and analogous cases):

- Y is an irreducible normal algebraic surface over a finite field $\mathbb{F}_q$,
- X is an arithmetic surface, i.e. a two-dimensional normal integral scheme with projective and surjective morphism to $\operatorname{Spec} \mathbb{Z}$.

Then for the rings of adeles we have:

$$\mathbb{A}_Y \subset \prod_{x \in D} K_{x,D} \qquad \mathbb{A}_X \subset \prod_{x \in D} K_{x,D},$$

where $x \in D$ is a pair with a closed point $x \in X$ and an irreducible curve $D \subset Y$ (or an integral closed one-dimensional subscheme $D \subset X$).

Local components

The ring

$$K_{x,D} = \prod_{1 \le i \le l} K_i \,,$$

where K_i is the completion of the field $\mathrm{Frac}\, \hat{\mathcal{O}}_x$ with respect to $D_i \subset \mathrm{Spec}\, \hat{\mathcal{O}}_x$, where $D \mid_{\mathrm{Spec}\, \hat{\mathcal{O}}_x} = \bigcup_{1 \le i \le l} D_i$ and D_i is irreducible.

In other words, K_i is associated with every branch of D at the formal neighbourhood of $x \in X$ (or $x \in Y$).

Two-dimensional local fields

The field K_i is a two-dimensional local field:

- $K_i \simeq \mathbb{F}_{q^r}((u))((t))$ in the geometric case,
- $K_i \supset \mathbb{Q}_p((t))$ or $K_i \supset \mathbb{Q}_p\{\{t\}\}$ in the arithmetic case,

where $f = \sum\limits_{-\infty}^{+\infty} a_i t^i \in \mathbb{Q}_p\{\{t\}\}$ (with $a_i \in \mathbb{Q}_p$) iff

- there is $c_f \in \mathbb{Z}$ such that $\nu_p(a_i) > c_f$ for all i,
- and $\lim\limits_{i \to -\infty} \nu_p(a_i) = +\infty$.

Algebraic surfaces

Let us give an exact definition of $\mathbb{A}_Y$.
For any irreducible curve C on Y let $t_C = 0$ be an equation of C on some affine open subset of Y. We note that

$$\mathbb{A}_C((t_C)) \subset \prod_{x \in C} K_{x,C} .$$

Then $g \in \mathbb{A}_Y \subset \prod_{x \in D} K_{x,D}$ iff

- $g \in \prod_{D \subset X} \mathbb{A}_D((t_D))$,
- and for almost all $D \subset X$ the restriction of g to D-component belongs to $\mathbb{A}_D[[t_D]]$.

This definition does not depend on the choice of t_D.

Subgroups of the ring of adeles

Similarly to $\mathbb{A}_Y$ one defines the ring $\mathbb{A}_X$ (with more technical details).
Let $\mathbb{A}_{012} = \mathbb{A}_Y$ (or $\mathbb{A}_{012} = \mathbb{A}_X$).
For any $D \subset Y$ (or $D \subset X$) let K_D be the completion of the field of rational functions $\mathbb{F}_q(Y)$ (or $\mathbb{Q}(X)$) with respect to the discrete valuation given by D. We have $K_D \subset \prod_{x \in D} K_{x,D}$.

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For any $x \in Y$ (or $x \in X$) let $K_x = \hat{\mathcal{O}}_x \cdot L$, where $L = \mathbb{F}_q(Y)$ (or $L = \mathbb{Q}(X)$). We have $K_x \subset \prod_{D \ni x} K_{x,D}$.

We define inside $\prod_{x \in D} K_{x,D}$:

$$\mathbb{A}_{01} = \mathbb{A}_{012} \cap \prod_D K_D \,, \qquad \mathbb{A}_{02} = \mathbb{A}_{012} \cap \prod_x K_x \,.$$

Adelic quotient group for algebraic surface

We fix an ample divisor C on Y. Then the complement $U = Y \setminus \operatorname{supp} C$ is affine.

Theorem

There is an exact sequence of compact groups for $\mathbb{A}_{012} = \mathbb{A}_Y$:

$$0 \longrightarrow G_1 \longrightarrow \mathbb{A}_{012}/(\mathbb{A}_{01} + \mathbb{A}_{02}) \longrightarrow G_2 \longrightarrow 0\,,$$

$$\text{where} \qquad G_1 \ \simeq \ \left(\prod_{D \subset Y, D \not\subset C} \left(\left(\prod_{x \in D}{}' \mathcal{O}_{x,D} \right) / \hat{\mathcal{O}}_D \right) \right) / \left(\prod_{x \in U} \hat{\mathcal{O}}_x \right)\,,$$

and $\prod'$ means an adelic (on algebraic surface) product, the ring $\mathcal{O}_{x,D}$ is the product of discrete valuation rings from the ring $K_{x,D}$ (for example, if $K_{x,D} = \mathbb{F}_{q^r}((u))((t))$, then $\mathcal{O}_{K_{x,D}} = \mathbb{F}_{q^r}((u))[[t]]$), the ring $\hat{\mathcal{O}}_D$ is the discrete valuation ring in K_D.

Groups G_1 and G_2

The group G_1 is compact, since the group

$$\hat{\mathcal{O}}_x \simeq \mathbb{F}_{q^r}[[u, t]]$$

is compact, and for a fixed $D \subset Y$ the group

$$\left({\prod_{x \in D}}' \mathcal{O}_{x,D} \right) / \hat{\mathcal{O}}_D \simeq \left(\mathbb{A}_D / \mathbb{F}_q(D) \right) [[t_D]]$$

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The group G_2 has also an explicit presentation. We note only that in the simplest case

$$Y = \mathbb{P}^1_{\mathbb{F}_q} \times_{\mathbb{F}_q} \mathbb{P}^1_{\mathbb{F}_q}, \qquad C = \mathbb{P}^1_{\mathbb{F}_q} \times y + y \times \mathbb{P}^1_{\mathbb{F}_q}$$

(y is a fixed $\mathbb{F}_q$-rational point on $\mathbb{P}^1_{\mathbb{F}_q}$) we have

$$G_2 \simeq \mathbb{F}_q((u))((t)) / \left(\mathbb{F}_q[u^{-1}]((t)) + \mathbb{F}_q((u))[t^{-1}] \right) .$$

Arithmetic surfaces

An arithmetic surface X (with morphism $X \to \operatorname{Spec} \mathbb{Z}$) is “not compact” (since $\operatorname{Spec} \mathbb{Z}$ is “not compact”).
To have the good analogy with projective algebraic surface Y we have “ to compactify” X.
For this goal we have to take into account the fibre over the ∞-point of $\operatorname{Spec} \mathbb{Z}$, which corresponds to the Archimedean valuation of $\mathbb{Q}$.

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Let a curve $X_{\mathbb{Q}} = X \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{Q}$ be the fibre over the generic point of $\operatorname{Spec} \mathbb{Z}$.
We define a ring of arithmetic adeles

$$\mathbb{A}_X^{\mathrm{ar}} = \mathbb{A}_X \oplus \left(\mathbb{A}_{X_{\mathbb{Q}}} \hat{\otimes}_{\mathbb{Q}} \mathbb{R} \right),$$

where the ring

$$\mathbb{A}_{X_{\mathbb{Q}}} \hat{\otimes}_{\mathbb{Q}} \mathbb{R} = \varinjlim_{D_1 \in \operatorname{Div}(X_{\mathbb{Q}})} \varprojlim_{D_2 \leq D_1} \prod_{u \in X_{\mathbb{Q}}} \left(\hat{\mathcal{O}}_u(D_1) / \hat{\mathcal{O}}_u(D_2) \right) \otimes_{\mathbb{Q}} \mathbb{R}.$$

Subgroups of the group of arithmetic adeles

We define subgroups of the group $\mathbb{A}_{012}^{\mathrm{ar}} = \mathbb{A}_{\mathrm{X}}^{\mathrm{ar}}$ as

$$\mathbb{A}_{01}^{\mathrm{ar}} = \mathbb{A}_{01} \, , \qquad \mathbb{A}_{02}^{\mathrm{ar}} = \mathbb{A}_{\mathrm{X},02} \oplus (\mathbb{Q}(\mathrm{X}) \otimes_{\mathbb{Q}} \mathbb{R}) \, ,$$

where the group in the first formula is naturally embedded into the both summands for $\mathbb{A}_{012}^{\mathrm{ar}}$,

the first summand in the second formula is naturally embedded into the first summand (scheme part) for $\mathbb{A}_{012}^{\mathrm{ar}}$,

the second summand in the second formula is naturally embedded into the second summand (Archimedean part) for $\mathbb{A}_{012}^{\mathrm{ar}}$.

Quotient group

We fix a “horizontal divisor” C on X. Then $U = X \setminus C$ is affine, and “C plus Archimedean fibre” is an analog of an ample divisor on a projective algebraic surface.

Theorem

There is an exact sequence of compact groups:

$$0 \longrightarrow \tilde{G}_1 \longrightarrow \mathbb{A}_{012}^{\mathrm{ar}}/(\mathbb{A}_{01}^{\mathrm{ar}} + \mathbb{A}_{02}^{\mathrm{ar}}) \longrightarrow \tilde{G}_2 \longrightarrow 0\,,$$

where $\tilde{G}_1$ is written analogously to the group G_1 for an algebraic surface (but we have to take into account the fields $\mathbb{R}((t))$ and $\mathbb{C}((t))$ which come from the Archimedean part of $\mathbb{A}_{012}^{\mathrm{ar}}$).

The group $\tilde{G}_2$

Let us give the description of the group $\tilde{G}_2$

Example

In the simplest case when $X = \mathbb{P}^1_{\mathbb{Z}}$ and C is a hyperplane section, we have that

$$\tilde{G}_2 \simeq \mathbb{R}((t))/\left(\mathbb{Z}((t)) + \mathbb{R}[t^{-1}]\right) .$$

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Proposition

For an arithmetic surface X we have $\tilde{G}_2 \simeq \varprojlim_{m \geq 0} \Theta_m$, where the group

$$\Theta_m \simeq H^1(X, \mathcal{O}_X(-mC)) \otimes_{\mathbb{Z}} \mathbb{T} \simeq \mathbb{T}^{\mathrm{rank}(H^1(X, \mathcal{O}_X(-mC)))}$$

is a finite direct product of copies of $\mathbb{T}$ (we recall that $\mathbb{T} \simeq \mathbb{R}/\mathbb{Z}$).