Irreducible representations of finitely generated nilpotent groups

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Induced representations

Let G be a group and V a (possibly, infinite-dimensional) vector space over $\mathbb{C}$.

Let H be a subgroup of a group G and $\rho : H \to \mathrm{Aut}_{\mathbb{C}}(V)$ a representation of H.

For an element $g \in G$, let $H^g \subset G$ be the conjugate subgroup $H^g = gHg^{1}$ and ρ^g the representation of H^g defined by the formula $\rho^g(ghg^{1}) = \rho(h)$ for $h \in H$.

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We may finitely induce representation ρ of H on G. Then the representation space of $\mathrm{ind}_H^G(\rho)$ is the following.

Definition. $\mathrm{ind}_H^G(\rho) = \{f : G \to V \mid f(hg) = \rho(h)f(g)$, $supp(f)$ is a finite set of left cosets$\}$.

G action is defined by $g : f(g') \to f(g'g)$ (right translations).

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Classical results:

- Irreducible representations of finite nilpotent groups are monomial.
- (Kirillov, Dixmier 1959) Irreducible unitary (possibly, infinite-dimensional) representations of connected nilpotent Lie groups are monomial.

Definition. A representation π of a group G has finite weight if there is a subgroup $H \subset G$ and a character χ of H such that the vector space $\mathrm{Hom}_H(\chi, \pi|_H) = \{v \in V \mid \pi(h)v = \chi(h)v \text{ for any } h \in H\}$ is non-zero and finite-dimensional. In this case (H, χ) is called a weight pair for π.

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A category of unitary representations is too restrictive for arithmetic purposes. One should consider complex representations without any topological structure. It leads to numerous differences.

Motivation

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- There exists a theory of smooth representations of p-adic algebraic groups (Bernstein). This theory is also valid for a more general class of totally disconnected locally compact groups. Discrete groups are a simple particular case of this class of groups and the general theory delivers a reasonable class of representations, namely, representations on a vector space without any topology.

Parshin's conjecture

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Note, that if a representation π is monomial then the pair (H, χ) is defined not uniquely but still this condition is quite rigid (there are few pairs (H, χ) such that $\pi \simeq ind_H^G(\chi)$). On the other hand, the finite weight condition is much milder and easier to verify.

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A sharp distinction between Brown's setting and the conjecture is that Brown treats unitary representations, while the conjecture concerns complex representations without any topological structure. This leads to numerous differences, most notably, the following one.

- The category of unitary representations is semi-simple.
- On the other hand, there are non-trivial extensions between representations without a topological structure and, in general, the converse to Schurs lemma does not hold for such representations.

Particular cases of the conjecture

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- For finitely generated abelian groups the conjecture holds true, because all irreducible representations of such groups are just characters. It follows from Schur's lemma generalization for countable groups.
- For the next case of finitely generated nilpotent groups of nilpotency class two the conjecture was proved by Parshin.

Theorem (B., Gorchinskiy)

The conjecture is true in full generality both for finite and infinite-dimensional representations.

We prove a more general result on representations over an arbitrary field, which may be non-algebraically closed and may have a positive characteristic.

All the above theorems are particular cases of our theorem.

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It is easy to show one implication in the conjecture: if an irreducible complex representation is monomial, then it has finite weight.

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Proposition

Let H be a normal subgroup of an arbitrary group G. Let ρ be an irreducible complex representation of H such that the finitely induced representation $\mathrm{ind}_H^G(\rho)$ satisfies $\mathrm{End}_G\left(\mathrm{ind}_H^G(\rho)\right) = \mathbb{C}$. Then the representation $\mathrm{ind}_H^G(\rho)$ is irreducible.

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Proposition

Let H be a normal subgroup of an arbitrary group G. Let ρ be an irreducible complex representation of H such that the finitely induced representation $\mathrm{ind}_H^G(\rho)$ *satisfies* $\mathrm{End}_G\left(\mathrm{ind}_H^G(\rho)\right) = \mathbb{C}$*. Then the representation* $\mathrm{ind}_H^G(\rho)$ *is irreducible.*

Corollary. Suppose that G is a nilpotent group, H is a subgroup of G and ρ are as in the proposition. Then $\mathrm{ind}_H^G(\rho)$ is irreducible.

Proof. $G \supset G_1 = < H, G > \supset < H, [G, G] > \supset < H, [[G, G], G] > \supset \ldots \supset G_n = < H, \{e\} > = H$. Then G_i is normal in G_{i-1}. We proceed by induction on n.

Definition. Let $rad(H) = \{g \in G$ such that there exists such $i \in \mathbb{N}$ that $g^i \in H\}$.

Theorem (Mal'cev, 1949)

If G is a nilpotent finitely generated group then for any subgroup $H \subset G$ the set $rad(H) \subset G$ is a subgroup and $[rad(H) : H] < \infty$.

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Theorem (B., Gorchinskiy)

Let (H_1, χ_1) and (H_2, χ_2) be weight pairs such that $\mathrm{ind}_{H_1}^G(\chi_1)$ and $\mathrm{ind}_{H_2}^G(\chi_2)$ are irreducible. Then the following conditions are equivalent:

- $\mathrm{ind}_{H1}^G(\chi_1) \sim ind_{H2}^G(\chi_2)$
- *there exist $g \in G$ such that $(rad(H_2))^g := g\ rad(H_2)\ g^{-1} = rad(H_1)$ and $\chi_2^g|_{H_2^g \cap H_1} = \chi_1|_{H_2^g \cap H_1}$.*

Moduli space problem

Further study concerns the moduli space problem of irreducible representations with finite weight of finitely generated nilpotent groups.

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The moduli space problem is related to considering the following space

$$\mathbb{M}_G = \bigcup_{H \subset G} \mathbb{T}_{H/[H,H]} / \sim$$

where $\mathbb{T}_{H/[H,H]}$ denotes a complex torus of an abelinization of H and $\sim$ denotes factorisation over equivalence relation (isomorphisms of representations from previous slide).

Irreducible representations of the group of unipotent matrices of order 4 over integers

We construct a moduli space of irreducible representations with a finite weight of the group of unipotent 4×4 matrices over the ring of integers.
The moduli space has a natural iterated structure of a bundle over the space of admissible subgroups $H \subset G$.

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In particular, we obtain a full classification of pairs (H, χ) such that corresponding representations $\mathrm{ind}_H^G(\chi)$ are irreducible. Interestingly enough, the number of isomorphism classes of irreducible representations with a finite weight of discrete nilpotent groups increases very rapidly, while a nilpotency class increments only by one. For the Heisenberg group over the integers there are only two substantially different cases of pairs (H, χ), which correspond to irreducible monomial representations. In turn, there are over 50 different cases for the group G, which makes our classification quite technical and lengthy.

(See "Irreducible Representations of the Group of Unipotent Matrices of Order 4 over Integers", I. Beloshapka, IMRN, rny207, 2018 for details)

Thank you!