On the Rokhlin type formulae of Spin geometry

Haibao Duan

Institue of Mathematics, CAS

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The plan of the talk

The **special orthogonal groups** are the compact connected Lie groups with the matrix presentations

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Arrangement of the talk:

1. The Spin characteristic classes;
2. The classical Rokhlin formula;
3. The Rokhlin type formulae (in the Spin characteristic classes);
4. Applications to Spin geometry.

The Spin characteristic classes

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- Haefliger (1956) found that the second Stiefel Whitney class $w_2(M)$ is the only obstruction to the existence of a spin structure on an orientable Riemannian manifold M.
- This was extended by Borel and Hirzebruch (1958) to cases of vector bundles, and by Karoubi (1968) to the non-orientable pseudo-Riemannian manifolds.

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We can then introduce **the spin characteristic classes** for a spin vector bundle ξ over a space X with classifying map $f : X \to B_{Spin(n)}$ by setting

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Remark. The classical Chern characteristic classes for the complex vector bundles, and the Pontryagin characteristic classes for the oriented real vector bundles can be defined in the same way.

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- Thomas (1962) calculated the integral cohomology of $B_{Spin(\infty)}$ in the stable range, but the result was subject to the choice of two sets $\{\Phi_i, \Psi_i\}$ of indeterminants.
- In the context of Weyl invariants, a description of the integral cohomology $H^*(B_{Spin(n)})$ was formulated by Benson and Wood (1995), where explicit generators and relations are absent:

  "*We have not set about the rather daunting task of using this description to give explicit generators and relations*"

The Spin characteristic classes

for the respective algebras $H^*(B;Z)$ and $H^*(B;Z_2)$. Set

$$D^i = D \cap H^i(B;Z), D_2^i = D_2 \cap H^i(B;Z_2) \quad (i \geq 0).$$

We shall prove

THEOREM (1.2). *There are cohomology classes* $\{Q_i\}$, $\{\Phi_i\}$, $\{\Psi_i\}$ $(i \geq 1)$ *with the following properties:*

(1.3) $Q_i \in H^{4i}(\hat{B},Z), \Phi_i \in D^{4i} \subset H^{4i}(B;Z), \quad \Psi_i \in D_2^i \subset H^i(B;Z_2).$

(1.4) *If i is not a power of 2, then*

$$Q_i = \pi^* P_i, \quad \Phi_i = 0, \quad \Psi_i = 0.$$

Let $j = 2^r$, *for* $r = 0, 1, \cdots$. *Then*

(1.5) $$\pi^* P_{2j} = 2Q_{2j} + Q_j^2 - \pi^* \Phi_{2j}, \quad \pi^* P_1 = 2Q_1 ;$$

(1.6) $$\rho_2(Q_j) = \pi^*(W_{4j} + \Psi_{4j}), \quad \rho_2(\Phi_j) = (\Psi_{2j})^2.$$

Moreover,

(1.7) $$H^*(\hat{B};Z) = Z[Q_1, Q_2, \cdots] \oplus \hat{T}, \quad \textit{where } 2\hat{T} = 0.$$

Furthermore, if $\{Q_i'\}(i \geq 1)$ *is a second set of cohomology classes satisfying*

11. PIECING TOGETHER THE INTEGRAL COHOMOLOGY

We calculated $H^*(B\mathrm{Spin}(n); \mathbb{Z})/\mathrm{torsion}$ in the last section. We saw that the torsion is killed by multiplication by two, and so we have a pullback diagram

$$\begin{array}{ccc} H^*(B\,\mathrm{Spin}(n);\mathbb{Z}) & \rightarrow & H^*(B\,\mathrm{Spin}(n);\mathbb{Z})/\mathrm{torsion} \\ \downarrow & & \downarrow \\ \mathrm{Ker}\,\beta & \rightarrow & H_\beta. \end{array}$$

Thus we have the following.

THEOREM 11.1. *$H^*(B\,\mathrm{Spin}(n); \mathbb{Z})$ is isomorphic to the subring of*

$$\mathrm{Ker}\ \beta \oplus H^*(B\mathrm{Spin}(n);\mathbb{Z})/\mathrm{torsion}$$

consisting of pairs of elements with the same image in H_β.

We have not set about the rather daunting task of using this description to give explicit generators and relations.

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The Spin characteristic classes

Theorem (Duan, 2018). The integral cohomology of B_{Spin} has the presentation

$$H^*(B_{Spin}) = \mathbb{Z}[Q_1, Q_2, Q_3, \cdots] \oplus \pi^*\tau(B_{SO})$$

where the generators $Q_k, degQ_k = 4k$, are *characterized uniquely* by the following properties:

i) if $k > 1$ is not a power of 2, then $Q_k = \pi^* p_k$.
ii) if $k = 2^r$ then
$$\rho_2(Q_k) = \pi^*(w_2^{(k+1)}),\ 2Q_{2k} + Q_k^2 = \pi^* f(w_2^{(k+1)}).$$

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Definition. For a Spin bundle ξ over a space X with classifying map $f : X \to B_{Spin}$, **the k^{th} Spin classes** of ξ is

$$q_k(\xi) := f^*(Q_k),\ k \geq 1.$$

The classical Rokhlin formula

Theorem (Poincare duality) For any oriented $4m$ dimensional manifold M its intersection form

$$H^{2m}(M) \times H^{2m}(M) \to H^{4m}(M) = \mathbb{Z}$$

is an integral unimodular symmetric matrix A, whose signature will be denoted by σ_M.

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Remark. These results motivated the classifications on simply connected 4 dimensional manifolds due to Freedman **(Topological)** and Donaldson **(smooth)** in the 1980's.

The generalized Rokhlin formulae

For a smooth manifold M the total Pontryagin characteristic class $p(M)$ is defined to be that of the tangent bundle TM on M, and will be denoted by

$$p(M) := 1 + p_1 + \cdots + p_k,\ k = \left[\frac{\dim M}{4}\right].$$

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Similarly, if M is spin (i.e. $w_2(M) = 0$) the total Spin characteristic class is defined, and will be denoted by

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In addition, the relations on the ring $H^*(Spin(n))$ suffices for us to express the Pontryagin classes by the Spin classes, such as

$$p_1 = 2q_1,\ p_2 = 2q_2 - q_1^2,\ p_3 = q_3,\ p_4 = 2q_4 - q_2^2,\ \cdots.$$

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$$\alpha_m = a_m \cdot p_m + l_m(p_1, \cdots, p_{m-1}) \text{ and}$$
$$\tau_m = b_m \cdot p_m + k_m(p_1, \cdots, p_{m-1}),$$

where a_m and b_m are certain non–zero rationals.

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where a_m and b_m are certain non–zero rationals. These allow us to eliminate p_m to obtain the following expression of τ_m without involving p_m:

Rokhlin type formula in Pontryagin classes

$$\tau_m = \frac{b_m}{a_m}(\alpha_m - l_m(p_1, \cdots, p_{m-1})) + k_m(p_1, \cdots, p_{m-1}).$$

The generalized Rokhlin formulae

For $1 \leq m \leq 4$ the polynomials α_m and τ_m are

$$\alpha_1 = -\frac{1}{24} p_1;$$
$$\alpha_2 = \frac{1}{2^7 \cdot 3^2 \cdot 5}(-4p_2 + 7p_1^2)$$
$$\alpha_3 = \frac{1}{2^{10} \cdot 3^3 \cdot 5 \cdot 7}(-16p_3 + 44p_2 p_1 - 31p_1^3)$$
$$\alpha_4 = \frac{1}{2^{15} \cdot 5^2 \cdot 3^4 \cdot 7}(-192p_4 + 512 \cdot p_1 p_3 + 208p_2^2 - 904p_1^2 p_2 + 381p_1^4)$$

and

$$\tau_1 = \frac{1}{3} p_1;$$
$$\tau_2 = \frac{1}{3^2 \cdot 5}(7p_2 - p_1^2);$$
$$\tau_3 = \frac{1}{3^3 \cdot 5 \cdot 7}(62p_3 - 13p_2 p_1 + 2p_1^3);$$
$$\tau_4 = \frac{1}{3^4 \cdot 5^2 \cdot 7}(381p_4 - 71 \cdot p_1 p_3 - 19p_2^2 + 22p_1^2 p_2 - 3p_1^4),$$

respectively.

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$\tau_1 = -8 \cdot \alpha_1$ *(dim=4)*;
$\tau_2 = q_1^2 - 2^5 \cdot (2^3 - 1) \cdot \alpha_2$ *(dim=8)*;
$\tau_3 = \frac{2}{3}(q_1 q_2 - 2q_1^3) - 2^7 \cdot (2^5 - 1) \cdot \alpha_3$ *(dim=12)*;
$\tau_4 = \frac{2}{3 \cdot 5} q_1 q_3 + \frac{1}{3^2} q_2^2 - \frac{2 \cdot 5}{3^2} q_1^2 q_2 + \frac{2 \cdot 31}{3^2 \cdot 5} q_1^4 - 2^9 \cdot (2^7 - 1) \cdot \alpha_4$.

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Corollary. In accordance to $m = 1, 2, 3, 4$, the signature σ_M of an almost parallelizable smooth manifold M^{4m} is divisible, respectively, by

$$2^4,\ 2^5 \cdot (2^3 - 1),\ 2^8 \cdot (2^5 - 1),\ 2^9 \cdot (2^7 - 1).$$

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Remark. Atiyah-Singer Index Theorem, 1963: The $\widehat{A}$ genus α_m of a smooth spin manifold M^{4m} is equal to the index of its Dirac operator.

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i) $M - \{o\}$ *admits a smooth structure for some point* $o \in M$*;*
ii) there is a basis $\{x_1, \cdots, x_n\}$ *on* $H^4(M^8)$ *so that the corresponding intersection form is* A*;*

Applications

The formulae can also applied to study the existence problem of smooth structures on certain triangulable topological manifolds.

Definition. For a symmetric unimodular integral matrix $A = (a_{ij})_{n \times n}$ of rank n, and a sequence $b = (b_1, \cdots, b_n)$ of n integers, we say the pair (A, b) to be **a Wall pair** if the following constraints are satisfied:

$$a_{ii} \equiv b_i \ mod 2, \ 1 \leq i \leq n.$$

Theorem (C.T.C. Wall,1962). *For each Wall pair (A, b) there exists a unique 3-connected topological manifold M^8 that satisfies the following properties*

i) $M - \{o\}$ admits a smooth structure for some point $o \in M$;
ii) there is a basis $\{x_1, \cdots, x_n\}$ on $H^4(M^8)$ so that the corresponding intersection form is A;
iii) the first Spin characteristic class of M is

$$q_1 = b_1 x_1 + \cdots + b_n x_n \in H^4(M^8).$$

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Theorem 1. *Let M^8 be the manifold associated to a Wall pair (A, b). There exists a smooth structure on M^8 extending the one on $M - \{o\}$ if and only if*

$$sign(A) \equiv bAb^{\tau} \mod 2^5 \cdot (2^3 - 1),$$

where b^{τ} denotes the transpose of the row vector b.

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Theorem 2. *If* (A, b) *is a Wall pair so that*

$$sign(A) \neq bAb^{\tau} \ mod 2^5 \cdot (2^3 - 1),$$

then the corresponding manifold M *does not admit any smooth structure.*$\square$

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Theorem 3. *If the manifold M associated to a Wall pair (A, b) admits a smooth structure, then its total Spin characteristic class is*

$$q(M) = 1 + (b_1x_1 + \cdots + b_1x_n) + \frac{45 \cdot sign(A) + 11 \cdot bAb^{\tau}}{2 \cdot (2^3 - 1)} \cdot \omega_M$$

where $2 \cdot (2^3 - 1)$ *divides* $45 \cdot sign(A) + 11 \cdot bAb^{\tau}$.

Applications

Theorem 4. *For a manifold M associated to a Wall pair (A, b) the following statements are equivalent:*

i) M is smooth and admits a metric with positive scalar curvature
ii) $sign(A) = bAb^{\tau}$;
iii) $q_2(M) = 4sign(A) \cdot \omega_M$.$\square$

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i) M is smooth and admits a metric with positive scalar curvature
ii) $sign(A) = bAb^{\tau}$;
iii) $q_2(M) = 4sign(A) \cdot \omega_M$. $\square$

Therefore, the problem of finding all the 3 connected 8 dimensional smooth manifolds that have a metric with positive scalar curvature is equivalent to the number theoretical problem of finding those Wall pairs (A, b) satisfying the quadratic equation ii) above.

Thanks!