A global estimate of discrete Riemann mappings

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Holomorphic Functions

Holomorphic Functions

If $z = x + iy$, for any holomorphic function

$$f(z) = u(x, y) + iv(x, y)$$

by using the Cauchy-Riemann equation we have

$$f'(z) = u_x + iv_x = v_y - iu_y.$$

Considering f as a holomorphic function from U to V, where $U, V \subset \mathbb{C}$ are domains, then

$$Df = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix}.$$

If $f'(z) \neq 0$, then $u_x^2 + v_x^2 \neq 0$, Lengths are not usually preserved, but angles are. The action of the derivative at z_0 is multiplication by $f'(z_0)$.

Conversely, suppose that $f : U \to V$ is continuous, and continuously differentiable, and the derivative Df is invertible, has positive determinant and preserves angles.

Write $f(z) = (u(x, y), v(x, y))$. The Jacobi determinant

$$|Df| = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} > 0;$$

If angles are to be preserved then this must be of the form

$$Df = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{pmatrix}.$$

So the Cauchy-Riemann equations

$$u_x = v_y$$

$$v_x = -u_y,$$

are satisfied, and hence f is holomorphic, except possibly at finitely many points. But since the function f is continuous, any singularities are removable and f is holomorphic on the domain U.

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are satisfied, and hence f is holomorphic, except possibly at finitely many points. But since the function f is continuous, any singularities are removable and f is holomorphic on the domain U.

Therefore, we deduce that:

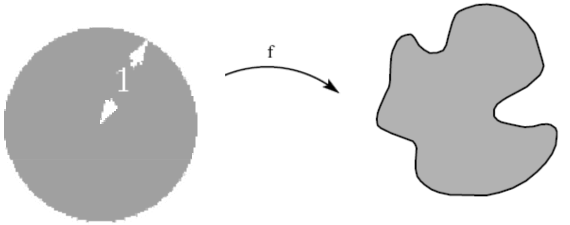
- A local holomorphic homeomorphism preserves angles.

- A holomorphic function maps each infinitesimal circle to an infinitesimal circle.

Riemann mapping theorem

Let $\Omega \subsetneq \mathbb{C}$ be a simply-connected domain of the complex plane with at least 2 boundary points, and $z_0 \in \Omega$. Then there is a unique conformal map $f : \{|z| < 1\} \to \Omega$ such that $f(0) = z_0$ and $f'(0) > 0$.

- Extended via "covering theory" to handle Riemann surfaces in full generality.
- A core topic in mathematics.
- Application in physics, engineering, visualization.



Measurable Riemann Mapping

The usual classical form of writing a Riemannian metric in the plane is

$$Edx^2 + 2Fdxdy + Gdy^2$$

where E, F, G are real-valued functions of (x, y), and the symmetric matrix

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

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$$E + G > 0, \;\; EG - F^2 > 0.$$

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The length of a curve $(x(t), y(t))$, where $t \in I$ is

$$L = \int_I \sqrt{E(\frac{dx}{dt})^2 + 2F(\frac{dx}{dt})(\frac{dy}{dt}) + G(\frac{dy}{dt})^2}$$

To introduce quasiconformal mappings, we write the metric $Edx^2 + 2Fdxdy + Gdy^2$ in another form:

$$Edx^2 + 2Fdxdy + Gdy^2 = \lambda(z)|dz + \mu d\bar{z}|^2, \quad z = x + yi,$$

where $\lambda > 0$ and $|\mu| < 1$.
The function μ is called the Beltrami differential (of the Riemannian metric).

Measurable Riemann Mapping Theorem (Gauss, Morrey, Ahlfors-Bers)

Suppose that $\mu \in L^\infty(\hat{\mathbb{C}})$ and $\|\mu\|_\infty < 1$, Then there exists a unique homeomorphism $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that f fixes $0, 1$ and ∞. It is differentiable a.e., with partial derivatives locally $\partial_z f, \partial_{\bar{z}} f \in L^2$, and

$$\partial_{\bar{z}} f = \mu(z) \cdot \partial_z f.$$

That is, for some function $\lambda(z) > 0$,

$$f^*(\lambda_0(w)|dw|^2) = \lambda(z)|dz + \mu d\bar{z}|^2,$$

where $\lambda_0(w)|dw|^2$ is the spherical metric in $\hat{\mathbb{C}}$.

Such a homeomorphism f is quasi-conformal. Its maximal dilatation is

$$K[f] = \frac{1 + ||\mu||_\infty}{1 - ||\mu||_\infty}.$$

In the case that the mapping $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is smooth, we consider the tangent map

$$f_* : T_p(\hat{\mathbb{C}}) = \mathbb{R}^2 \to T_f(p)(\hat{\mathbb{C}}) = \mathbb{R}^2.$$

This is a linear transformation. It maps a unit circle to an ellipse. Then

$$K_{f,p} = \frac{R}{r},$$

where R is the longest axis, and r is the shortest axis. Therefore

$$K[f] = \sup_{p \in \hat{\mathbb{C}}} K_{f,p}.$$

Circle Packing

The circle is arguably the most studied object in all of mathematics.

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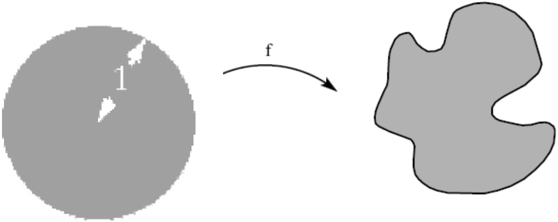
Recall that for any simply connected domain of the complex plane with at least 2 boundary points, we have the following classical Riemann mapping.



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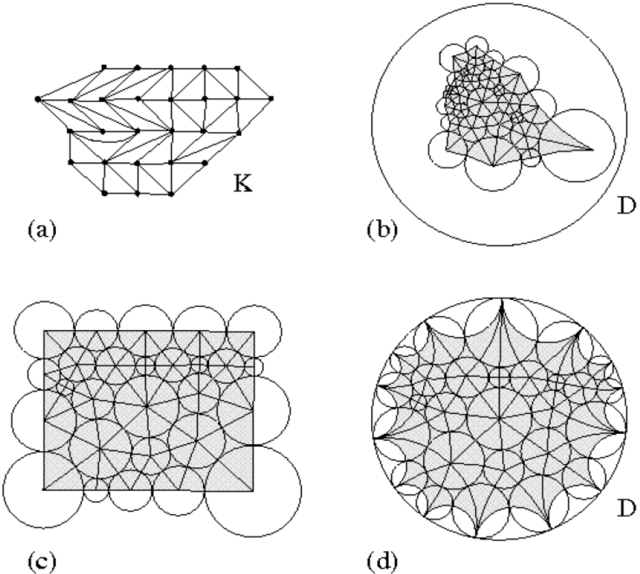
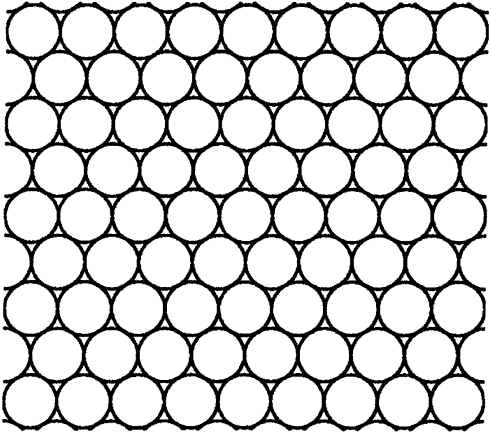
A finite circle packing P on the Riemann sphere $\hat{\mathbb{C}}$ is a configuration of circles with specified patterns of tangency.

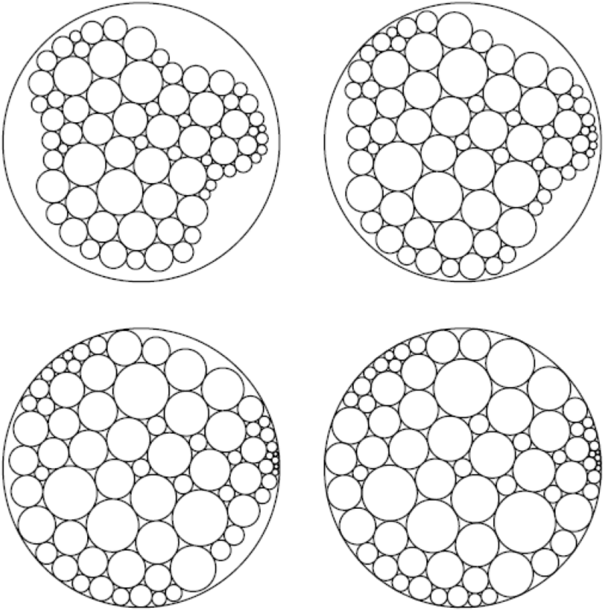
The contact graph $G_P = (V, E)$ of such a circle packing P is a graph whose vertices correspond to the circles in the packing, and an edge appears in G_P if and only if the corresponding circles are tangent to each other.

The following are circle packings and their contact graph. Please see some of the illustrations from the web-page of Prof. K. Stenphenson.



The following is a regular hexagonal packing of the complex plane $\mathbb{C}$.





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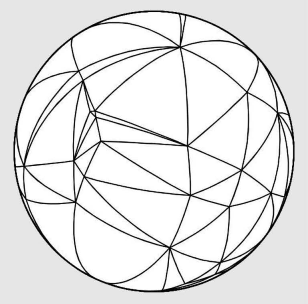
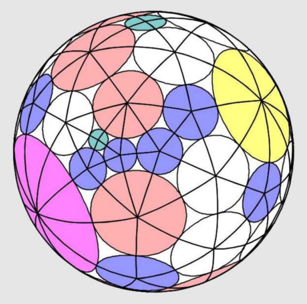
Theorem

For every triangulation G of the Riemann sphere, there is a circle packing P with graph (isomorphic to) G.

Moreover, P is unique up to Möbius transformation.

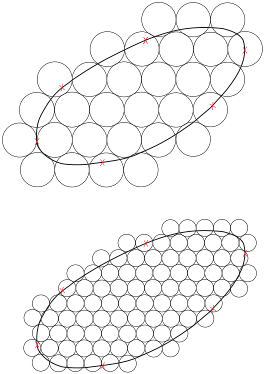


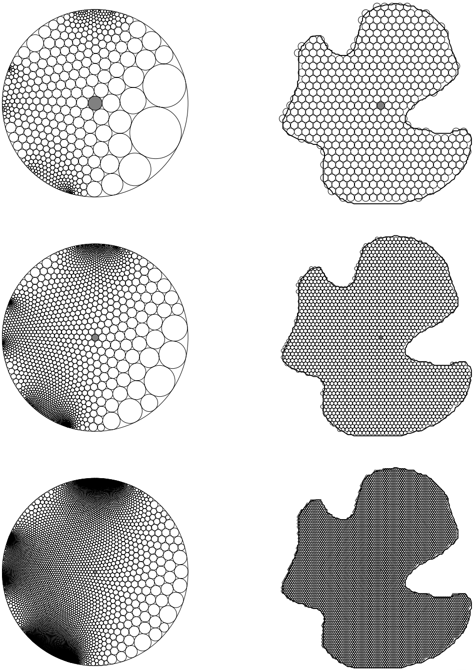


Discrete Riemann Mapping

For any domain $D \subset \mathbb{C}$ with at least 2 boundary points, lay down a regular hexagonal packing of circles in $\mathbb{C}$, say each of radius $1/n$.

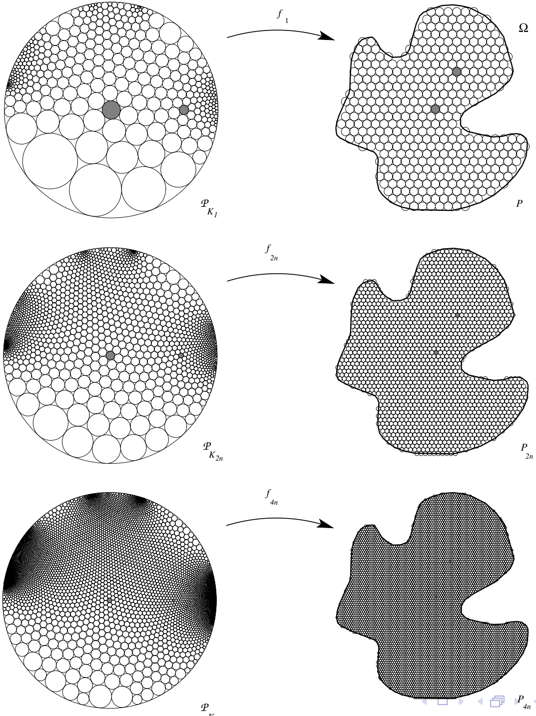
By using the boundary component ∂D like a cookie-cutter, we obtain a circle packing which consists of all the circles intersecting the closed region $\bar{D}$.





By using the results of Thurston, we obtain a circle packing of the Riemann sphere.

Therefore, we obtain a map between the unit disk and the given domain.


f_1
Ω
$\mathcal{P}_{K_1}$
P
f_{2n}
$\mathcal{P}_{K_{2n}}$
P_{2n}
f_{4n}
P_{4n}

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Rodin & Sullivan proved this result.

Rodin-Sullivan Theorem

Assume that the classical conformal mapping $f : \mathbb{D} \to \Omega$ and the discrete conformal mappings $f_n : \mathbb{D} \to \Omega$ are defined and normalized as described above. Then the mapping f_n converges uniformly on compact subset of $\mathbb{D}$ to f as $n \to +\infty$.

He-Schramm C^∞ convergence Theorem

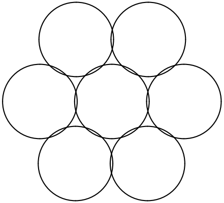
The discrete functions $f_n : \mathbb{D} \to \Omega$ converge in $C^\infty(\mathbb{D})$ to the classical conformal mapping $f : \mathbb{D} \to \Omega$.

Rigidity Constants s_n

The rigidity constants s_n plays an important role in the previous results.

An n-generations hexagonal circle packing H'_n is defined to be a circle packing combinatorially equivalent to the n-generations regular hexagonal packing H_n.

The following is H_1:



Let $c_k \in H_n, k = 1, 2, \cdots, 6$, be the 1st generation circles tangent to the center circle c_0 and let $c_1', c_2', \cdots, c_6'$ be the corresponding 1st generation circles in H_n'.
We define

$$s_n = \sup_{\{(H_n', c_0')\}} \max_{1 \le k \le 6} \left(\frac{\text{radius}(c_k')}{\text{radius}(c_0')} - 1 \right),$$

where $\{(H_n', c_0')\}$ runs over all n-generations hexagonal circle packings in $\mathbb{C}$. That is, they are combinatorially equivalent to H_n.

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The main known results about the rigidity constants s_n are summarized in the following results.
Theorem (Rodin-Sullivan). $s_n \to 0$ as $n \to \infty$.

Theorem (He). $s_n \le C/n$ for some constant C independent of n.

Theorem (Doyle-He-Rodin).

$$s_n = \frac{2\sqrt[3]{2}\Gamma^2(1/3)}{3\Gamma(2/3)} \frac{1}{n} + o\left(\frac{1}{n}\right).$$

.

We have

Theorem (He-L).

$$s_n = \frac{2\sqrt[3]{2}\Gamma^2(1/3)}{3\Gamma(2/3)}\frac{1}{n} + O\left(\frac{1}{n^2}\right).$$

Global Convergence

Suppose Ω is αJordan domain. By using the bycentric coordinates, we can construct an approximation map f_ϵ.

Theorem (He)

The circle packing solutions f_ϵ converge globally uniformly to the

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Furthermore, we suppose that Ω is a bounded and simply connected plane region with boundary $\partial\Omega \in C^\infty$. We can modify the approximation map f_ϵ.

Theorem (In preparation)

For the discrete conformal mappings $f_n : \mathbb{D} \to \Omega$, we have

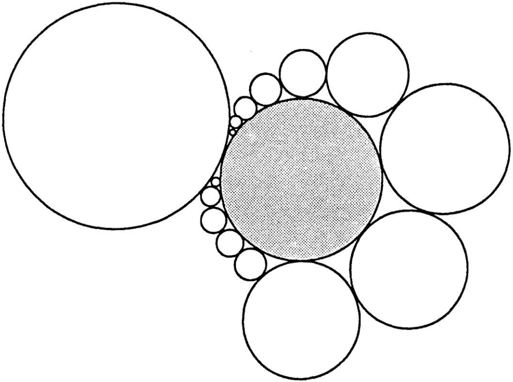
$$|f_n(z) - f(z)| = O\left(\frac{1}{\sqrt{n}}\right), \quad \forall z \in \mathbb{D}_n,$$

where $\mathbb{D}_n \subset \mathbb{D}$ and $d(\partial\mathbb{D}, \mathbb{D}_n) \leq \frac{1}{n}$.

Idea of the previous proofs

Ring Lemma (Rodin-Sullivan)

There is a constant r depending only on n such that if n circles surround the unit disk (i.e., they form a cycle externally tangent to the unit disk), then each circle has radius at least r.



Proof.
Fix n. There is a uniform lower bound for the radius of the largest outer circle c_1 (namely, that which occurs when all n outer circles are equal).

A circle c_2 adjacent to c_1 also has a uniform lower bound for its radius because if c_2 were extremely small then a chain of $n - 1$ circles starting from c_2 could not escape from the crevasse between c_1 and the unit circle.

Repeat this reasoning for the circle c_3 adjacent to c_2, and so on.

□

Lemma

Consider the rectangles $R = [0, m] \times [0, 1]$ *and* $R' = [0, m'] \times [0, 1]$. *Also we assume* $1/C < m,\ m' < C$ *for some* $C > 1$. *Let* $f : R \to R'$ *be a* K*-quasiconformal mapping with maximal dilatation* K*, which maps the corners of* R *to the corresponding corners of* R'.

If there exists an integer $n \geq 1$ *such that the Beltrami differential* $\mu = \mu_f$ *satisfies*

$$\iint_R |\mu(z)| dxdy \leq O(1/n),$$

then $|m - m'| \leq O(1/n)$.

Proof.
Let J_f be the Jacobian of f. For any $y \in [0, 1]$, then

$$m' = \int_0^m \frac{\partial f(x, y)}{\partial x} dx \le \int_0^m \left| \frac{\partial f(x, y)}{\partial x} \right| dx \le \int_0^m K_f^{1/2} J_f^{1/2} dx.$$

Squaring both sides of (**??**) and by applying the Schwartz inequality gives

$$\begin{aligned} (m')^2 &\le \left(\int_0^1 \int_0^m K_f^{1/2} J_f^{1/2} dxdy \right)^2 \\ &\le \int_0^1 \int_0^m K_f dxdy \cdot \int_0^1 \int_0^m J_f dxdy \\ &= m' \int_0^1 \int_0^m K_f dxdy. \end{aligned}$$

Since $K_f(z) - 1 = \dfrac{2|\mu(z)|}{1 - |\mu(z)|} \leq C_1|\mu(z)|$, we have

$$m' \leq \int_0^1 \int_0^m [1 + (K_f - 1)] dxdy \leq m + O(1/n).$$

Similarly, by considering the rectangles $[0,1] \times [0, \frac{1}{m}]$ and $[0,1] \times [0, \frac{1}{m'}]$,

$$1/m' \leq 1/m + O(1/n).$$

It follows by (**??**) and (**??**) that $|m - m'| \leq O(1/n)$. □

Lemma

Let f map $\mathbb{D}$ conformally onto the Jordan domain Omega with with smooth boundary. Then f' has a continuous extension to $\overline{\mathbb{D}}$ and,

$\forall \zeta, z \in \overline{\mathbb{D}}$,

$$\frac{f(\zeta) - f(z)}{\zeta - z} \to f'(z) \neq 0, \quad \zeta \to z.$$

Furthermore, $f^{(n)}$ has a continuous extension to $\overline{\mathbb{D}}$, where $n = 1, 2, \cdots$.

Thanks for your attention!