An additive problem with coefficients of automorphic functions and related problems

Irina Rezvyakova

Steklov Mathematical Institute, Moscow

The 4th Sino-Russian Conference in Mathematics
October, 15-19, 2018
HKU

Additive problem

The additive problem is the question of getting some nontrivial information about the sum

$$S = \sum_{\substack{m_2 n_2 - m_1 n_1 = l, \\ n_1 \le N}} \overline{a(n_2)} a(n_1)$$

when $N \to +\infty$, where

1. $a(\cdot)$ is an arithmetic function,
2. m_1, m_2, l are positive integers, $(m_1, m_2) = 1$.

In general we may assume that m_1, m_2, l can grow with N, i.e.

$$m_1 m_2 l \le N^{\delta} \quad \text{for some} \quad 0 < \delta < 1.$$

A trivial result

Some trivial result can be obtained by the Cauchy inequality using the estimate for the square mean of the coefficients $a(\cdot)$. Set for simplicity $m_1 = m_2 = l = 1$ to be fixed positive integers. Then

$$|S| \leq \sum_{\substack{n_2-n_1=1,\\ n_1\leq N}} |\overline{a(n_2)}a(n_1)| = \sum_{n\leq N} |\overline{a(n+1)}a(n)| \leq \sum_{n\leq N+1} |a(n)|^2$$

For example,

1. $\zeta^2(s) = \sum_{n=1}^{+\infty} \frac{d(n)}{n^s} \to \quad a(n) = d(n) := \sum_{d_1 d_2 = n} 1, \quad S \ll N \log^3 N,$
2. $L(s) = \sum_{n=1}^{+\infty} \frac{a(n)}{n^s} \to \quad a(n) \ll_\varepsilon n^\varepsilon \quad S \ll_\varepsilon N^{1+2\varepsilon}.$

The property

$$a(n) \ll_\varepsilon n^\varepsilon$$

is usually called the Ramanujan conjecture.

What is a nontrivial result

If one can estimate from above $|S|$ by

$$S \ll N^{1-\delta}, \quad \delta > 0$$

(or, to prove an asymptotic formula with such error term) then it is said that a *nontrivial* result is obtained.

Direct application

For simplicity, let us put aside the case with nonzero main term. Then the estimate $S \ll N^{1-\delta}$ gives an analytic continuation of the series

$$L(s) = \sum_{n=1}^{+\infty} \frac{\overline{a(n+1)}a(n)}{n^s}$$

to the region $\mathrm{Re}\, s > 1-\delta$ by the Abel's partial summation formula

$$\sum_{a<n\le b} c(n)F(n) = -\mathbb{C}(b)F(b) + \int_a^b \mathbb{C}(u)dF(u),$$

если $F \in C^1[a,b]$, $\mathbb{C}(u) = \sum_{a<n\le u} c(n)$.

Direct application

Let $\mathrm{Re}\, s > 1$, $M > 1$, $c(n) = \overline{a(n+1)}a(n)$, $F(u) = u^{-s}$. Then , $\mathbb{C}(u) = O(u^{1-\delta})$ and

$$\sum_{1\le n\le M} \frac{\overline{a(n+1)}a(n)}{n^s} = -\mathbb{C}(M)M^{-s} - s\int_1^M \frac{\mathbb{C}(u)}{u^{s+1}}du.$$

Now put $M \to +\infty$ to obtain that

$$L(s) = -s\int_1^{+\infty} \frac{\mathbb{C}(u)}{u^{s+1}}du,$$

is analytic in $\mathrm{Re}\, s > 1-\delta$.

Another application

For $L(s) = \sum_{n=1}^{+\infty} \frac{a(n)}{n^s}$, the problem of evaluating the integral

$$\int_0^T |L(1/2+it)|^2 dt$$

gives rise to an additive problem with the coefficients $a(n)$, since as in the case of the Riemann zeta-function we can (roughly speaking) use an approximation to $L(s)$ via finite Dirichlet series:

$$L(1/2+it) = \sum_{n \le A(t)} \frac{a(n)}{n^{1/2+it}} + \dots$$

Exploring the mean-value estimates

If we know the asymptotic formula like

$$\int_0^T |L(1/2+it)|^2 dt = TP(\log T) + O(H^{1+\varepsilon}), H = T^{1-\delta}$$

then we can get a bound

$$\int_T^{T+H} |L(1/2+it)|^2 dt = O(H^{1+\varepsilon})$$

and by the following formula

$$|L(1/2+iT)|^2 \ll \log T \left(1 + \int_{T-H}^{T+H} |L(1/2+it)|^2 dt \right)$$

the estimate

$$L(1/2+it) \ll |t|^{\frac{1-\delta}{2}+\varepsilon}.$$

Exploring the mean-value estimates

For example, for $\zeta(s)$ we have from the mean-value formula the bound

$$\zeta(1/2+it) \ll |t|^{1/6+\varepsilon}$$

which is by now improved not much using the estimates for special exponential sums as (J. Bourgain – N. Watt, 2017)

$$\zeta(1/2+it) \ll |t|^{1/6-0.01\ldots}$$

though the Riemann hypothesis implies the estimate $\zeta(1/2+it) \ll |t|^{\varepsilon}$.

Exploring the mean-value estimates

Estimates for the number of zeros of analytic functions in certain domains are also very important. By Littlewood's lemma, the mean-value estimate is connected to the number of zeros of $L(s)$ in a rectangle.

Let $L(\sigma+it)$ be analytic and nonzero on rectangle γ with verticies $\sigma_0, \sigma_1, \sigma_1+iT, \sigma_0+iT$. Then

$$2\pi \sum_{\rho\in\gamma} dist(\rho) = \int_0^T \log|L(\sigma_0+it)|dt - \int_0^T \log|L(\sigma_1+it)|dt$$
$$+\int_{\sigma_0}^{\sigma_1} \arg L(\sigma+iT)d\sigma - \int_{\sigma_0}^{\sigma_1} \arg L(\sigma)d\sigma.$$

But we can write a bound

$$\int_0^T \log|L(\sigma_0+it)|dt = \frac{1}{2}\int_0^T \log|L(\sigma_0+it)|^2 dt \le \frac{T}{2}\log\left(\frac{1}{T}\int_0^T |L(\sigma_0+it)|^2\right).$$

The Riemann zeta-function

The Riemann zeta-function

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \quad \mathrm{Re}\, s > 1$$

For $\mathrm{Re}\, s > 1$

$$\zeta^{-1}(s) = \prod_p \left(1 - \frac{1}{p^s}\right) = \sum_{n=1}^{+\infty} \frac{\mu(n)}{n^s}, \quad \mu(n) = \begin{cases} 1, & n = 1, \\ 0, & p^2 | n, \\ (-1)^t, & n = p_1 \cdot \ldots \cdot p_t. \end{cases}$$

The Riemann zeta-function

Therefore, for $\mathrm{Re}\, s = \sigma > 1$

$$|\zeta^{-1}(s)| < \sum_{n=1}^{+\infty} \frac{1}{n^{\sigma}} < 1 + \int\limits_{1}^{+\infty} \frac{du}{u^{\sigma}} = \frac{\sigma}{\sigma - 1}$$

,

and, consequently,

$$|\zeta(s)| > \frac{\sigma - 1}{\sigma} \neq 0.$$

The Riemann zeta-function

$\zeta(s)$ has meromorphic continuation to the whole complex-plane and satisfies the functional equation

$$\xi(s) := s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s) = \overline{\xi(1-\bar{s})},$$

where $\xi(0) = \xi(1) = 1$.
The Riemann zeta-function has a simple pole at $s = 1$.
$\zeta(s)$ has *trivial zeros* at $s = -2, -4, \ldots$.
nontrivial zeros of the Riemann zeta-function are the zeros of $\xi(s)$. They lie in the *critical strip* $0 \leq \mathrm{Re}\, s \leq 1$ and symmetric over the real axis and the line $\mathrm{Re}\, s = 1/2$ (*the critical line*).

The Riemann zeta-function

The Riemann hypothesis asserts that all nontrivial zeros of $\zeta(s)$ lie on the critical line $\mathrm{Re}\, s = 1/2$.

(I.M. Vinogradov) There are no zeros of $\zeta(\sigma + it)$ in the region

$$\sigma \geq 1 - O\left((\log t)^{-2/3}(\log\log t)^{-1/3}\right), t \geq 10.$$

Selberg class

In 1989 A. Selberg introduced a class of Dirichlet series $L(s)$ which satisfy the following properties:

Selberg class

1. the series $L(s) = \sum_{n=1}^{+\infty} \frac{a(n)}{n^s}$ are absolutely convergent for $\mathrm{Re}\, s > 1$,
2. $(s-1)^m L(s)$ is an entire function of finite order for some integer $m \geq 0$,
3. $a(1) = 1$, $a(n) \ll_\varepsilon n^\varepsilon$ for every $\varepsilon > 0$ and $n > 1$,
4. (Euler product) $\log L(s) = \sum_{n=1}^{+\infty} \frac{b(n)}{n^s}$, where $b(n) = 0$ except when n is of the form $n = p^r$ and, also, $b(n) = O(n^\delta)$ for some $\delta < 1/2$,
5. (Functional equation) $L(s)$ has a functional equation of the form

$$\Lambda(s) := H_L(s)L(s) = \overline{\Lambda(1-\overline{s})},$$

where $H_L(s) = \theta A^s \prod_{j=1}^{k} \Gamma(\lambda_j s + \mu_j)$ and $|\theta| = 1$, $A > 0$, $\lambda_j > 0$, $\mathrm{Re}\, \mu_j \geq 0$.

Degree of an element from Selberg class

Definition

$d = d_L := 2 \sum_{j=1}^{k} \lambda_j$ is the degree of $L(s)$.

Examples

1. $d = 1 : \zeta(s),\ L(s,\chi) = \sum_{n=1}^{+\infty} \frac{\chi(n)}{n^s}$.
2. $d = 2 : L(s,\psi)$, where ψ is the Hecke ideal calss group character for $\mathbb{Q}(\sqrt{-D})$.

Theorem

1. (H.E. Richert, 1957) There are no elements in the Selberg class for $0 < d < 1$.
2. (A. Perelli, J. Kaczorowski, 1999) The Selberg class of degree 1 consists only of $\zeta(s)$ and $L(s + iA, \chi)$.
3. (A. Perelli, J. Kaczorowski, 2002, 2011) The Selberg class for $1 < d < 2$ is empty.

Automorphic forms

The Ramanujan conjecture was proved by P. Deligne for a class of L-functions, namely, for $L_f(s) = \sum\limits_{n=1}^{+\infty} \frac{a_f(n)}{n^s}$ attached to a (holomorphic) automorphic cusp form $f(z)$ of integral weight $k \geq 1$ for the group $\Gamma_0(D) = \left\{\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 (\bmod D)\right\}$, and a character χ modulo D (briefly, $f \in S_k(\Gamma_0(D), \chi)$), which is an eigenfunction of all the Hecke operators. To be more precise, f is a holomorphic function on the $\mathbb{H} = \{\mathrm{Im}\, z > 0\}$ which vanishes at every cusp of $\Gamma_0(D)$ and

$$f(\gamma z) = \chi(\gamma)(cz+d)^k f(z) \quad \text{for every } \gamma \in \Gamma_0(D),$$

$$\text{where } \gamma z = \frac{az+b}{cz+d}, \quad \chi(\gamma) = \chi(d).$$

$$f(z) = \sum_{n=1}^{+\infty} a_f(n) n^{\frac{k-1}{2}} e^{2\pi i n z}, \quad \mathrm{Re}\, z > 0.$$

Automorphic forms

The Hecke operators T_n for $n = 1, 2, \ldots$ are defined by

$$T_n f(z) = \frac{1}{n} \sum_{ad=n} \chi(a) a^k \sum_{0 \le b < d} f\left(\frac{az+b}{d}\right).$$

Assume that f is an eigenfunction of all the Hecke operators and that $a(1) = 1$. Then for $L_f(s) = \sum_{n=1}^{+\infty} \frac{a_f(n)}{n^s}$ we have:

Euler product

$$L_f(s) = \prod_p \left(1 - \frac{a_f(p)}{p^s} + \frac{\chi(p)}{p^{2s}}\right)^{-1}$$

with $|a_f(n)| \le d(n)$.

Automorphic forms

$L_f(s)$ satisfies the following functional equation

$$\Lambda(s) = \theta \cdot \overline{\Lambda(1-\bar{s})},$$

where $\Lambda(s)$ is an entire function,

$$\Lambda(s) = \left(\frac{2\pi}{\sqrt{D}}\right)^{-s-\frac{k-1}{2}} \Gamma\left(s + \frac{k-1}{2}\right) L_f(s),$$

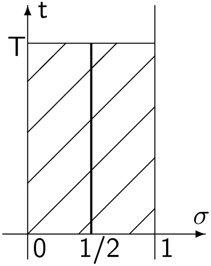
and $|\theta| = 1$.

Corollary

$L_f(s)$ is a function from Selberg class of degree 2.

Positive proportion theorem

The estimate for the second moment of $L(s)$ appears in the problems which are quite close to the RH.
Selberg conjectured that for $L(s)$ from class S an analogue of the Riemann hypothesis holds, i.e. all its nontrivial zeros lie on the critical line.



$$N(T) = \sum_{\substack{\rho=\sigma+it, \zeta(\rho)=0 \\ 0<t\leq T}} 1 \sim \frac{T}{2\pi} \ln T.$$

$$N_0(T) = \sum_{\substack{\rho=\frac{1}{2}+it,\ \zeta(\rho)=0 \\ 0<t\leq T}} 1,$$

G. Hardy, J. Littlewood (1921):

$N_0(T) \gg T.$

Positive proportion theorem

A. Selberg, 1942: a positive proportion of nontrivial zeros of $\zeta(s)$ lie on the critical line $\mathrm{Re}\, s = \frac{1}{2}$ (the same holds for $L(s, \chi)$).

N. Levinson (1974): more than one-third of zeros of Riemann's zeta-function are on $\sigma = 1/2$.

J.B. Conrey (1989): More than 2/5 of nontrivial zeros of the Riemann zeta-function lie on the critical line.

H.Bui, J.B.Conrey, M.Young and Sh. Feng (2010): More than 41% of the zeros of the zeta function are on the critical line.

Positive proportion theorem

$$\xi^{(k)}(s) = (-1)^k \xi^{(k)}(1-s).$$

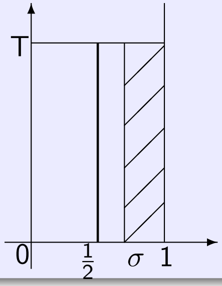
If $\xi(s) \neq 0$ for $\mathrm{Re}\, s > \sigma_0$ $(\sigma_0 \leq 1)$, then $\xi^{(k)}(s) \neq 0$ also.

J.B. Conrey (1983):

For $\xi^{(k)}(s)$ the estimate $N_0(T) = 1 - O(k^{-2})$ holds for $k \to +\infty$.

Selberg density theorem

A. Selberg (1946):



Let $N(\sigma, T)$ be the number of zeros of $\zeta(s)$ in the region $\mathrm{Re}\, s \geq \sigma$, where $1/2 \leq \sigma \leq 1$. Then there exists $a > 0$ such that the estimate

$$N(\sigma, T) \ll T^{1-a\left(\sigma-\frac{1}{2}\right)} \log T$$

holds.

Corollary:

almost all nontrivial zeros of $\zeta(\sigma + it)$ lie in the region $|\sigma - 1/2| \leq \frac{\Phi(t)}{\log(|t|+2)}$, where $\Phi(t) \to +\infty$ with $|t| \to +\infty$.

Value-distribution for $\zeta(1/2+it)$

Let $\varkappa_{a,b}$ be the characteristic function of the interval (a,b). Then

$$\int_{T}^{2T} \varkappa_{a,b}\left(\frac{\log|\zeta(1/2+it)|}{\sqrt{\pi \log\log T}}\right) dt = T \int_{a}^{b} e^{-\pi u^2} du + o(T).$$

A. Ghosh (1983):

the error term can be bounded as $O(T(\log\log\log T)^{-1/2})$.

K.-M. Tsang (1984):

the error term can be bounded as $O\left(T\frac{(\log\log\log T)^2}{\sqrt{\log\log T}}\right)$.

Selberg's theorem for degree 2 L-functions

J.L. Hafner, 1983: Let $L(s,f)=\sum_{n=1}^{+\infty}\frac{a(n)}{n^s}$ be an L-function attached to a holomorphic cusp form $f(z)=\sum_{n=1}^{+\infty}a(n)n^{\frac{k-1}{2}}e^{2\pi inz}$ of an even integral weight k for the full modular group with trivial character which is an eigenfunction of all the Hecke operators. Then a positive proportion of nontrivial zeros of $L(s,f)$ lie on the critical line.

I. Rezvyakova:

the same statement holds true for $f\in S_k(\Gamma_0(D),\chi)$, $k\geq 1$.

Density theorem for degree 2 L-functions

Selberg's density theorem holds for $L_f(s)$, where $f \in S_k(\Gamma_0(D), \chi)$.

Therefore, the value-distribution result holds for $L_f(1/2 + it)$:

Value-distribution result for $L_f(1/2+it)$

$$\int_T^{2T} \varkappa_{a,b}\left(\frac{\log|L_f(1/2+it)|}{\sqrt{\pi\log\log T}}\right)dt = T\int_a^b e^{-\pi u^2}du + o(T),$$

and (in case of independent $L_{f_1}(s)$ and $L_{f_2}(s)$)

$$\int_T^{2T} \varkappa_{a,b}\left(\frac{\log|L_{f_1}(1/2+it)| - \log|L_{f_2}(1/2+it)|}{\sqrt{2\pi\log\log T}}\right)dt$$

$$= T\int_a^b e^{-\pi u^2}du + o(T).$$

Scheme of the proof of Selberg's positive proportion theorem

Idea of detecting zeros of a continuous function

Let $F(t)$ be a real-valued function. Let's define

$$J(t) = \int_t^{t+H} |F(u)| du, \qquad I(t) = \left| \int_t^{t+H} F(u) du \right|.$$

If $J(t) > I(t)$ then F changes its sign on $(t, t+H)$.

Scheme of the proof of Selberg's positive proportion theorem

Set $E = \{t \in (T, 2T) : J(t) > I(t)\}$. Then it is easy to show that F has $\gg \mu(E)H^{-1}$ zeros on $(T, 2T)$.

For L-functions we want to establish that

$$\mu(E) \gg T \quad \text{for} \quad H \asymp (\log T)^{-1}.$$

Scheme of the proof of Selberg's positive proportion theorem

Define

$$M(t) = \int_t^{t+H} (F(u) - 1)\, du.$$

Then

$$J(t) = \int_t^{t+H} |F(u)| du \geq H - |M(t)|.$$

We want to show that for a subset of positive measure on $(T, 2T)$ (and $H \asymp \log^{-1} T$)

$$|M(t)| < H/3, \quad |I(t)| < H/3.$$

Scheme of the proof of Selberg's positive proportion theorem

Suppose that we have proved the estimates

$$\int_T^{2T} |I(t)|^2 dt = O\left(T\frac{H}{\ln T}\right), \int_T^{2T} |M(t)|^2 dt = O\left(T\frac{H}{\log T}\right).$$

Then

$$|I(t)| \leq \frac{H}{3} \quad \text{and} \quad |M(t)| \leq \frac{H}{3}$$

outside the subset of measure $O\left(\frac{T}{H\log T}\right)$. Choosing $H = \frac{A}{\log T}$, we get $\mu E \gg T$.

Scheme of the proof of Selberg's positive proportion theorem

To associate with $L(1/2+it)$ a real-valued function, we recall the functional equation

$$\Lambda(s) = \theta\overline{\Lambda(1-\overline{s})}$$

and take $F(t) = \theta^{-1/2}\Lambda(1/2+it)$, for example. The second moment gives rise to the series

$$D_{m_1,m_2}(s,l) = \sum_{n=1}^{+\infty} \frac{a(n)\overline{a(\frac{m_1 n + l}{m_2})}}{(m_1 n + \frac{l}{2})^s}$$

which we can continue to the left of the line $\mathrm{Re}\, s = 1$ using the nontrivial estimates in additive problem.

The idea of the proof for density theorem

The main inequlaity one should prove is the following

$$N := \int_{\sigma}^{1} (N(x, 2T) - N(x, T))dx \ll T^{1-a(\sigma-1/2)}.$$

By the lemma of Littlewood:

$$N = \frac{1}{2\pi} \int_{T}^{2T} (\log|F(\sigma+it)| - \log|F(3+it)|)dt + \frac{1}{2\pi} \int_{\sigma}^{3} (\arg|F(\sigma+2iT)| - \arg|F(3+iT)|)d\sigma.$$

By the inequality $\log x \le \frac{1}{2}(x^2-1)$, our task is to estimate

$$\int_{T}^{2T} (|F(\sigma+it)|^2 - 1)dt \ll T^{1-a(\sigma-1/2)}.$$

The idea of the proof for density theorem

Using Selberg's idea, for $\mathrm{Re}\, s > 1$ we can find Dirichlet polynomial $\eta(s)$ (which approximates $L^{-1}(s)$) such that

$$F(s) = L(s)\eta(s) = 1 + \sum_{n > T^{a_1}} \frac{\dots}{n^s},$$

An easy integration on the line $\sigma = 3/2$ gives

$$\int_T^{2T} (|F(\sigma + it)|^2 - 1)dt \ll T^{1-a(\sigma-1/2)}$$

.

Combining with the estimate on the line $\sigma = 1/2$, Gabriel's theorem gives us necessary bound

$$\int_T^{2T} (|F(\sigma + it)|^2 - 1)dt \ll T^{1-a(\sigma-1/2)}.$$

Distribution theorem

A. Selberg proved how accurate is the approximation

$$\log \zeta(1/2+it) = \sum_{p\le y} \frac{1}{p^{1/2+it}} + r_y(t)?$$

The result is

$$\int_T^{2T} |r_y(t)|^{2k} dt \ll T(Ak)^k, \int_T^{2T} |\operatorname{Re} \sum_{p\le y} \frac{1}{p^{1/2+it}}|^{2k} dt \asymp T(\log\log y)^k$$

Here the dencity theorem is used, and all the steps of the proof are similar to general L-function with Euler product.

Distribution theorem

The distribution theorem can be proved now by the method of characteristic functions and the Esseen inequality

$$\sup_x |G_1(x) - G_2(x)| \ll \int_T^{2T} \left| \frac{g_1(t) - g_2(t)}{t} \right| dt + \sup_x |G_2'|,$$

using for the characteristic function

$$g(t) = \frac{1}{T} \int_T^{2T} e^{it\omega_T(x)} dx, \text{ where } \omega_T(x) = \frac{\log \zeta(1/2 + it)}{\sqrt{\log \log T}},$$

the formula

$$e^{ix} = \sum_{j=0}^{N-1} \frac{(ix)^j}{j!} + O\left(\frac{|x|^N}{N!} \right).$$

Linear combinations of L-functions from Selberg class

All the previous theorems allows us to understand better the RH. If we consider general linear combination of L-functions from Selberg class that still satisfy a functional equation of Riemann type, then an analogue of RH is not true for them.

Examples

1. (Davenport-Heilbronn function) $g(s) = \varkappa_1 L(s, \chi_1) + \overline{\varkappa_1} L(s, \overline{\chi_1})$, where $\varkappa_1 \in \mathbb{C}$, χ_1 is the Dirichlet character mod 5, $\chi_1(2) = i$.
2. (The Epstein zeta-function) $\zeta(s, Q)$, cprresponding to positive definite form $Q(m, n) = am^2 + bmn + cn^2$ with integer coefficients of the fundamental discriminant $-D$ and the class number $h = h(-D) > 1$.

Linear combinations of L-functions from Selberg class

H. Davenport, H. Heilbronn (1936) and S.M. Voronin (1976)

For $\sigma > \frac{1}{2}$ let $N(\sigma, T)$ be the number of zeros of $\zeta(s, Q)$ in the region $\mathrm{Re}\, s > \sigma$, $0 < \mathrm{Im}\, s \leq T$. Then

$$N(\sigma, T) \asymp T.$$

Similar result holds for Davenport-Heilbronn function.

Though, if $N(T)$ is the number of zeros of $\zeta(s, Q)$ in the region $0 < \mathrm{Im}\, s \leq T$,then as in the case of the Riemann zeta-function

$$N(T) \asymp T \log T.$$

Linear combinations of L-functions from Selberg class

There is a

Hypothesis

Almost all nontrivial zeros of such linear combinations lie on the critical line.

E. Bombieri, D. Hejhal (1987):

Assuming GRH for $L(s, \psi)$ (where ψ is the ideal class group character for $\mathbb{Q}(\sqrt{-D})$) and pair correlation conjecture for the ordinates of zeros of $L(s, \psi)$, it is true that almost all zeros of $\zeta(s, Q)$ lie on the critical line.

Linear combinations of L-functions from Selberg class

Finally, if $N_0(T,L)$ is the number of L-function on $\{\frac{1}{2}+it, T<t\leq 2T\}$ then

For Davenport-Heilbronn function

A. Selberg, 1998-99: $N_0(T,g) \gg T\log T$.

I. Rezvyakova:

Positive proportion of nontrivial zeros of $\zeta(s,Q)$ lie on the critical line.

Additive problem for modular forms

Additive problem

$$S = \sum_{m_2 n_2 - m_1 n_1 = l, n_1 \le N} \overline{a(m)} a(n), \quad m_1 m_2 l \le N^{\delta}.$$

Additive problem for modular forms

$$\sum_{m-n=l, n \le N} \overline{a(m)} a(n) \ll R.$$

History:

A. Good (1982): $R \ll N^{2/3+\varepsilon}$ (holomorphic modular forms).
T. Meurmann (1987): $R \ll N^{2/3+\varepsilon}$ (Maass forms).
H. Iwaniec, J.B. Conrey (2001): $R \ll N^{8/9+\varepsilon}$ cusp forms for congruence groups.

The case am-bn = l:

J.L. Hafner (1983-87): both cases (implicit result).
V. Blomer, G. Harcos, P. Sarnak, P. Michel. E. Kowalski, J. Vanderkam (2001-2007): congruence subgroups.

Additive problem

Theorem

Let $N \gg 1$, $(m_1, m_2) = 1$, $m_1^9 m_2^{11} \le N^2$, $l \le N^{11/13}$. For any $\varepsilon > 0$ the following estimate holds:

$$S = O_\varepsilon \left(N^{\frac{5}{6} + \frac{1}{78} + \varepsilon} m_1^{9/13} m_2^{-2/13} \right),$$

Circle method

Theorem

Let $\delta(k) = \begin{cases} 1, & \text{if } k = 0, \\ 0, & \text{else.} \end{cases}$

Then

$$\delta(k) = \sum_{q \le Q < a^* \le q+Q} \sum^{*} \int_0^{\frac{1}{qa^*}} 2\cos(2\pi k(a/q - \alpha))d\alpha,$$

where $aa^* = 1(\mod q)$, $Q \ge 1$.

The idea of the circle method contains in the following formula:

$$\delta(k) = \int_0^1 e^{2\pi ikx}dx.$$

Estimate via spectral methods

For $f \in S_k(\Gamma_0(D), \chi_0)$ and $k > 2$

$$S = \sum_{1/2 < s_j < 1} c_j(l, m_1, m_2) N^{s_j} + O(N^{2/3+\varepsilon})$$

Therefore, we have $s_j \leq 11/13$ (as well as the lower bound for $\lambda_j = s_j(1 - s_j) \geq 22/169$), but this result is worse than the result of A. Selberg (1965):

$$\lambda_j \geq 3/16.$$

By the latest bound of H. Kim – P. Sarnak (2002):

$$s_j < 2/3.$$