Some recent results on multiplier ideal sheaves

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Relation between Complex Differential Geometry and Complex Analytic/Algebraic Geometry

Two basic problems in complex geometry :

- Which kind of complex manifolds is Stein (a closed complex submanifold in $\mathbb{C}^n$) ? (Levi problem)

- Which kind of complex manifolds is projective algebraic (a closed complex submanifold in $\mathbb{P}^n$) ?

Stein manifold : holomorphically convex and holomorphically separable complex manifold

Every Kähler metric on a complex manifold has locally a smooth strictly psh function as the potential,

• Bergman metric is a special Kähler metric which has Bergman kernel as the potential.

Positive holomorphic line bundle on a complex manifold, i.e., a smooth hermitian metric with positive curvature form,

has a smooth strictly psh function as the weight of the metric w.r.t. the local trivialization.

✓ Oka-Grauert's solution of Levi problem :
Kähler exhaustion implies Steinness
(existence of a Kähler metric with exhaustion global potential)

✓ Kodaira's embedding theorem :
for holomorphic line bundles L, positive ($c_1(L) > 0$) implies ample

✓ for the projective category, GAGA (W.L. Chow, Serre) :
complex analytic geometry = complex algebraic geometry

relation between several complex variables
and complex algebraic geometry :

✓ compact Riemann surface is projective algebraic,

✓ noncompact Riemann surface is Stein

► Relation between Stein manifolds and projective algebraic manifolds :
a projective manifold \ the zero set of a nontrivial holomorphic section of the positive line bundle is a Stein manifold

• Both L^2 extension theorems across and from the submanifolds play a connection role

• Philosophy behind

▶ Grauert's solution of Levi problem on complex manifolds,

▶ Kodaira embedding theorem,

▶ Hömander's L^2 existence theorem with psh weights,

▶ Lu Qikeng's Theorem : a bounded domain in $\mathbb{C}^n$ with complete Bergman metric of constant holomorphic sectional curvature is biholomorphic to the unit ball

▶ properties of multiplier ideal sheaves :

▶ differential geometric conditions imply complex analytic conclusion

▶ existence or construction of specified holomorphic objects is reduced to the construction of specified plurisubharmonic functions (real-valued, may take $-\infty$)

✓ Lelong, Oka introduced the concept of the psh functions

• "hard" objects (rigid) : holomorphic functions, sections

• "soft" objects (flexible) : plurisubharmonic functions, currents

- singular hermitian metric on a holomorphic line bundle : locally $e^{-\varphi}, \varphi \in L^1_{loc}$
  curvature $\Theta = i\partial\bar{\partial}\varphi$ in the sense of currents
  pseudoeffective line bundle : $\Theta \geq 0$, i.e., φ is psh.
  special case : positive line bundle : φ is smooth strictly psh
- psh and positive closed (1,1)-current :
  φ is psh, then $i\partial\bar{\partial}\varphi$ is a positive closed (1,1)-current
  a positive closed (1,1)-current has locally a psh potential
  special case : Kähler form (metric)

• philosophy :

▶ singularities play important roles in various branches of math. :

✓ Green (Green function), Riemann, H. Weyl, Dirac, ...

▶ singular integral, Newtonian potential, fundamental (weak) solution, generalized function (distribution), currents, Poincaré-Lelong equation, ...

▶ singularities of psh functions and currents play a key role in several complex variables and complex geometry :

• Techniques : create singularities, use singularities

▶ Hörmander L^2 method using weights :
weight regarded as a singular metric on (trivial) line bundles

Hömander's L^2 existence theorem with psh weights :
Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$, φ be a plurisubharmonic function on D, then one can solve $\bar{\partial}u = v$ with L^2 estimate $||u||_\varphi := \int_D |u|^2 e^{-\varphi} \leq C||v||_\varphi$, for some constant C

- ▶ complex Monge-Amperé : pluripotential theory
- ▶ Kähler-Einstein metric : ω-plurisubharmonic functions
- ▶ ω-psh functions on Hermitian manifold (M, ω)
- • definition domains of Mabuchi functional, K-energy,... :
- • smooth strictly ω-psh functions
- ▶ singular Hermitian metric on holomorphic vector bundle

- singularity of a psh $\varphi : \varphi(z) = -\infty$
  e.g., for $\varphi = c\, log(|f_1|^2 + \cdots + |f_k|^2)$ is psh, where $c > 0$
- relation between a complex analytic subset and a pluripolar set :
  $f_1^{-1}(0) \cap \cdots \cap f_k^{-1}(0) = \varphi^{-1}(-\infty)$
- psh with analytic singularities

equivalence of singularities

- φ is more singular than ψ, denoted by $\varphi \preccurlyeq \psi$: if $\varphi \leq \psi + O(1)$
- φ and ψ have equivalent singularities : if $\varphi \preccurlyeq \psi$ and $\psi \preccurlyeq \varphi$

invariants :

- Lelong number : $v(\varphi, x) := \liminf_{z\to x} \frac{\varphi(z)}{ln|z-x|}$
- complex singularity exponent (log canonical threshold) $c_x(\varphi) = sup\{c \geq 0 : \exp^{-2c\varphi}$ is L^1 w.r.t. the Lebesgue measure on $\mathbb{C}^n$ on a neighborhood of $x\}$
- multiplier ideal sheaf : $\{f = 0| \int |f|^2 e^{-\varphi} < \infty\} = \{e^{-\varphi}$ not locally integrable $\}$ $\subset \{\varphi = -\infty\}$

multiplier ideal sheaf

▶ Definition : to a plurisubharmonic function φ, is associated an ideal subsheaf $\mathcal{I}(\varphi)$ of $\mathcal{O}$: germs of holomorphic functions $f \in \mathcal{O}_x$ such that $|f|^2 e^{-2\varphi}$ is locally integrable.

▶ L^p multiplier ideal sheaf

• origin goes back to Hörmander, Bombieri, Skoda

• multiplier ideal sheaves could be used to give a unified treatment of

✓ Grauert's solution to Levi problem,

✓ Cartan's global theory on Stein manifolds and

✓ Kodaira embedding theorem

• multiplier ideal sheaves could be regarded as a mixture of L^2 method for $\bar\partial$ and sheaf cohomology method

First Properties

- Nadel theorem : $\mathcal{I}(\varphi)$ is coherent
- Theorem : multiplier ideal sheaf is integrally closed, i.e., the integral closure of $\mathcal{I}(\varphi)$ is itself
- Nadel vanishing theorem : Let $(L, e^{-\varphi})$ be a big line bundle on a compact Kähler manifold X (i.e., the curvature current Θ of the singular Hermitian metric is a Kähler current : $\exists \epsilon > 0$, s.t., $\Theta \geq \epsilon\omega$). Then

$$H^q(X, K_X \otimes L \otimes \mathcal{I}(\varphi)) = 0,$$

for any $q \geq 1$.

Siu's Theorem on global generation of multiplier ideal sheaves :

Let $(L, e^{-\varphi})$ be a pseudoeffective line bundle on an n dimensional compact complex manifold X, A be an ample line bundle over X such that for every point $x \in X$ there are a finite number of elements of $\Gamma(X, A)$ which all vanish to order at least $n+1$ at x and which do not simultaneously vanish outside x.
Then $\Gamma(X, (K_X + A + L) \otimes \mathcal{I}(\varphi))$ generates $(K_X + A + L) \otimes \mathcal{I}(\varphi)$ at every point of X.

For properties and applications of the multiplier ideal sheaves, you're referred to

• Y.T. Siu, Multiplier ideal sheaves in complex and algebraic geometry. Sci. China Ser. A 48 (2005), suppl., 1–31.
• Y.-T. Siu, Some recent transcendental techniques in algebraic and complex geometry, Proc. of ICM. 2002.
• J.P. Demailly, Kähler manifolds and transcendental techniques in algebraic geometry. Proc. of ICM. 2006.

Strong openness conjecture

Denote by

$$\mathcal{I}_+(\varphi) := \cup_{\varepsilon>0}\mathcal{I}((1+\varepsilon)\varphi).$$

Demailly's strong openness conjecture :
For any plurisubharmonic function φ on X, one has

$$\mathcal{I}_+(\varphi) = \mathcal{I}(\varphi).$$

• J-P. Demailly, Multiplier ideal sheaves and analytic methods in algebraic geometry. School on Vanishing Theorems and Effective Results in Algebraic Geometry (Trieste, 2000), 1–148, ICTP Lect. Notes, 6, Trieste, 2001.

The meaning of the conjecture :

✓ $|f|^2e^{-\varphi}$ is locally integrable, then there exists an $\varepsilon_0 > 0$ s.t. $|f|^2e^{-(1+\varepsilon_0)\varphi}$ is also locally integrable

✓ $\{p \in \mathbb{R} : |f|^2e^{-p\varphi}$ is locally integrable$\}$ is open

origin from calculus :

$\{p \in \mathbb{R} : 1/|x|^{pc} = e^{-p\varphi}$ is locally integrable at the origin$\}$ is open, where $\varphi = c\ log|x|, c > 0$

$\{p \in \mathbb{R} : f(x)/|x|^{pc} = f(x)e^{-p\varphi}$ is locally integrable at the origin$\}$ is open

• Y.-T. Siu, Invariance of plurigenera and torsion-freeness of direct image sheaves of pluricanonical bundles. Finite or infinite dimensional complex analysis and applications, 45-83, Adv. Complex Anal. Appl., 2, Kluwer Acad. Publ., Dordrecht, 2004.

• J.-P. Demailly, Analytic Methods in Algebraic Geometry, Higher Education Press, Beijing, 2010

• J-P. Demailly, J. Kollár, Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds. Ann. Sci. École Norm. Sup. (4) 34 (2001), no. 4, 525–556.

✓ Openness conjecture : under the assumption that $e^{-2\varphi}$ is locally integrable, it was proved by Berndtsson (arXiv :1305.5781)

✓ $dimX < 3$, proved by Favre, Jonsson, Mustată

• C. Favre and M. Jonsson, Invent. Math. and JAMS 2005 ;

• M. Jonsson and M. Mustată, Ann. de L'Inst. Fourier, 2012.

solution of Demailly's strong openness conjecture

Theorem. (Guan, Zhou, Ann. of Math. 2015) Demailly's strong openness conjecture holds.

• Q.A. Guan, X.Y. Zhou, Strong openness conjecture for plurisubharmonic functions, arXiv :1311.3781.
• Q.A. Guan, X.Y. Zhou, A proof of Demailly's strong openness conjecture, Ann. of Math.(2) 182 (2015), no. 2, 605–616.

consequences of the strong openness conjecture

Corollary : General vanishing theorem on compact Kähler manifolds.
Let (L, φ) be a pseudo-effective line bundle on a compact Kähler manifold X of dimension n, and $nd(L, \varphi)$ be the numerical dimension of (L, φ).

$$H^q(X, K_X \otimes L \otimes \mathcal{I}(\varphi)) = 0,$$

for any $q \geq n - nd(L, \varphi) + 1$.

• This was conjectured by Junyan Cao in Compositio Math. 2014.

It was originally asked by Demailly

Corollary : relation between invariants for psh singularities

- $\mathcal{I}(c\varphi) = \mathcal{I}(c\psi)$, for any $c > 0$
- Lelong numbers up to proper modifications are the same :
  for all proper modifications $\pi : X \to \mathbb{C}^n$ above 0 and all points $p \in \pi^{-1}(0)$, we have $v(\varphi \circ \pi, p) = v(\psi \circ \pi, p)$

• This was conjectured by Boucksom-Favre-Jonsson

Corollary : For a big line bundle L, the equality $\mathcal{I}(||mL||) = \mathcal{I}(h_{min}^m)$ holds for every integer $m > 0$.

• This was conjectured by Demailly, Ein, and Lazarsfeld.
• also listed as an open problem in
Lazarsfeld : Positivity in algebraic geometry, Vol 2.

▶ Def. asymptotic multiplier ideal $\mathcal{I}(||L||)$:
maximal member of the family of the ideals $\{\mathcal{I}(1/k \cdot |kL|)\}$ for k large

Theorem (Demailly) : If L is a pseudo-effective line bundle, then there exists a unique (up to equivalence) singular Hermitian metric h_{min} with minimal singularities.

Corollary : Strong openness also holds for L^p multiplier ideal sheaves for $0 < p < \infty$

Siu's Conjecture : strong openness of L^p multiplier ideal sheaves for $1 \leq p < \infty$
Corollary : L^p multiplier ideal sheaves are coherent

see Fornaess (ArXiv 1507.00562).

the Kähler condition in Nadel vanishing theorem is not necessary for holomorphically convex manifolds.

Theorem (Meng, Zhou, 2016, to appear in JAG)
Let (X,ω) be a Hermitian holomorphically convex manifold, and let L be a holomorphic line bundle over X equipped with a singular Hermitian metric h. Assume that $i\Theta_{L,h} \geq \varepsilon\omega$ for some continuous positive function ε on X. Then

$$H^q(X, \mathscr{O}(K_X \otimes L) \otimes \mathcal{I}(h)) = 0 \quad for \quad q \geq 1$$

.

using the strong openness property of the multiplier ideal sheaves, we obtain the following vanishing theorem

Theorem (Meng, Zhou, 2016, to appear in JAG)
Let X be a strongly 1-convex manifold and (L, h) be a psef line bundle over X ($i\Theta_{L,h} \geq 0$ in the sense of current). Then

$$H^q(X, K_X \otimes L \otimes \mathcal{I}(h)) = 0 \quad for \quad q \geq 1$$

Case of the semipositive line bundles : Ohsawa, Peternell

Theorem (Meng, Zhou, 2016, to appear in JAG)
Let X be a holomorphically convex manifold, and let L be a holomorphic line bundle over X equipped with a singular hermitian metric h. Assume the curvature current $i\Theta_{L,h} \geqslant \gamma$ for some continuous real $(1,1)$-form γ on X and γ is strictly positive outside a compact subset K of X. Then we have

$$\dim H^q(X, K_X \otimes L \otimes \mathcal{I}(h)) < +\infty \quad \text{for} \quad q \geqslant 1,$$

and the corresponding restriction maps are bijective.

smooth metric case : Ohsawa

Siu's lemma plays important roles in Siu's study of Fujita conjecture and Phong-Sturm's study of holomorphic stability problems.

Siu's lemma *Let $\varphi(z)$ be a nonpositive plurisubharmonic function on $\mathbb{B}_r^1 \times \mathbb{B}_r^{n-1}$ such that*

$$I_\varphi := \int_{(z_2,\cdots,z_n)\in\mathbb{B}_r^{n-1}} e^{-\varphi(0,z_2,\cdots,z_n)} d\lambda_{n-1} < +\infty,$$

Assume that $r_1 \in (0,r)$. Then there exists a positive number C independent of φ, such that

$$\varliminf_{z_1\to 0} \int_{(z_2,\cdots,z_n)\in\mathbb{B}_{r_1}^{n-1}} e^{-\varphi(z_1,z_2,\cdots,z_n)} d\lambda_{n-1} \leq C I_\varphi.$$

• D.H. Phong and J. Sturm, Algebraic estimates, stability of local zeta functions, and uniform estimates for distribution functions, Ann. of Math., 2000, 277-329.

Using L^2 extension theorem movably, Zhou and Zhu prove :
Generalized Siu's lemma. (Zhou, Zhu, MRL 2017)
Let $\varphi(z', z'')$ be a plurisubharmonic function, h be a nonnegative continuous function on $\mathbb{B}_r^m \times \mathbb{B}_r^{n-m}$ $(1 \leq m \leq n)$.

$$\lim_{\varepsilon \to 0^+} \frac{1}{\mu(\mathbb{B}_\varepsilon^1)} \int_{\mathbb{B}_\varepsilon^1 \times \mathbb{B}_r^{n-1}} h(z', z'') e^{-\varphi(z', z'')} dV_n = \int_{z'' \in \mathbb{B}_r^{n-1}} h(0, z'') e^{-\varphi(0, z'')} dV_{n-1}$$

By Fatou's lemma, we may get Siu's Lemma from this.

In this way, we give another short proof of the openness conjecture.

In general, one couldn't expect to have

$$\lim_{z'\to 0}\int_{z''\in\mathbb{B}_r^{n-1}} e^{-\varphi(z',z'')}dV_{n-1} = \int_{z''\in\mathbb{B}_r^{n-1}} e^{-\varphi(0,z'')}dV_{n-1}$$

Using the strong openness property of multiplier ideal sheaves, Zhou and Zhu prove :
Generalized Siu's lemma (Zhou, Zhu 2017)

Assume that

$$I_{f,\varphi} := \int_{z'' \in \mathbb{B}_r^{n-m}} |f(z'')|^2 e^{-\varphi(0,z'')} d\lambda_{n-m} < +\infty$$

Assume that $\varepsilon, r_1, r_2 \in (0, r)$ and $r_1 < r_2$.

Then there exists a holomorphic function $F(z', z'')$ on $\mathbb{B}^m_{r_2} \times \mathbb{B}^{n-m}_{r_2}$ such that $F(0, z'') = f(z'')$ on $\mathbb{B}^{n-m}_{r_2}$,

$$\int_{\mathbb{B}^m_{r_2} \times \mathbb{B}^{n-m}_{r_2}} |F(z', z'')|^2 e^{-\varphi(z', z'')} d\lambda_n < +\infty,$$

and

$$\lim_{\varepsilon \to 0^+} \frac{1}{\lambda(\mathbb{B}^m_\varepsilon)} \int_{\mathbb{B}^m_\varepsilon \times \mathbb{B}^{n-m}_{r_1}} h(z', z'')|F(z', z'')|^2 e^{-\varphi(z', z'')} d\lambda_n$$
$$= \int_{z'' \in \mathbb{B}^{n-m}_{r_1}} h(0, z'')|f(z'')|^2 e^{-\varphi(0, z'')} d\lambda_{n-m}$$

Guan, Zhou, A solution of an optimal L^2 extension problem and applications, Ann. of Math., 2015

✓ based on a series of works in Liouville's J. 2012, Comptes Rendus 2012, Science China Math. 2015

✓ as corollaries, solved several open problems, including Suita's conjecture, several problems posed by Ohsawa, et al

a relation to Berndtsson's theorem

- M a pseudoconvex domain in $\mathbb{C}^{n+m}$ with coordinate (z,t), Y is a domain in $\mathbb{C}^m$ with coordinate t,

$$p(z,t) = t;$$

- M is a projective manifold, and Y is a complex manifold, and p is a fibration.
- Let e be the local frame of L. Let κ_{M_t} be the Bergman kernel of $K_{M_t} \otimes L$ on M_t, and $\kappa_{M_t} := B_t(z) dz \wedge d\bar{z} \otimes e \otimes \bar{e}$ locally

"Guan-Zhou method" (called in Ohsawa's recent book) :
✓ We found in our Annals paper that
our optimal L^2 extension theorem implies
Theorem (Berndtsson). $\log B_t(z)$ is a plurisubharmonic function with respect to (z, t),

✓ implies Griffiths positivity of the relative canonical bundle (Guan, Zhou, Sci. China Math. 2017)

- B. Berndtsson, Curvature of vector bundles associated to holomorphic fibrations, Ann. of Math, 169 (2009), 531–560.

▶ log-plurisubharmonicity has an important relation with KE metrics on Fano manifolds recently

▶ Berndtsson-Lempert (JMSJ 2016) studied the converse

✓ equivalence between the above optimal L^2 extension theorem and Griffiths positivity of the relative canonical bundle

✓ Ingredients of the proof :

- optimal L^2 extension ;
- extremal property of the Bergman kernel ;
- submean property of psh functions ;
- $exp(\int f) \leqq \int(exp(f))$, since the function e^x is convex

in our optimal L^2 theorems,
the manifold M of the pair (M, S) is assumed to be almost Stein, i.e., contain a Stein manifold as a Zariski open subset, including :

- Stein manifold and its complex subvarieties ;
- projective manifold and its complex subvarieties ;
- projective family and its complex subvarieties

when M is just weakly pseudoconvex Kähler manifold (having a smooth psh exhaustion function),
✓ an optimal L^2 extension theorem holds.
• Zhou, Zhu, An optimal L^2 extension theorem on weakly pseudoconvex Kähler manifolds, JDG, Vol. 110 (2018), 135-186.

difficulties : lacks of Cartan's extension theorem and regularization of psh functions

✓ Application : pseudoeffectivity of twisted relative canonical bundles in the Kähler setting

✓ An important topic in complex geometry is the positivity of twisted relative pluricanonical bundles and its direct images.

✓ by establishing optimal $L^{\frac{2}{m}}$ extension theorem for twisted pluricanonical bundles in the Kähler setting (derived from

• the generalized Siu's lemma
• optimal L^2 extension theorem in the Kähler setting)

✓ and using the "Guan-Zhou method" mentioned above,
✓ we prove the following result for the positivity of twisted relative pluricanonical bundles for Kähler fibrations.

Theorem (Zhou, Zhu 2017)

Let $\pi : X \longrightarrow Y$ be a surjective proper holomorphic map from a Kähler manifold (X, ω) to a complex manifold Y, and (L, h) be a holomorphic line bundle over X equipped with a singular Hermitian metric h such that the curvature current $\sqrt{-1}\Theta_{L,h} \geq 0$.

Denote by Y_0 the set of all points in Y which are regular values of π.

Let $X_y := \pi^{-1}(y)$, $L_y := L|_{X_y}$, $h_y := h|_{X_y}$,
$Y_h := \{y \in Y_0;\ h_y \not\equiv +\infty\}$ and

$$Y_{m,\text{ext}} := \Big\{y \in Y_0;\ \dim H^0(X_y, mK_{X_y} + L_y) = \operatorname{rank} \pi_*(mK_{X/Y} + L)\Big\}.$$

Assume that there exists a point $y' \in Y_h \cap Y_{m,\text{ext}}$ such that

$$H^0\big(X_{y'}, (mK_{X_{y'}} + L_{y'}) \otimes \mathcal{I}_m(h_{y'}^{\frac{1}{m}})\big) \neq 0.$$

Then the relative m-Bergman kernel metric $(B^o_{m,X/Y})^{-1}$ on $(mK_{X/Y} + L)|_{X_{m,\text{ext}}}$ is a singular Hermitian metric with semipositive curvature current, where $X_{m,\text{ext}} := \pi^{-1}(Y_{m,\text{ext}})$. Moreover, $(B^o_{m,X/Y})^{-1}$ extends across $X \setminus X_{m,\text{ext}}$ uniquely to a singular Hermitian metric $(B_{m,X/Y})^{-1}$ on $mK_{X/Y} + L$ with semipositive curvature current on all of X.

✓ answering a problem by Berndtsson, Păun

✓ solving a conjecture by Berndtsson, Păun and Takayama

• B. Berndtsson and M. Păun, *Bergman kernels and the pseudoeffectivity of relative canonical bundles*, Duke Math. J. **145** (2008), 341–378.
• B. Berndtsson and M. Păun, *Bergman kernels and subadjunction*, arXiv :1002.4145v1.
• M. Păun and S. Takayama, *Positivity of twisted relative pluricanonical bundles and their direct images*, JAG 2018 (arXiv :1409).
• C. D. Hacon, M. Popa and C. Schnell, *Algebraic fiber spaces over abelian varieties : around a recent theorem by Cao and Păun*, arXiv :1611.08768v2.

There have been various applications of

✓ the optimal L^2 extension theorem (Guan-Zhou)

✓ geometric meaning of the optimal L^2 extension theorem ("Guan-Zhou method") and

✓ the solution of Demailly's strong openness conjecture

in several complex variables and complex geometry by the following authors :

Demailly,
Ohsawa,
Berndtsson-Lempert,
Paun-Takayama,
Paun,
Hacon-Popa-Schnell
......

Thank You !