Kloosterman sums over prime numbers

Maxim A. Korolev[*]

[*]Steklov Mathematical Institute (Moscow, Russia)

Hong Kong University
15-19 October, 2018

Given $q \geqslant 2$, $(n, q) = 1$, by n^* we denote inverse residue to n modulo q, that is, the solution of the congruence

$$n^* n \equiv 1 \pmod{q}.$$

Other notations:

$$\overline{n}, \quad 1/n.$$

For integer a, b, *complete Kloosterman* sum $S(q; a, b)$ is defined as

$$\sum_{\substack{n=1 \\ (n,q)=1}}^{q} \exp\left(2\pi i \, \frac{an^* + bn}{q}\right) = \sum_{n \in \mathbb{Z}_q^*} e_q\big(an^* + bn\big).$$

Here, as usual, $\mathbb{Z}_q^*$ denotes reduced residual system modulo q.

Trivial cases: $a \equiv 0 \pmod{q}$ (or $b \equiv 0 \pmod{q}$). Indeed, one has

$$S(q;a,0) = \sum_{n\in\mathbb{Z}_q^*} e_q(an^*) = \sum_{n\in\mathbb{Z}_q^*} e_q(an) = c_q(a)$$

– *Ramanujan sum*,

$$c_q(a) = \frac{\varphi(q)}{\varphi(q/(q,a))}\,\mu\Big(\frac{q}{(a,q)}\Big),$$

$$\varphi(k) = \sum_{\substack{n=1\\(n,k)=1}}^{k} 1 \quad - \quad \textit{Euler totient function}$$

$$\mu(n) = \begin{cases} 1, & n=1, \\ 0, & n=p^2m, p>2 \\ (-1)^k, & n=p_1\cdots p_k \end{cases} \quad - \quad \textit{Möbius function}$$

Incomplete Kloosterman sum:

$$S(q;a,b;\mathcal{A}) = \sum_{n\in\mathcal{A}} e_q(an^* + bn), \quad \mathcal{A} \subset \mathbb{Z}_q^*, \quad \mathcal{A} \neq \mathbb{Z}_q^*.$$

Typical cases:

$$\mathcal{A} = \mathbb{Z}_q^* \bigcap [1, N], \quad \mathcal{A} = \mathbb{Z}_q^* \bigcap [M, M+N]$$

where $1 < N < q$.
More "exotic" case:

$$\mathcal{A} = \mathbb{Z}_q^* \bigcap \{p : p \leqslant N\}.$$

Applications:

Circle method (H.D. Kloosterman ect.):

$$N = ax_1^2 + bx_2^2 + cx_3^2 + dx_4^2, \quad x_j \in N.$$

Sieve methods (C. Hooly etc.):

$$\sum_{n \leqslant N} d(n^2 + a), \quad \prod_{n \leqslant N} (n^2 + 1), \quad \ldots$$

Continued fractions, Farey fractions (H. Heilbronn, A. Ustinov, D.A. Frolenkov etc.)

$$\sum_{a=1}^{q} s(a/q), \quad s(a/q) \quad - \quad \text{length of the expansion to c.f.}$$

Main problem: to obtain the inequality

$$|S(q; a, b; \mathcal{A})| \leqslant |\mathcal{A}|\Delta,$$

where $\Delta = \Delta(q; \mathcal{A}) \to 0$ as $q \to +\infty$.

Multiplicativity property : given $a, b, q = q_1 q_2$ s.t. $(q_1, q_2) = 1$ then

$$S(q; a, b) = S(q_1; a_1, b)S(q_2; a_2, b)$$

for some a_1, a_2.

Hence, it is enough to estimate for $q = p^n$, p – prime.

If $n \geqslant 2$, one has for $(a, q) = 1$:

$$|S(q; a, b)| \leqslant d(q)\sqrt{q}$$

The main difficulty: case $q = p$.

H.D. Kloosterman (1926), I.M. Vinogradov (1933), D.I. Tolev (2010), elementary method:

$$|S(p; a, b)| \leqslant 3^{1/4} p^{3/4}.$$

By standard trick and multiplicativity, this gives an estimate for the incomplete sum to composite modulo q:

$$\Bigl|\sum_{n \leqslant N} e_q(an^* + bn)\Bigr| \leqslant q^{3/4+\varepsilon}.$$

The last bound is non-trivial for $N \geqslant q^{3/4+\varepsilon}$.

H. Salie (1932), H. Davenport (1933): $3/4 \mapsto 2/3$.

A. Weil (1948): $|S(p; a, b)| \leqslant 2\sqrt{p}$ and hence

$$\left|\sum_{n \leqslant N} e_q(an^* + bn)\right| \leqslant q^{1/2+\varepsilon},$$

which is non-trivial for $N \geqslant q^{1/2+\varepsilon}$.

A. Weil's bound is unimprovable: for any $\delta > 0$, there exist infinitely many triples p, a, b s.t. $(p, ab) = 1$ and

$$|S(p; a, b)| \geqslant (2 - \delta)\sqrt{p}.$$

Of course, this does not mean that there is no non-trivial bounds for the case $N \leqslant \sqrt{q}$.

But the exponent $1/2$ was a barrier for a long time.

This barrier was broken in 1955 by A.G. Postnikov for special case

$$q = p^n, \quad p \text{ is prime and } \quad n \to +\infty.$$

Strictly speaking, A.G. Postnikov studied character sum

$$\sum_{M<n\leqslant M+N} \chi(k),$$

where χ denotes Dirichlet's character modulo $q = p^n$. But his method is also applicable to the short Kloosterman sum

$$\sum_{M<n\leqslant M+N} e_q(an^* + bn).$$

What is the reason?

Well-known formula for geometric progression:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots, \quad |x| < 1.$$

p-adic analogue of this formula has the form

$$(1+px)^* \equiv$$
$$1 - px + (px)^2 - (px)^3 + \ldots + (-1)^{n-1}(px)^{n-1} \pmod{p^n}$$

Thus Kloosterman sum becomes the exponential sum with polynomial and can be treated by methods of H. Weyl or I.M. Vinogradov.

Theorem (I.E. Shparlinski – S.A. Stepanov, 1988). *For any* $n \geqslant n_0$, $q = p^n$, *for any rational function*

$$R(x) = \frac{f(x)}{g(x)} \equiv f(x)(g(x))^* \pmod{q},$$

where

$$f(x) = a_k x^k + \ldots + a_1 x + a_0,$$
$$g(x) = b_\ell x^\ell + \ldots + b_1 x + b_0,$$

the following estimate holds:

$$S = \sum_{1 \leqslant n \leqslant N}{}' e_q\bigl(R(n)\bigr) \ll_{k,\ell} N \exp\left(-c\,\frac{(\log N)^3}{(\log q)^2}\right)$$

This bound is non-trivial for very small N, namely, for

$$N \geqslant \exp\left(c_1(\log q)^{2/3}\right)$$

In particular, if $N \asymp q^{\varepsilon}$ then

$$S \ll N^{1-c_0\,\varepsilon^3}$$

This method can be generalized to *powerful moduli*.
Given q, we define the *radical* of q as

$$\mathrm{rad}(q) = \prod_{p\mid q} p.$$

The modulus q is said to be *powerful*, if its radical is small (in logarithmic scale) in comparison with q. Simplest case: $q = p^n$.

Theorem (M.K., 2016). *Suppose* $q \geqslant q_0$, $d = rad(q)$, $c_1 = 900$, $c_2 = 160^{-4}$ *and let*

$$\max\left(d^{15}, e^{c_1(\log q)^{2/3}}\right) \leqslant N \leqslant \sqrt{q}.$$

Then, for any a, b, c *such that* $(a, q) = 1$, *one has*

$$\Bigl|\sum_{c<n\leqslant c+N} e_q(an^* + bn)\Bigr| \leqslant N \exp\left(-c_2\,\frac{(\log N)^3}{(\log q)^2}\right).$$

Very recently (Oct. 2018) this result was used by G. Ricotta, E. Royer and I.E. Shparlinski to establish the convergence-in-law of ***Kloosterman paths*** in Banach space $C^0([0,1], \mathbb{C})$.

In 1993-1996 A.A. Karatsuba invented new originally method of estimating of very short Kloosterman sums with arbitrary moduli q. His method is based on *"Mean value theorem"*, that is, the estimate for the number of solutions of the congruence

$$x_1^* + \cdots + x_k^* \equiv y_1^* + \cdots + y_k^* (\mathrm{mod}\, q)$$

where

$$X < x_1, \ldots, y_k \leqslant 2X, \quad X^{2k-1} \ll q.$$

This theorem shows (roughly speaking) that the most part of the solutions are "trivial", that is, y_j are the permutations of x_j. A.A. Karatsuba constructed first examples of subsets $\mathcal{A} \subset \mathbb{Z}_q^*$ such that $|\mathcal{A}| \asymp q^{\varepsilon}$ and Kloosterman sum $S_q(\mathcal{A}; a, b)$ has non-trivial estimate.

In particular, his method leads to the solution of one problem of P. Erdös and R.L. Graham (1980):

Given $\varepsilon > 0$; then there exists $k = k(\varepsilon)$ such that the congruence

$$x_1^* + \cdots + x_k^* \equiv a (\mathrm{mod}\, q)$$

has at least one solution $1 \leqslant x_j \leqslant q^{\varepsilon}$ for any $a \in \mathbb{Z}_q$

This was done by I.E. Shparlinski (2002) with $k \sim 4\,\varepsilon^{-3}$

(improved by A.A. Glibichuk (2006) to: $k \sim 8\,\varepsilon^{-2}$).

Hypothesis (A.A. Karatsuba): $k \asymp \varepsilon^{-1}$.

Further development of Karatsuba's method leads to the estimates of very short Kloosterman sums with prime moduli $q = p$ (A.A. Karatsuba, M.K., J. Bourgain and M.Z. Garaev).

For example, one can show that

$$\Bigl|\sum_{1\leqslant n\leqslant N} e_q(an^* + bn)\Bigr| \ll ND^{-3/4},$$

$$D = \frac{\log N}{(\log q)^{2/3}(\log\log q)^{1/3}}.$$

This bound is non-trivial for

$$N \geqslant e^{c(\log q)^{2/3}(\log\log q)^{1/3}}, \quad c > 0$$

If n runs through very short segment $1 \leqslant n \leqslant N$, $(n, q) = 1$ then $an^* + bn$ is uniformly distributed modulo q.

Suppose that $\mathcal{A}$ is the set of primes. Then the corresponding sum has the form

$$W_q(N) = \sum_{\substack{p \leqslant N \\ p \nmid q}} e_q(ap^* + bp)$$

One more reason why these sums are interesting:

Using *General Riemann Hypothesis* for all $L(s, \chi)$, $\chi \bmod q$, one can obtain a non-trivial estimate for $W_q(N)$

only for the case $X \geqslant q^{1+\varepsilon}$,

i.e. for quite long sum.

More convenient is the sum

$$T_q(N) = \sum_{\substack{n \leqslant N \\ (n,q)=1}} \Lambda(n) e_q(an^* + bn).$$

Here $\Lambda(n)$ is *von Mangoldt function*, that is

$$\Lambda(n) = \begin{cases} \log p, & \text{if} \quad n = p^k, \quad p \text{ is prime}, \quad k \geqslant 1 \\ 0, & \text{otherwise.} \end{cases}$$

$$\sum_{p \leqslant x} 1 \sim \frac{x}{\log x}, \quad \sum_{n \leqslant x} \Lambda(n) \sim x$$

Theorem (E. Fouvry, P. Michel, 1998). *Given* $\varepsilon > 0$, *let* $q \geqslant q_0(\varepsilon)$ *be prime number,* $(a, q) = 1$. *Then, there exists* $\delta = \delta(\varepsilon)$ *such that*

$$T_q(N) \ll_\varepsilon Nq^{-\delta} \quad \text{for} \quad q^{3/4+\varepsilon} \leqslant N \leqslant q.$$

Theorem (M.Z. Garaev, 2010). *Under the same conditions, if* $b = 0$ *then*

$$T_q(N) \ll_\varepsilon \bigl(N^{15/16} + N^{2/3}q^{1/4}\bigr)q^{\varepsilon}.$$

For example, if $N \asymp q$ then $T_q(N) \ll N^{1-1/16+\varepsilon}$.

Theorem (E. Fouvry, I.E. Shparlinski, 2011). *If* $b = 0$, *then previous estimate is valid for any composite* q *and*

$$q^{3/4+\varepsilon} \leqslant N \leqslant q^{4/3-\varepsilon}.$$

Applications:

In 1987, P. Erdös, A.M. Odlyzko and A. Sarközy, considered the congruence

$$p_1p_2 \equiv a \pmod{q} \quad \text{in primes} \quad p_1, p_2 \leqslant N.$$

Question: does this congruence have solutions for any $a \in \mathbb{Z}_q^*$ for $q^{1-c} \leqslant N \leqslant q$?

Modification: the congruence

$$p_1(p_2 + p_3) \equiv a \pmod{q} \quad \text{in primes} \quad p_j \leqslant N.$$

is solvable in primes for

$c = \frac{1}{39}$ J.B.Fridlander, P.Kurlberg, I.E.Shparlinski, 2007

$c = \frac{1}{17}$ M.Z. Garaev, 2010

Theorem (J. Bourgain, 2005): *Given $\varepsilon > 0$, $q \geqslant q_0(\varepsilon)$ is prime; then there exists $\eta = \eta(\varepsilon)$ such that*

$$\boldsymbol{T_q(N) \ll Nq^{-\eta}} \quad \textit{for} \quad \boldsymbol{q^{1/2+\varepsilon} \leqslant N \leqslant q}$$

Theorem (R. Baker, 2012): *Suppose that* ***squarefull part*** *of q is $\leqslant q^{1/4}$, then*

$$\boldsymbol{T_q(N) \ll Nq^{-\eta}} \quad \textit{for} \quad \boldsymbol{q^{1/2+\varepsilon} \leqslant N \leqslant q} \quad \textit{and} \quad \boldsymbol{\eta = \frac{\varepsilon^4}{2000}}.$$

Squarefool part $\boldsymbol{v}$ of $\boldsymbol{q}$ is defined by the unique representation $\boldsymbol{q = uv}$, where $\boldsymbol{(u, v) = 1}$, $\boldsymbol{u}$ is squarefree, $\boldsymbol{v}$ is squarefool, that is,

$$\boldsymbol{v = p_1^{\alpha_1} \cdots p_r^{\alpha_r}, \quad \alpha_j \geqslant 2}.$$

I.M. VINOGRADOV's identity in the form of R.C. VAUGHAN: for any $V \leqslant \sqrt{N}$, one has

$$\sum_{n \leqslant N} \Lambda(n)\Phi(n) = S_1 - S_2 - S_3 + S_4,$$

$$S_1 = \sum_{k \leqslant V} \mu(k) \sum_{n \leqslant Nk^{-1}} (\log n)\Phi(kn),$$

$$S_2 = \sum_{k \leqslant V^2} a_k \sum_{n \leqslant Nk^{-1}} \Phi(kn) = \sum_{k \leqslant V} + \sum_{V<k \leqslant V^2},$$

$$S_3 = \sum_{V<k \leqslant NV^{-1}} b_k \sum_{V<n \leqslant Nk^{-1}} \Lambda(n)\Phi(kn) = O(V),$$

$$S_4 = \sum_{n \leqslant V} \Lambda(n)\Phi(n).$$

Here $|a_k|, |b_k| \leqslant \max\{\tau(k), \log k\}$.

We set $\Phi(n) = e_q(an^* + bn)$.

By A. WEIL's bound, the sums with $k \leqslant V$ contributes at most $Vq^{1/2+\varepsilon}$. The rest sums reduce to *bilinear forms* of the type

$$S(X,Y) = \sum_{X<x\leqslant 2X} \sum_{Y<y\leqslant 2Y} A_x B_y e_q(a(xy)^* + bxy),$$

Here $A_x, B_y \ll q^{\delta}$ for any fixed $\delta > 0$, $XY \leqslant N$, $X \geqslant V$ and

$$X \leqslant V^2 \quad \text{for} \quad S_2 \quad \text{and} \quad X \leqslant NV^{-1} \quad \text{for} \quad S_3.$$

Suppose first that $b \equiv 0 \pmod q$ (this case in more simple).

By standard technic ("Hölder in. + Hölder in. + Cauchy in.") one gets:

$$|S(X,Y)|^{2ks} \ll (XY)^{2ks} \cdot \frac{qI_k(X)I_s(Y)}{(XY)^{ks}}.$$

Here $I_r(Z)$ is the number of solutions of the congruence

$$x_1^* + \cdots + x_r^* \equiv x_{r+1}^* + \cdots + x_{2r}^* \pmod q$$

with $Z < x_j \leqslant 2Z$.

The appearance of $I_r(X)$ is quite natural:

$$\left(\sum_{X<x\leqslant 2X} e_q(ax^*)\right)^k = \\ = \sum_{x_1,\ldots,x_k} e_q(a(x_1^*+\ldots+x_k^*)) = \\ = \sum_{\lambda=1}^{q} j_k(\lambda)e_q(a\lambda),$$

where $j_k(\lambda)$ is the number of solutions of

$$x_1^*+\ldots+x_k^* \equiv \lambda \pmod q.$$

Moreover,

$$\sum_{\lambda=1}^{q} j_k^2(\lambda) = I_k(X)$$

Then we choose parameters k, s and use different estimates for I_k, I_s. This leads to the results.

There are very good estimates for the case when q is prime. In particular, we have the estimate of A.A. Karatsuba - J. Bourgain - M.Z. Garaev (1995; 2014):

$$I_k(X) \ll N^k\left(1 + \frac{N^{2k-1}}{q}\right)(\log q)^c$$

Using this, we get

Theorem 1 (M.K., 2018): *Given* $\varepsilon > 0$, *prime* $q \geqslant q_0(\varepsilon) > 0$, *for* $q^{1/2+\varepsilon} \leqslant N \leqslant q$ *we have for* $b = 0$:

$$\sum_{n \leqslant N} \Lambda(n) e_q(an^*) \ll N q^{-\eta}, \quad \eta = \frac{\varepsilon^2}{20}.$$

This improves slightly the results of J.Bourgain and R. Baker (where $\eta \asymp \varepsilon^4$).

Next, if $k = 2$ then we have an estimate of
D.R. Heath-Brown (1978): for arbitrary q, the number of solutions of

$$x_1^* + x_2^* \equiv x_3^* + x_4^* \pmod{q}, \quad X < x_j \leqslant 2X,$$

is

$$\ll_\varepsilon X^2\left(\frac{X^{3/2}}{\sqrt{q}} + 1\right)q^{\varepsilon}$$

In particular, if $X \ll \sqrt[3]{q}$ then $I_2(X) \ll X^2 q^{\varepsilon}$.

Using this, we get:

Theorem 2 (M.K., 2017). *Given* ε, *arbitrary composite* $q \geqslant q_0(\varepsilon)$, *then*

$$\sum_{n \leqslant N} \Lambda(n) e_q(an^*) \ll N(q^{7/10} N^{-1})^{5/37} q^{\varepsilon}$$

for any $q^{7/10+\varepsilon} \leqslant N \leqslant q$.

Theorem 3 (M.K., 2018). *Given* ε*, arbitrary composite* $q \geqslant q_0(\varepsilon)$*, then*

$$\sum_{n \leqslant N} \Lambda(n) e_q(an^*) \ll N\Delta q^{\varepsilon},$$

where

$$\Delta \ll \begin{cases} (q^{5/8}N^{-1})^{1/5}, & \text{for} \quad q^{5/8} \leqslant N \leqslant q^{85/96}, \\ (q^{1/16}N^{2/5})^{-1/8}, & \text{for} \quad q^{85/96} \leqslant N \leqslant q^{107/96}, \\ (Nq^{-7/4})^{1/10}, & \text{for} \quad q^{107/96} \leqslant N \leqslant q^{7/4}. \end{cases}$$

This bound is non-trivial for $q^{5/8+\varepsilon} \leqslant N \leqslant q^{7/4-\varepsilon}$.

Case $b \not\equiv 0 \pmod q$ (more complicated).
In this case, one has

$$|S(X,Y)|^8 \ll (XY)^8 \cdot \frac{qY I_2(X) \cdot J_2(Y)}{(XY)^4},$$

where $I_2(X)$ is defined as above, and $J_2(Y)$ denotes the number of solutions of

$$\begin{cases} y_1^* + y_2^* \equiv y_3^* + y_4^* \pmod q \\ y_1 + y_2 \equiv y_3 + y_4 \pmod q \\ Y < y_j \leqslant 2Y. \end{cases}$$

Lemma (M.K., 2018). *For any composite* q, *any* $Y \leqslant q$, *one has*

$$J_q(Y) \ll 2^{\omega(q)}\tau_3(q)Y^2 \ll Y^2 q^{\varepsilon}.$$

($\omega(q)$ denotes the number of distinct prime divisors of q).

Theorem 4 (M.K., 2018). *Given* $\varepsilon > 0$, *any composite* $q \geqslant q_0(\varepsilon)$, *one has*

$$\sum_{n \leqslant N} \Lambda(n) e_q(an^* + bn) \ll N q^{\varepsilon} \Delta,$$

where

$$\Delta \ll \begin{cases} (q^{3/4} N^{-1})^{1/7}, & \text{for} \quad q^{3/4} \leqslant N \leqslant q^{7/8}, \\ (q^{2/3} N^{-1})^{3/35}, & \text{for} \quad q^{7/8} \leqslant N \leqslant q. \end{cases}$$

This bound is non-trivial for $q^{3/4+\varepsilon} \leqslant N \leqslant q^{1-\varepsilon}$.

6. Applications

1. Suppose q is prime. Then the congruence

$$p_1(p_1 + p_2 + p_3) \equiv a \pmod{q}$$

has solutions in primes $1 < p_j \leqslant N$ for any $a \in \mathbb{Z}_q$ if

$$q^{1-1/38+\varepsilon} \leqslant N \leqslant q.$$

2. Suppose that $k \geqslant 3$, $\varepsilon > 0$, $q \geqslant q_0(k, \varepsilon)$ is prime and $g(x) \equiv x + \frac{1}{x} \equiv x + x^* (\bmod q)$. Then the congruence

$$g(p_1) + \cdots + g(p_k) \equiv a \pmod{q}$$

has solutions in primes $1 < p_j \leqslant N$ for any a, if

$$q^{c_k+\varepsilon} \leqslant N \leqslant q, \quad \text{where}$$

$$c_k = \frac{2k+31}{3k+29} \quad \text{for} \quad 3 \leqslant k \leqslant 9, \qquad c_k = \frac{3k+22}{4(k+5)} \quad \text{for} \quad k \geqslant 10.$$

2a. Given $\mathbf{0} < \boldsymbol{\varepsilon} < \mathbf{0.01}$, $\boldsymbol{q} \geqslant \boldsymbol{q_0(\varepsilon)}$, and let $\boldsymbol{q}^{\mathbf{3/4}+\boldsymbol{\varepsilon}} \leqslant \boldsymbol{N} \leqslant \boldsymbol{q}$. Then the congruence

$$\boldsymbol{g(p_1) + \cdots + g(p_k) \equiv a} \pmod{\boldsymbol{q}}$$

has solutions $\mathbf{1} < \boldsymbol{p_j} \leqslant \boldsymbol{N}$ for any $\boldsymbol{a} \in \mathbb{Z}_{\boldsymbol{q}}$, if

$$\boldsymbol{k} \geqslant \left[\frac{\mathbf{7}}{\mathbf{4}\,\boldsymbol{\varepsilon}}\right] + \mathbf{1}.$$

3. Suppose $X \to +\infty$. Then, for any fixed $N \geqslant 0$,

$$\sum_{X<p_j\leqslant 2X} \tau(p_1p_2 + p_1p_3 + p_2p_3) =$$

$$= 2A\pi_1^3(X)\bigl(\ln X + B + \gamma + \ln(2\sqrt{3})\bigr) -$$

$$- \frac{AX^3}{(\ln X)^3}\sum_{\nu=0}^{N}\frac{C_\nu}{(\ln X)^\nu} + O_N\left(\frac{\pi_1^3(X)}{(\ln X)^{N+1}}\right)$$

where γ is Euler constant,

$$A = \prod_p\left(1 - \frac{1}{p(p-1)^2}\right), \quad B = \sum_p \frac{\ln p}{p(p-1)^2 - 1},$$

and C_ν are some explicit constants, $\pi_1(X) = \pi(2X) - \pi(X)$.

If

$$d|(p_1p_2+p_1p_3+p_2p_3) \quad \text{and} \quad (d, p_1p_2p_3) = 1$$

then

$$p_1^* + p_2^* + p_3^* \equiv 0 \pmod{d}.$$

Also, one deep result of E.Fouvry and I.E. Shparlinski (2011) was used (a kind of E. Bombieri - A.I. Vinogradov theorem).

THANK YOU FOR ATTENTION!