Limiting forms of difference Little Picard theorems [1]

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The Fourth Sino-Russian Conference in Mathematics

The University of Hong Kong (15th – 19th, October)

19th October 2018

[1]Research partially supported by Hong Kong Research Grant Council

Outline

Painlevé Property

- Let $R(z, \cdot, \cdot)$ be rational in y, y', and the coefficients are analytic in z. Picard proposed to determine the forms of R in

$$\frac{d^2y}{dz^2} = R(z, y, y') = \frac{P(z, y, y')}{Q(z, y, y')}$$

such that ALL solutions have *only* fixed critical points, where P, Q are of the forms $\sum_{I=(i_1, i_2)} a_I y^{i_1} (y')^{i_2}$.

- The above assumption is equivalent to: All solutions are single-valued around all movable singularities.

- This is the criterion used by Painlevé, Gambier *et al* (1893–1906) to find all the possible R. The criterion is now called the Painlevé property.

- Accordingly, 50 classes of R: 6 of them are new.

Painlevé Property

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- Accordingly, 50 classes of R: 6 of them are new.

Painlevé equations

- $\mathrm{P_I}$: $y'' = 6y^2 + z$
- $\mathrm{P_{II}}$: $y'' = 2y^3 + zy + \alpha$
- $\mathrm{P_{III}}$: $y'' = \dfrac{y'^2}{y} - \dfrac{1}{z}y' + \dfrac{1}{z}(\alpha y^2 + \beta) + \gamma y^3 + \dfrac{\delta}{y}$
- $\mathrm{P_{IV}}$: $y'' = \dfrac{y'^2}{2y} + \dfrac{3y^3}{2} + 4zy^2 + 2(z^2 - \alpha)y + \dfrac{\beta}{y}$
- $\mathrm{P_V}$: $$y'' = [\frac{y'^2}{2y} + \frac{y'^2}{y-1}] - \frac{y'}{z} + \frac{(y-1)^2}{z^2}(\alpha + \frac{\beta}{y}) + \frac{\gamma y}{z} + \frac{\delta y(y+1)}{y-1}$$
- $\mathrm{P_{VI}}$: $$y'' = \frac{1}{2}(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t})(y')^2 - (\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t})y' + \frac{y(y-1)(y-t)}{t^2(t-1)^2}(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2})$$

Which are Integrable discrete Eqns?

- What are *integrable difference equations*

$$x_{n+1}+x_{n-1}=R(n;\,x_{n-1},\,x_n)=\frac{P(n,\,x_{n-1},\,x_n)}{Q(n,\,x_{n-1},\,x_n)}?$$

where R is rational in $x_{n-1},\,x_n$, coefficients in n

- Grammaticos, Ramani and Papageorgiou (e.g. Phys. Rev. Lett. 1991): *Singularity Confinement Property* that successfully identified a large number of integrable difference equations that arise from physical applications (e.g. 2D-quantum field theory)
- **E.g.** It is generally regarded that the Eqn

$$x_{n+1}+x_{n-1}=\frac{(\alpha n+\beta)x_n+\gamma}{1-x_n^2},$$

denoted by $\mathrm{dP_{II}}$, is a discrete analogue of $\mathrm{P_{II}}$

Some discrete Painlevé equations

- $\mathrm{dP_I}$: $x_{n+1} + x_n + x_{n-1} = \dfrac{\alpha n + \gamma(-1)^n}{x_n} + \delta$;
- $\mathrm{dP_{VI}}$: $(x_{n+1} + x_n)(x_n + x_{n-1}) = \dfrac{(x_n^2 - \kappa^2)(x_n^2 - \mu^2)}{(x_n + \alpha n + \beta)^2 - \gamma^2}$
- $\mathrm{a - dP_I}$: $\dfrac{(\alpha n + \beta)}{x_{n+1} + x_n} + \dfrac{(\alpha(n-1) + \beta)}{x_n + x_{n-1}} = -x_n^2 + \gamma$;
- $\mathrm{a - dP_I}$: $x_{n+1} + x_{n-1} = \dfrac{\alpha n + \beta}{x_n} + \dfrac{\gamma}{x_n^2}$;
- $\mathrm{a - dP_{II}}$:
  $$\frac{\alpha n + \beta}{x_{n+1}x_n + 1} + \frac{\alpha(n-1) + \beta}{x_n x_{n-1} + 1} = -x_n + \frac{1}{x_n} + (\alpha n + \beta) + \gamma.$$

Discrete to continuous

- Philosophy: In general, consider

$$y(z+1) - y(z) = h\,F(y(z)).$$

- Change of variables:

$$y(z) = u(x), \quad x = hz.$$

So

$$\frac{u(x+h) - u(x)}{h} = F(u(x)).$$

Letting

$$h \to 0 \quad \Rightarrow \quad \frac{du}{dx} = F(u(x)).$$

- But $x = hz$, so $h \to 0 \Rightarrow z \to \infty$.
- So *local property (finite difference)* is being "transferred" to ∞: Nevanlinna theory applies.

Discrete Eqns verse Functional Eqns

- $$\text{Discrete variable} \longrightarrow \text{Continuous variable}$$
- $\mathrm{dP_{II}}$
$$y_{n+1} + y_{n-1} = \frac{(\alpha n + \beta) y_n + \gamma}{1 - y_n^2}$$
- $\longrightarrow$
$$y(z+1) + y(z-1) = \frac{(\alpha z + \beta) y(z) + \gamma}{1 - y(z)^2}.$$

Difference Painlevé test

- Grammaticos, Ramani & Papageorgiou (1991): *Singularity Confinement Property*: $a, b, \infty, c, d, \infty, \infty, e, f \cdots$. If *finite values always return*, then it is *integrable*.
- Conte and Mussette (1996): *Discrete Painlevé Test*.
- Veselov (1992): The *integability* has an essential correlation with the *weak growth* of certain characteristics. (Arnold (1991)).
- Ablowitz, Halburd and Herbst (2000): Finite order of growth at infinity via the Nevanlinna Theory.
- Shimomura (1981), Yanagihara (1985): there are large classes of 1st- and 2nd-order difference equations that admit (global) meromorphic solutions.

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Malmquist's theorem

- L. Fuchs (1884): Let the Eqn.

$$y'(z) = \frac{P(z, y)}{Q(z, y)} \qquad (1)$$

where P, Q are polynomials in y with coefficients analytic in z. If *all the solutions* of (1) have only fixed critical points (i.e., Painlevé property), then it must reduce to a Riccati Eqn.:

$$y'(z) = p_2(z)y^2 + p_1(z)y + p_0(z). \qquad (2)$$

- Malmquist (1913): If the DE (1) admits *a transcendental meromorphic solution*, then it reduces to a Riccati eqn (1)
- K. Yosida (1933) gave a very simple "Nevanlinna proof".

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Nevanlinna characteristics

- Nevanlinna introduces the *Nevanlinna characteristics* $T(r,\, f)$ to replace *maximum modulus function* $M(r,\, f) = \max_{|z|=r} |f(re^{i\theta})|$, and $T(r, f) \sim \log M(r, f)$ for f entire.
- $n(r, f) := \#\ \big(\text{poles of } f(z) \text{ in } |z| < r\big).$
- $$\begin{aligned} T(r,f): &= m(r,f) + N(r,f) \\ &:= \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})|d\theta + \int_0^r \frac{n(t,f)}{t}dt. \end{aligned}$$
- Abbreviation: for arbitrary $a \in \mathbb{C}$
$$N(r,\, a) = N\Big(r,\, \frac{1}{f-a}\Big)$$
- $T(r,\, f)$ is a convex function of $\log r$, $T(r,\, f) \uparrow \infty$ as $r \uparrow \infty$.

Nevanlinna order

- Let f be entire,

$$T(r,f) \sim \log M(r,f).$$

- When f is meromorphic, its order $\sigma(f)$ is defined by

$$\sigma(f) = \overline{\lim_{r\to\infty}} \frac{\log T(r,f)}{\log r} = \begin{cases} 0 & \text{(zero order) e.g., polynomial} \\ < +\infty & \text{(finite order) e.g.,} e^{\text{polynomial}} \\ +\infty & \text{(infinite order) e.g., } e^{e^z} \end{cases}$$

2nd order difference Malmquist's Thm

Theorem (Halburd-Korhonen (2007))

If the Eqn.

$$y(z+1)+y(z-1)=R(z,\,y)$$

where R is rational in y and polynomial in z, has a finite order transcendental meromorphic solution., then either y satisfies $\overline{y}=(\overline{p}y+q)/(y+p)$ or after a linear transformation

- $\overline{y}+y+\underline{y}=\dfrac{\pi_1 z+\pi_2}{y}+\kappa_1;$

  $\overline{y}-y+\underline{y}=\dfrac{\pi_1 z+\pi_2}{y}+(-1)^z\kappa_1;$

- $\overline{y}+\underline{y}=\dfrac{\pi_1 z+\pi_3}{y}+\pi_2; \qquad \overline{y}+\underline{y}=\dfrac{\pi_1 z+\kappa_1}{y}+\dfrac{\pi_2}{y^2};$

- $\overline{y}+\underline{y}=\dfrac{(\pi_1 z+\kappa_1)y+\pi_2}{(-1)^{-z}-y^2}; \quad \overline{y}+\underline{y}=\dfrac{(\pi_1 z+\kappa_1)y+\pi_2}{1-y^2};$

- $\overline{y}y+y\underline{y}=p$; $\overline{y}+\underline{y}=py+q$ *where p, q are polynomials and π_k, κ_k are periodic of period $k\,(=1,2,3)$.*

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- $\overline{y}y+y\underline{y}=p;\ \overline{y}+\underline{y}=py+q$ *where* $p,\ q$ *are polynomials and* $\pi_k,\ \kappa_k$ *are periodic of period* $k\,(=1,2,3)$.

A discrete Clunie lemma

- Theorem (Halburd-Korhonen (2006))

  Let $f(z)$ is a finite order σ meromorphic solution to the difference equation

$$f^n P(z, f) = Q(z, f)$$

  where both P and Q are difference polynomials of $f(z)$ and its shifts such that the total degree of $Q \leq n$. Then for each $\varepsilon > 0$

$$m(r, P) = O(r^{\sigma - 1 + \varepsilon})$$

  holds for all r without exceptional set (written in C.-Feng format).

  - This lemma is crucial in establishing the conjecture of Ablowitz, Halburd & Herbst.
  - The lemma is refined by Laine and Yang (2007). There are several versions now.

A logarithmic difference lemma

- Theorem (Halburd-Korhonen (2006), C.-Feng (2008))
  Let $f(z)$ be a meromorphic function of finite order σ. Then for each $\varepsilon > 0$, we have

$$m\Big(r, \frac{f(z+1)}{f(z)}\Big) = O(r^{\sigma-1+\varepsilon}). \tag{3}$$

  - This is generally *not true* for infinite order functions.
  - This essentially means that

$$m(\frac{f(z+1)}{f(z)}, r) = o(T(r, f)) \quad \text{a.e.} \tag{4}$$

  - Nevanlinna's original estimate:

$$f(z+1)/f(z) \longrightarrow f'(z)/f(z),$$

  so that (4) holds without growth order restriction !!

Little Picard's Theorem

- Theorem (Picard (1879))

  An entire function f *assumes every value in* $\mathbb{C}$, *except perhaps for at most one exception*

  (E.g. $f(x) = e^x$.)

  - Method: Elliptic modular functions and Liouville's theorem.
  - Thus for an non-constant meromorphic function f

  $$f(\mathbb{C}) = \hat{\mathbb{C}} \setminus \{\text{at most two points}\}.$$

  That is, a meromorphic function that omits three points must reduce to a *constant*.

  - We say points in $\hat{\mathbb{C}}$ that are missed or assumed only finitely many times by f a *Picard exceptional values*.

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Nevanlinna Theory I

- Key inequality I: Given $a_1, a_2 \in \mathbb{C}$,

$$T(r, f) < N(r, f) + N(r, a_1) + N(r, a_2) - N_1(r, f) \qquad (5)$$
$$+ O\big(r \log T(r, f)\big), \quad r \to \infty \ (\notin E)$$

where

$$N_1(r, f) = N(r, 1/f') + 2N(r, f) - N(r, f').$$

- z_0 is a pole of f:

$$\text{contrib. of } N(r, f) - N_1(r) = N(r, f) - 2N(r, f) + N(r, f')$$
$$= -N(r, f) + N(r, f') = 1;$$

- z_0 is a a_j-point ($j = 1, 2$) of f:

$$\text{contrib. of } N(r, a_j) - N_1(r) = N(r, a_j) - N(r, 1/f')$$
$$= N(r, a_j) - N(r, 1/(f - a_j)') = 1;$$

Nevanlinna Theory II

- Key inequality II: Given $a_1, a_2 \in \mathbb{C}$,

$$\begin{aligned} T(r, f) < \overline{N}(r, f) + \overline{N}(r, a_1) + \overline{N}(r, a_2) \\ + O\big(r \log T(r, f)\big), \quad r \to \infty \ (\notin E) \end{aligned} \tag{6}$$

where

$$\begin{aligned} \overline{N}(r, f) &= \text{counts each pole with multiplicity } 1, \\ \overline{N}(r, a_j) &= \text{counts each } a_j\text{-point with multiplicity } 1 \end{aligned}$$

- Multiply $\frac{-1}{T(r,f)}$ and add 3 on both sides:

$$\Big(1 - \frac{\overline{N}(r, f)}{T(r, f)}\Big) + \Big(1 - \frac{\overline{N}(r, a_1)}{T(r, f)}\Big) + \Big(1 - \frac{\overline{N}(r, a_2)}{T(r, f)}\Big) + o(1) \le 3 - 1$$

$r \to \infty \ (\notin E)$

Nevanlinna Theory III

- $$\Big(1-\frac{\overline{N}(r,\,f)}{T(r,\,f)}\Big)+\Big(1-\frac{\overline{N}(r,\,a_1)}{T(r,\,f)}\Big)+\Big(1-\frac{\overline{N}(r,\,a_2)}{T(r,\,f)}\Big)+o(1)\le 2$$

  $r\to\infty\ (\not\in E)$
- If f misses $\infty,\ a_1,\ a_2$, then the above becomes

  $$3+o(1)\approx\big(1-o(1)\big)+\big(1-o(1)\big)+\big(1-o(1)\big)\le 2.$$

  A contradiction and thus proves the Little Picard Theorem.
- Nevanlinna deficiency at a:

  $$0\le\Theta(a)=1-\limsup_{r\to\infty}\frac{N(r,\,a)}{T(r,\,f)}\le 1$$

Difference Variations

- We re-interpret the followings:

(i) constants belong to $\ker\left(\frac{d}{dx}\right)$

(ii) f has three Picard values a, b, c means

$$f^{-1}(a) = \emptyset, \quad f^{-1}(b) = \emptyset, \quad f^{-1}(c) = \emptyset.$$

$\rightarrow$

(I) functions belong to $\ker\,(\text{some difference operator})$

(II)

$$f^{-1}(a) \neq \emptyset, \quad f^{-1}(b) \neq \emptyset, \quad f^{-1}(c) \neq \emptyset.$$

but each lies on a *specific periodic sequences*.

- Halburd-Korhonen (2006): $\Delta f(x) = f(x+1) - f(x)$,
- Chiang-Feng (2008, 2018): *Askey-Wilson operator* $\mathcal{D}_q f(x)$;
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Difference-type Picard theorem

Theorem (Halburd-Korhonen (2006))

If f is a finite-order meromorphic function that admits three difference Picard values with separation η, then $0 \equiv \Delta f(z) := f(z+\eta) - f(z)$. That is, f is a periodic function of period η.

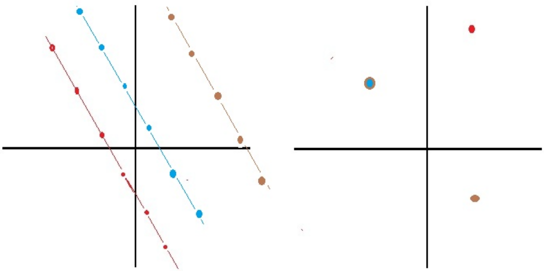

Figure: The left-side represents three preimages (any two consecutive points differ by η) of the right-side.

Varying-step difference operator

It is natural to ask if we could recover the classical Picard theorem by letting the steps/periods

- $\omega \to \infty$ so that the periodic sequences (the preimage) of difference Picard theorem become sparse and "finally almost disappeared" when interpreted appropriately:

  $$\Delta_\omega f(x) = f(x+\omega) - f(x)$$

  where $\omega \to \infty$;
- or $c \to 0$ so that the period of the periodic function from

  $$\Delta_c f(x) = f(x+c) - f(x)$$

  shrinks to zero so the a periodic function of zero-period is a constant.

Proximity function with Infinite periods

Theorem (C.-Luo (2017))

Let $f(z)$ be a meromorphic function of finite order σ, $0 < \beta < 1$ and $0 < |\omega| < r^{\beta}$. Then given $0 < \varepsilon < (1-\beta)/(2-\beta)$, we have

$$\begin{aligned} m\Big(r, \frac{f(z+\omega)}{f(z)}\Big) &= O\big(r^{\sigma-(1-\beta)(1-\varepsilon)+\varepsilon}\big) \\ &= o\big(T(r, f)\big) \end{aligned} \tag{7}$$

holds for r outside a set of finite logarithmic measure.

Infinite period theory I

- Key inequality I': Given $a_1, a_2 \in \mathbb{C}$. The log-difference lemma above leads to

$$T(r, f) < N(r, f) + N(r, a_1) + N(r, a_2) - N_{\Delta_\omega}(r, f) \quad (8)$$
$$+ O\big(r^{\sigma-(1-\beta)(1-\varepsilon)+\varepsilon}\big), \quad r \to \infty$$

where

$$N_{\Delta_\omega}(r, f) = N(r, 1/\Delta_\omega f) + 2N(r, f) - N(r, \Delta_\omega f).$$

- The main task here is to find an analogue $\widetilde{N}_{\Delta_\omega}(r, f)$ for $\overline{N}(r, f)$ for the $\Delta_\omega-$operator.

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Infinite period theory II

- Our aim is to find a correct $\widetilde{N}_{\Delta_\omega}(r, f)$ so that

$$T(r, f) < \widetilde{N}_{\Delta_\omega}(r, f) + \widetilde{N}_{\Delta_\omega}(r, a_1) + \widetilde{N}_{\Delta_\omega}(r, a_2) + O\big(r^{\sigma-(1-\beta)(1-\varepsilon)+\varepsilon}\big), \quad r \to +\infty,$$

where the *varying steps-integrated counting fns* are defined by

-
$$\widetilde{N}_{\Delta_\omega}(r, a) = \widetilde{N}_{\Delta_\omega}\Big(r, \frac{1}{f-a}\Big) = \int_0^r \frac{\tilde{n}_{\Delta_\omega}(t, a)}{t}\, dt,$$

and

$$\widetilde{N}_{\Delta_\omega}(r, \infty) := \widetilde{N}_{\Delta_\omega}(r, f) = \int_0^r \frac{\tilde{n}_{\Delta_\omega}(t, f)}{t}\, dt.$$

The above are the analogues for the $\overline{N}(r, a)$ and $\overline{N}(r, f)$ respectively.

Infinite period theory II

- Our aim is to find a correct $\widetilde{N}_{\Delta_\omega}(r,f)$ so that

$$T(r,\, f) < \widetilde{N}_{\Delta_\omega}(r,f) + \widetilde{N}_{\Delta_\omega}(r,a_1) + \widetilde{N}_{\Delta_\omega}(r,a_2) + O\big(r^{\sigma-(1-\beta)(1-\varepsilon)+\varepsilon}\big), \quad r \to +\infty,$$

  where the *varying steps-integrated counting fns* are defined by

- $$\widetilde{N}_{\Delta_\omega}(r,a) = \widetilde{N}_{\Delta_\omega}\Big(r,\, \frac{1}{f-a}\Big) = \int_0^r \frac{\tilde{n}_{\Delta_\omega}(t,a)}{t}\, dt,$$

  and

$$\widetilde{N}_{\Delta_\omega}(r,\infty) := \widetilde{N}_{\Delta_\omega}(r,\, f) = \int_0^r \frac{\tilde{n}_{\Delta_\omega}(t,f)}{t}\, dt.$$

  The above are the analogues for the $\overline{N}(r,\, a)$ and $\overline{N}(r,\, f)$ respectively.

Varying-step $a-$points counting functions I

We define the *Verying steps counting function* of f

$$\tilde{n}_{\Delta_\omega}(r, a) = \tilde{n}_{\Delta_\omega}\Big(r, \frac{1}{f-a}\Big) = \sum_{\substack{|x|<r,\\ h=\text{ multiplicity of } f(x)=a,\\ k=\text{ multiplicity of } \Delta_\omega f(x)=0}} (h-k)$$

over all x in $\{|x|<r\}$ where $h=h(x)$ is the *multiplicity* of the $a-$points of $f(x)$, and $k=k(x)$ is the *multiplicity* of the $0-$point of $\Delta_\omega f(x)$, respectively.

Varying-steps type pole counting functions II

Similarly, we define

$$\begin{aligned}\tilde{n}_{\Delta_\omega}(r,\infty) &= \tilde{n}_{\Delta_\omega}\Big(r,\frac{1}{f}=0\Big)\\ &= \sum_{\substack{|x|<r,\\ h=\text{ multiplicity of } 1/f(x)=0,\\ k=\text{ multiplicity of } \Delta_\omega(1/f)(x)=0}} (h-k)\end{aligned}$$

over all x in $\{|x|<r\}$, where $h=h(x)$ is the *multiplicity* of the zeros of $1/f(x)$, and $k=k(x)$ is the *multiplicity* of zeros of $\Delta_\omega(1/f)(x)$.

Varying-step Nevanlinna Deficiency

- We have

$$0 \le \Theta_{\Delta_\omega}(a) = 1 - \varlimsup_{r\to\infty} \frac{\widetilde{N}_{\Delta_\omega}(r,\, a)}{T(r,\, f)} \le 1$$

- Let $f(z)$ be a meromorphic function of finite order σ, we call $a \in \widehat{\mathbb{C}}$ to be a *Picard exceptional value for a varying-steps difference operator with infinite period* if $\widetilde{N}_{\Delta_\omega}(r,\, 1/(f-a)) = O(1)$
- If a is a *Picard-value for a varying-steps difference operator with infinite periods*, then $\Theta_{\Delta_\omega}(a) = 1$,

Varying-step Nevanlinna Deficiency

- We have

$$0 \le \Theta_{\Delta_\omega}(a) = 1 - \overline{\lim_{r\to\infty}} \frac{\widetilde{N}_{\Delta_\omega}(r, a)}{T(r, f)} \le 1$$

- Let $f(z)$ be a meromorphic function of finite order σ, we call $a \in \widehat{\mathbb{C}}$ to be a *Picard exceptional value for a varying-steps difference operator with infinite period* if $\widetilde{N}_{\Delta_\omega}(r, 1/(f-a)) = O(1)$
- If a is a *Picard-value for a varying-steps difference operator with infinite periods*, then $\Theta_{\Delta_\omega}(a) = 1$,

Difference Picard theorem with infinite periods

Theorem (C.-Luo (2017))

Let $f(z)$ be a meromorphic function of finite order σ. Suppose f has three Picard exceptional values with respect to a varying-steps difference operator Δ_ω with infinite period. Then $f(z)$ is a constant.

Proximity function with vanishing periods

Theorem (C.-Luo (2017))

Let $f(z)$ *be a meromorphic function in* $\mathbb{C}$ *and* $r=|z|$ *be fixed. We have*

$$\lim_{\eta\to 0} m_\eta\left(r,\frac{f(z+\eta)}{f(z)}\right)=0. \tag{9}$$

Moreover, if we further assume $0<|\eta|<\alpha_1(r)$, *where*

$$\alpha_1(r)=\min\left\{\log^{-\frac{1}{2}} r, 1/\left(n(r+1)\right)^2\right\},\ \ n(r)=n(r,f)+n\left(r,1/f\right). \tag{10}$$

Then

$$\lim_{r\to\infty} m_\eta\left(r,\frac{f(z+\eta)}{f(z)}\right)=0. \tag{11}$$

Difference Picard theorem with vanishing periods

Definition

We call $a \in \widehat{\mathbb{C}}$ is a *Picard exceptional value for varying-steps difference operator with vanishing period* of $f(z)$ if there is a sequence $\eta_n \to 0$ as $n \to \infty$ such that $\widetilde{N}_{\Delta_{\eta_n}}(r, 1/(f-a)) = O(1)$.

Theorem ((C.-Luo (2017))

Let $f(z)$ be a meromorphic function having three Picard exceptional values for varying-steps difference operator Δ_η with vanishing period. Then $f(z)$ is a constant.

Re-formulation of logarithmic derivative lemma

We give an alternative derivation of Nevanlinna's original logarithmic derivative lemma for finite order meromorphic function

$$m\Big(r, \frac{f'}{f}\Big) = O(\log r)$$

for all $r \to \infty$ via a formal limiting process:

$$m\Big(r, \frac{1}{\eta}\Big(\frac{f(z+\eta)}{f(z)} - 1\Big)\Big) \longrightarrow m\Big(r, \frac{f'}{f}\Big), \qquad \eta \to 0.$$

Limits

Theorem (C.- Luo (2017))

Let $f(z)$ be a meromorphic function of finite order σ, then

$$\lim_{r\to\infty}\lim_{\eta\to 0} m_\eta\Big(r, \frac{1}{\eta}\Big(\frac{f(z+\eta)}{f(z)}-1\Big)\Big) = O(\log r), \quad when \quad \sigma \geq 1; \tag{12}$$

$$\lim_{r\to\infty}\lim_{\eta\to 0} m_\eta\Big(r, \frac{1}{\eta}\Big(\frac{f(z+\eta)}{f(z)}-1\Big)\Big) = O(1), \quad when \quad \sigma < 1. \tag{13}$$

These results can be considered as Nevanlinna's original estimate for finite order meromorphic functions.

Thank you for your attention !!

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