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Holomorphic maps of domains with rich symmetry groups

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Reinhardt domains

$\Omega \subset \mathbb{C}^n$ is a Reinhardt domain if
$z = (z_1, \dots, z_n) \in \Omega \;\Rightarrow\; (e^{i\theta_1} z_1, \dots, e^{i\theta_n} z_n) \in \Omega$
for all real $(\theta_1, \dots, \theta_n)$.

Examples:

a ball $\{z \in \mathbb{C}^n : \sum |z_j|^2 < 1\}$,
a polydisc $\{z \in \mathbb{C}^n : |z_j| < 1, j = 1, \dots, n\}$.

The diagram of absolute values:
$D(\Omega) = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n_{+,\geq} : (x_1, \dots, x_n) \in \Omega\}$.

The logarithmic diagram:
$LD(\Omega) = \{\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n : (e^{\xi_1}, \dots, e^{\xi_n}) \in \Omega\}$.

Tube domains

$\Omega \subset \mathbb{C}^n$ is a tube domain if
$z = (z_1, \ldots, z_n) \in \Omega \;\Rightarrow\; (z_1 + i\theta_1, \ldots, z_n + i\theta_n) \in \Omega$
for all real $(\theta_1, \ldots, \theta_n)$.

The base:
$B(\Omega) = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : (x_1, \ldots, x_n) \in \Omega\}$.

Examples:

a ball $\{z = (x_1 + iy_1, \ldots, x_n + iy_n) \in \mathbb{C}^n : \sum_2^n x_j^2 < x_1\}$,
a polydisc $\{z \in \mathbb{C}^n : |x_j| < 1, j = 1, \ldots, n\}$,
the future tube
$\{z = (x_1 + iy_1, \ldots, x_n + iy_n) \in \mathbb{C}^n : x_1 > 0, \sum_2^n x_j^2 < x_1^2\}$.

If Ω is a tube domain with base B, then $\{(e^{z_1}, \ldots, e^{z_n}) : (z_1, \ldots, z_n) \in \Omega\}$ is a Reindardt domain with logarithmic diagram B.

Ω is a domain of holomorphy $\Leftrightarrow$ $B(\Omega)$ is convex.

Ω is strictly pseudoconvex $\Leftrightarrow$ $B(\Omega)$ is convex.

A Reinhardt domain $\widetilde{\Omega}$ is a domain of holomorphy $\Leftrightarrow$ $LD(\widetilde{\Omega})$ is convex.

$\widetilde{\Omega}$ is strictly pseudoconvex away from the coordinate axes $\Leftrightarrow$ $LD(\widetilde{\Omega})$ is strictly convex.

Biholomorphic automorphisms of Reinhardt domains

Ball:
linear fractional transformations; $\mathbf{Aut}\,\Omega = SU(n+1,1)/$centre.

Polydisc:
linear fractional transformations;
$\mathbf{Aut}\,\Omega = SU(1,1)/Z_2 \times \cdots \times SU(1,1)/Z_2 \times S_n$.

Thullen domains (1927) $\{|z_1|^2 + |z_2|^{2/\alpha} < 1\}$, $\alpha > 0 \neq 1$:
$z_1' = e^{i\theta_1} \frac{z_1 - a}{1 - \bar{a} z_1}$,

$z_2' = e^{i\theta_1} (1 - |a|^2) \frac{z_2}{(1 - \bar{a} z_1)^\alpha}; \theta_j \in \mathbb{R}, |a| < 1$.

'Exponential domains' $\{z_2 > e^{|z|^2}\}$: $z_1' = e^{i\theta_1}(z_1 + b)$,

$z_2' = e^{i\theta_2} z_2 e^{2\bar{b} z_1 + |b|^2}; \theta_j \in \mathbb{R}, b \in \mathbb{C}$.

Remark. Both Thullen domains and 'exponential domains' have spherical boundaries.

$\boldsymbol{n > 2}$

Sunada: bounded Reinhardt domains containing the origin (1978);

K. and Shimizu: bounded or Kobayashi-hyperbolic Reinhardt domains (1988-89).

'Monomial maps'

$$\begin{gathered}
\boldsymbol{z_1' = \lambda_1 z_1^{\kappa_{11}} \dots z_n^{\kappa_{1n}},} \\
\boldsymbol{\dots} \\
\boldsymbol{z_n' = \lambda_n z_1^{\kappa_{11}} \dots z_n^{\kappa_{1n}};} \\
\boldsymbol{\kappa_{ij} \in \mathbb{Z}, \ \det \begin{pmatrix} \kappa_{11} \dots \kappa_{1n} \\ \dots \\ \kappa_{n1} \kappa_{nn} \end{pmatrix} = \pm 1.}
\end{gathered}$$

Aut $\boldsymbol{\Omega}$ is generated by its identity component and monomial biholomorphisms of $\boldsymbol{\Omega}$.

Corollary 1. *If $\boldsymbol{\Omega}$ does not intersect coordinate hyperplanes then it has only monomial biholomorphisms.*

Proposition *If* Ω *is a Kobayashi hyperbolic Reinhardt domain, then the rotations* $(e^{i\theta_1}z_1, \ldots, e^{i\theta_n}z_n)$ *form a maximal abelian subgroup of* $\mathbf{Aut}\,\Omega$.

Corollary 2. *If* $\Omega_1 \cong \Omega_2$, *then* $\Omega_1 \overset{monom}{\cong} \Omega_2$.

Theorem. (Soldatkin, 2002) *Two Reinhardt domains in* $\mathbb{C}^2$ *are biholomorphically equivalent if and only if there exists a monomial biholomorphism between them.*

Biholomorphic maps between tube domains

Tube domains in $\mathbb{C}^2$ with nonaffine automorphisms:

a ball $\{x_2^2 < x_1\}$;

a bidisc $\{|x_{1,2}| < 1\}$;

(i) $\{x_2 > e^{x_1}\} \cong \{x_2 > -\log\sin(x_1)\}$;

(ii) $\{he^{x_1} > x_2 > e^{x_1}, h > 1\} \cong \{-\log\sin(x_1) + \log h > x_2 > -\log\sin(x_1)\}$;

(iii) $\{e^{x_1} > x_2\} \cong \{x_1 < -\log\sin(x_1)\}$.

Theorem. (K., Soldatkin, 2006) *If two tube domains in $\mathbb{C}^2$ are biholomorphically equivalent then either their bases are affinely equivalent or, after affine transformations, their bases belong to one of the following five lists of domains*: (i), (ii), (iii),
(iv) $\{x_1 > 0, x_2 > 0\} \cong \{0 < x_1 < \pi, x_2 > 0\} \cong \{0 < x_1 < \pi, 0 < x_2 < \pi\}$,
(v) $\{x_1 > 0\} \cong \{0 < x_1 < \pi\}$

Kobayashi pseudodistance

Let Ω be a connected complex manifold. We say that two points $a, b \in \Omega$ are connected by a holomorphic disc if there exist two points $w_1, w_2 \in \Delta = \{|w| < 1\}$ and a holomorphic map $\varphi : \Delta \to \Omega$ such that $\varphi(w_1) = a$ and $\varphi(w_2) = b$. We call the distance between w_1 and w_2 measured with respect to the Poincaré metric in Δ the distance between a and b along $\varphi(\Delta)$.
We say that $a, b \in \Omega$ are connected by a string of holomorphic discs if there exists a string of points $Z_0 = a, \ldots, Z_k = b$ such that every pair Z_j, Z_{j+1} is connected by a holomorphic disc. We sum the distances between all pairs of Z_j and Z_{j+1} along the corresponding discs and call this the distance between a and b along the string.

The *Kobayashi pseudodistance* between a and b is the infimum of the distances between a and b along all strings of holomorphic discs connecting them.

The Kobayashi pseudodistance is a biholomorphic invariant, which decreases under holomorphic maps.

A complex manifold is said to be *Kobayashi-hyperbolic* if the invariant Kobayashi pseudometric is a metric.

Problem. *Characterize the bases of Kobayashi-hyperbolic tube domains.*

The answer is unknown even for $\boldsymbol{n = 2}$.

Proper maps between Reinhardt domains

A map $f : \Omega_1 \to \Omega_2$ is *proper* if for each compact subset K of Ω_2 the inverse image $f^{-1}(K)$ is also compact.

A classical result due to Remmert: *a holomorphic map f is proper* $\Leftrightarrow$ *f is a branched finite covering map of Ω_2.*

Theorem (Isaev, K., 2006): A description of proper holomorphic maps between pairs of bounded Reinhardt domains in $\mathbb{C}^2$.
Exceptional case: Ω_1 and Ω_2 have piecewise Levi-flat boundaries and Ω_1 is foliated by complex (one-dimensional) annuli.

Corollary 1. *If there exists a proper holomorphic $f : \Omega_1 \to \Omega_2$, and the case is not exceptional, then there exists a proper monomial map $g : \Omega_1 \to \Omega_2$. In particular, $LD(\Omega_1)$ and $LD(\Omega_2)$ are affinely equivalent.*

Corollary 2. *If there exists a proper holomorphic $f : \Omega_1 \to \Omega_2$, and the case is not exceptional, then $f = g_1 \circ F \circ g_2$, where the g_j are monomial proper maps and F is a non-monomial biholomorphism of an intermediate domain.*

The first steps of the proof consist in

1. going to the envelopes of holomorphy.

Kerner: *a proper map between Riemann domains over Stein manifolds extends to a proper map between their envelopes of holomorphy.*

2. Extending to the boundary

Barrett: *a proper map between bounded pseudoconvex Reinhardt domains extends to a neighbourhood of the boundary of the source domain (away from the coordinate axes).*

After that we obtain a correspondence between pieces of the boundaries of Ω_1 and Ω_2, which are real hypersurfaces with large abelian local groups of CR-symmetries.

However, if the multiplicity of the branched cover is > 1, this gives us also a correspondence between different pieces of the boundary of Ω_1, from which we can make similar conclusions.
That is, if we could extend a proper holomorphic map to the boundary of Ω_1, then we would not need the Reinhardt structure in Ω_2 to analyse the structure of f.

Let Ω be a bounded Reinhardt domain of dimension 2, M be a complex 2-dimensional manifold, and let $f : \Omega \to M$ be proper holomorphic.

Generalized Kerner's Theorem. Let Ω_1 be a Riemann domain over an n-dimensional Stein manifold S and Ω_2 be an n-dimensional complex manifold. Let $f : \Omega_1 \to \Omega_2$ be proper holomorphic. Let $\widehat{\Omega}_1 \supset \Omega_1$ be the envelope of holomorphy. Then there exists a Stein space $\widehat{\Omega}_2 \supset \Omega_2$ that is the envelope of holomorphy of Ω_2 and a proper holomorphic map $\hat{f} : \widehat{\Omega}_1 \to \widehat{\Omega}_2$ such that $\hat{f}|_{\Omega_1} = f$.

If Ω_1 is Reinhardt, then $\widehat{\Omega}_1$ is too. Thus we can assume that Ω and M are Stein.

For each $z \in \Omega$ set $F_z = \{w \in \Omega : f(w) = f(z)\}$. Then $F = \cup F_z$ is an analytic subset of $\Omega \times \Omega$.
F is the graph of a *proper holomorphic correspondence*.

F extends to $\partial\Omega$ outside an analytic subset just as in Barrett's theorem.

Theorem. *Under the above assumptions one of the following holds.*
1. f *is the quotient map by a finite group of rotations (so that* M *is a Reinhardt domain).*
2. f *is the quotient map by a finite group of monomial transformations of order* 2, 3, 4, *or* 6 *which are distinct from rotations.*
3. f *is a composition of a monomial proper map and a quotient map as in case* 2.
4. f *is a composition of a monomial proper map onto a Thullen or an exponential domain* Ω' *and a quotient map of* Ω' *by a finite subgroup of automorphisms.*
5. Ω *is piecewise Levi-flat.*