RC positivity and Yau's rational connectedness conjecture

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(1) Background materials and Overview;

(2) A weak dual of Cartan-Serre-Grothendieck-Kodaira theorem;

(3) A solution to Yau's conjecture on rationally connected manifolds;

(4) RC-positivity, rigidity of harmonic maps and holomorphic maps.

- X: a compact complex manifold
- $(E, h) \rightarrow X$: a Hermitian holomorphic vector bundle
- There exists a unique connection (called Chern connection) compatible with the complex structure on (X, E) and also the Hermitian metric h
- In local coordinates {zⁱ} on X and local frames {e_α} over E, the Chern curvature tensor R^E ∈ Γ(X, Λ^{1,1}T^{*}_X ⊗ End(E)) of (E, h, ∇) is given by

$$R^{E} = \sqrt{-1}R^{\gamma}_{i\bar{j}\alpha}dz^{i} \wedge d\bar{z}^{j} \otimes e^{\alpha} \otimes e_{\gamma}, \qquad (0.1)$$

where $R_{i\overline{j}\alpha}^{\gamma} = h^{\gamma\overline{\beta}}R_{i\overline{j}\alpha\overline{\beta}}$ and $R_{i\overline{j}\alpha\overline{\beta}} = -\frac{\partial^2 h_{\alpha\overline{\beta}}}{\partial z^i \partial \overline{z}^j} + h^{\gamma\overline{\delta}}\frac{\partial h_{\alpha\overline{\delta}}}{\partial z^i}\frac{\partial h_{\gamma\overline{\beta}}}{\partial \overline{z}^j}.$

Background materials: positivity of curvatures

• Let (X, ω_g) be a compact Hermitian manifold with $\omega_g = \sqrt{-1}g_{i\bar{j}}dz^i \wedge d\bar{z}^j$. The curvature tensor of (T_X, ω_g) is

$$R_{i\overline{j}k\overline{\ell}} = -\frac{\partial^2 g_{k\overline{\ell}}}{\partial z^i \partial \overline{z}^j} + g^{p\overline{q}} \frac{\partial g_{k\overline{q}}}{\partial z^i} \frac{\partial g_{p\overline{\ell}}}{\partial \overline{z}^j}.$$

• (X, ω) has positive holomorphic bisectional curvature (HBSC)

$$R_{i\overline{j}k\overline{\ell}}\xi^{i}\overline{\xi}^{j}\eta^{k}\overline{\eta}^{\ell} > 0$$

for non-zero ξ , η .

• (X, ω) has positive holomorphic sectional curvature (HSC)

$$R_{i\overline{j}k\overline{\ell}}\xi^i\overline{\xi}^j\xi^k\overline{\xi}^\ell>0$$

for non-zero ξ .

• (X, ω) has positive (Chern) Ricci curvature if

$$R_{i\overline{j}} = g^{k\overline{\ell}} R_{i\overline{j}k\overline{\ell}} = -\frac{\partial^2 \log \det(g)}{\partial z^i \partial \overline{z}^j}$$

is positive definite. The Ricci curvature represents the first Chern class $c_1(X)$ of the complex manifold X.

• (X, ω) has positive (Chern) scalar curvature if the scalar curvature function

$$s=g^{i\bar{j}}R_{i\bar{j}}>0.$$

Background materials: positivity of line bundles I

• $(L, h) \rightarrow X$: a Hermitian holomorphic line bundle. The curvature of (L, h)

$$R^L = -\sqrt{-1}\partial\overline{\partial}\log h$$

• *L* is called positive (or ample), if there exists a smooth Hermitian metric *h* such that

$$R^{L} = -\sqrt{-1}\partial\overline{\partial}\log h > 0$$

as a smooth (1, 1)-form.

- L is called nef, if for any $\varepsilon > 0$ and metric ω , there exists a smooth metric h_{ε} such that $-\sqrt{-1}\partial\overline{\partial}\log h_{\varepsilon} \ge -\varepsilon\omega$.
- *L* is called pseudo-effective, if there exists a possibly singular metric *h* such that $-\sqrt{-1}\partial\overline{\partial} \log h \ge 0$ in the sense of current.

Background materials: positivity of line bundles II

• Let X be a smooth projective variety. There are many equivalent definitions for ampleness of a line bundle L. The Nakai-Moishezon-Kleiman criterion asserts that: a line bundle L is ample if and only if

$$L^{\dim Y} \cdot Y > 0$$

for every positive-dimensional irreducible subvariety $Y \subset X$.

• Similarly, a line bundle L is nef if and only if

$$L \cdot C \geq 0$$

for every irreducible curve $C \subset X$.

• A line bundle L is said to be strictly nef, if for any irreducible curve C in X,

$$L \cdot C > 0.$$

• It is easy to see

$$\mathsf{ample} \subsetneq \mathsf{strictly} \ \mathsf{nef} \subsetneq \mathsf{nef}$$

 A vector bundle V is said to be ample (resp. nef, strictly nef) if the tautological line bundle O_V(1) of P(V) → X is ample (resp. nef, strictly nef).

- differential geometry: positivity or negativity of
 - (1) holomorphic bisectional curvature HBSC,
 - (2) holomorphic sectional curvature HSC,
 - (3) Ricci curvature,
 - (4) scalar curvature.

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- algebraic geometry: ampleness (resp. nefness, pseudo-effectiveness) of
 - (1) T_X or T_X^* ;
 - (2) K_X or K_X^{-1} .

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- algebraic geometry: ampleness (resp. nefness, pseudo-effectiveness) of (1) T_X or T^{*}_X;
 (2) K_X or K⁻¹_X.
- more positivity in algebraic geometry:
 - (1) uniruled, i.e. covered by rational curves;

(2) rationally connected, i.e. any two points can be connected by some rational curve;

(3) Fano, i.e. $c_1(X) > 0$.

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Question

Relations between geometric positivity and algebraic positivity?

Theorem (Siu-Yau, Mori)

TX ample \iff *X* has a (Kähler) metric with HBSC > 0. Indeed, $X \cong \mathbb{P}^n$.

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• Siu-Yau's solution to the Frankel conjecture: (X, ω) Kähler with HBSC > 0 $\Longrightarrow X \cong \mathbb{P}^n$. (analytical method.)



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TX ample \iff *X* has a (Kähler) metric with $\operatorname{HBSC} > 0$. Indeed, $X \cong \mathbb{P}^n$.

• Siu-Yau's solution to the Frankel conjecture: (X, ω) Kähler with HBSC > 0 $\Longrightarrow X \cong \mathbb{P}^n$. (analytical method.)

• Mori's solution to the Hartshorne conjecture: TX ample $\implies X \cong \mathbb{P}^n$. (characteristic *p*.)

Background materials: HBSC, tangent bundle

- Further uniformization and structure theorems: there are more than 100 mathematicians who contributes significantly along this line:
- N.Mok: uniformization theorem for Kähler manifolds with $HBSC \ge 0$;
- Chau, Chen, Fang, Feng, Gu, Liu, Ni, Tam, Zhang, Zhu and etc..
- Zheng (n = 3), Demailly-Peternell-Schneider(general): structure theorem for projective manifolds with T_X nef.
- There are also generalized structured theorems for K_X^{-1} nef due to Campana-Demailly-Peternell, Junyan Cao, A. Horing.

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- There are also generalized structured theorems for K_X^{-1} nef due to Campana-Demailly-Peternell, Junyan Cao, A. Horing.
- Recently, by using algebraic methods and analytical methods, we obtain a generalization of Mori's theorem:

Theorem (Li-Ou-Y., 2018)

Let X be a projective manifold. If T_X is strictly nef, then $X \cong \mathbb{P}^n$.

It is well-known that the Ricci curvature represents the first Chern class of the manifold, i.e.

$$[Ric] = 2\pi \cdot c_1(X) = 2\pi \cdot c_1(K_X^{-1}) = -2\pi \cdot c_1(K_X).$$

 Kodaira Embedding theorem: a line bundle *L* is ample ⇐⇒ *L* has a smooth metric with positive curvature.

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- Theorem(Aubin-Yau). K_X is ample $\iff X$ has a Kähler metric ω with $Ric(\omega) < 0$.

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- Theorem(Yau). K_X^{-1} is ample $\iff X$ has a Kähler metric ω with $Ric(\omega) > 0$.

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- Theorem(Yau). K_X^{-1} is ample $\iff X$ has a Kähler metric ω with $Ric(\omega) > 0$.
- Theorem(Yau). K_X is flat $\iff X$ has a Kähler metric ω with $Ric(\omega) = 0$.

Conjecture (Campana-Peternell 91')

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- We have some progress

Theorem (Li-Ou-Y., 2018)

If $\Lambda^r T_X$ is strictly nef for some $1 \le r \le \dim X$, then X is rationally connected.

In particular, if K_X^{-1} is strictly nef, then X is rationally connected.

holomorphic sectional curvature (HSC) $\iff K_X$ or K_X^* ?

Conjecture (Yau, 1970s)

Let X be a compact Kähler manifold. If X has a Kähler metric with HSC < 0, then X is a projective manifold and K_X is ample.

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• Breakthrough by Damin Wu and S.T. Yau:

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- Ideas of their proof:
 - 1. Schwarz Lemma: X has $HSC \leq 0 \implies X$ has no rational curve;
 - 2. Mori: Projective manifold X + no rational curve $\Longrightarrow K_X$ nef.
 - 3. Yau: Monge-Ampere equation method $\implies K_X$ is big, i.e. $c_1^n(K_X) > 0$;
 - 4. $K_X big$ and X has no rational curve $\implies K_X$ ample.

• Using Wu-Yau's idea and by passing Mori's theory:

Theorem (Tosatti-Y., JDG 2017)

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- Further works: Zheng-Y., Heier, Lu, Wong, Zheng, Nomura and etc..
- By using similar ideas

Theorem (Chen-Y., Math. Ann. 2018)

Let X be a compact Kähler manifold. If X is homotopic to a compact Riemannian manifold with negative sectional curvature,

 \implies X is a projective manifold and K_X is ample.

In his "Problem section", Yau proposed the following conjecture (Problem 47):

Conjecture (Yau, 1982)

Let X be a compact Kähler manifold. If X has a Kähler metric with HSC > 0, then X is a projective manifold and X is rationally connected, i.e. any two points can be connected by some rational curve.

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 Recently, I confirmed this conjecture by introducing a new concept called "RC-positivity":

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Background materials: Yau's conjecture on positive HSC II

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Let (X, ω) be a compact Kähler manifold with HSC > 0,

 \implies X is a projective manifold and X is rationally connected.

Note that the Hirzebruch surface P(O ⊕ O(10)) has HSC > 0, but the anticanonical bundle K_X⁻¹ is not nef and so K_X⁻¹ is not ample.

Let X be a projective manifold.

Theorem (Siu-Yau, 1980)

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Let (X, ω) be a compact Kähler manifold with positive holomorphic bisectional curvature, then X contains a rational curve. Indeed, $X \cong \mathbb{P}^n$.

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Theorem (Mori, 1982)

If the canonical bundle K_X is not nef, then X contains a rational curve.

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• The following fundamental result is the key in characterizing uniruled manifold.

Theorem (Boucksom-Demailly-Paun-Peternell, 2012)

X is uniruled if (and only if) K_X is not pseudo-effective

RC-positivity and uniruled manifolds

• A line bundle *L* is called RC-positive if it has a smooth metric *h* such that its Ricci curvature

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• By using algebraic geometry and transcendental method (e.g. Calabi-Yau theorem), we obtain a differential geometric characterization of uniruled manifold (which was also a conjecture of Yau)

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The following statements are equivalent for algebraic manifold X.

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The following statements are equivalent for algebraic manifold X.

(1) X is uniruled, i.e. covered by rational curves;

- (2) $c_1(X)$ is *RC*-positive,
 - i.e. there exists a Hermitian metric $\boldsymbol{\omega}$ such that

 $\operatorname{Ric}(\omega)$ has at least one positive eigenvalue at each point.

• RC-positive vector bundles.

Definition

A Hermitian holomorphic vector bundle (E, h) over a complex manifold X is called RC-positive, if at point $q \in X$, and for each nonzero vector $v \in E_q$, there exists some nonzero vector $u \in T_qX$ such that

$$R^{E}(u,\overline{u},v,\overline{v}) > 0. \tag{0.2}$$

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• There is a slightly stronger notion called "uniformly RC-positivity"

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• For a Hermitian line bundle (L, h),

RC-positive \iff uniformly RC-positive

• Examples of RC-positive tangent bundles.

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 - Fano manifolds;
 - manifolds with positive second Chern-Ricci curvature;
 - Hopf manifolds $\mathbb{S}^1 \times \mathbb{S}^{2n+1}$;
 - complex manifolds with HSC > 0;
 - products of manifolds with $TM >_{\rm RC} 0$.

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- Examples of uniformly RC-positive tangent bundles.
 - Kähler manifolds with HSC > 0;
 - Fano manifolds with $HSC \ge 0$ (Mok);
 - Hopf manifolds $\mathbb{S}^1 \times \mathbb{S}^{2n+1}$;
 - products of manifolds with $TM >_{\rm URC} 0$.

• First result

Theorem (Y., 2018)

If a compact Kähler manifold X has uniformly RC-positive tangent bundle, then X is projective and rationally connected.

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• We also have a conjecture.

Conjecture

Let X be a projective manifold. We expect the equivalence:

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Let X be a projective manifold. We expect the equivalence:

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- (2) T_X is uniformly RC-positive.

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- (2) T_X is uniformly RC-positive.
- Here, we have proved $(2) \Longrightarrow (1)$.
- For $(1) \Longrightarrow (2)$, we have some progress

Theorem

Let X be a rationally connected manifold. Then the tautological line bundle $\mathcal{O}_{T_X^*}(-1)$ is uniformly RC-positive (e.g. there exists RC-positive Finsler metric on X).

Geometry of vector bundles, and using techniques in non-Kähler geometry/functional analysis/ $\overline{\partial}$ -estimate to investigate problems in algebraic geometry!

• Let's recall the Cartan-Serre-Grothendieck-Kodaira theorem

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Let X be a projective manifold with $\dim_{\mathbb{C}} X = n$. Then the following statements are equivalent

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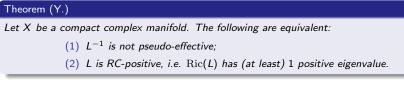
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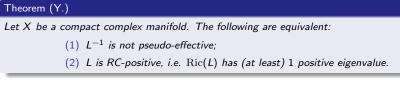


• Application: a weak dual of Cartan-Serre-Grothendieck-Kodaira theorem

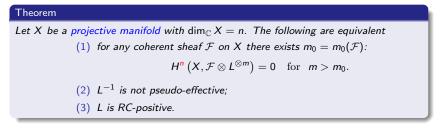
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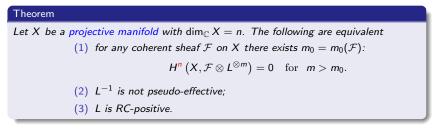
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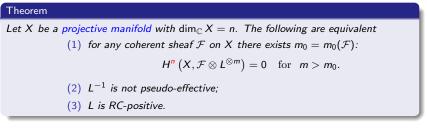


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- Andreotti-Grauert (1964'): (3) \Longrightarrow (1). Demailly-Peternell-Schneider, Demailly, Turtaro, Ottem, Matsumura and etc..
- Ideas of the proof: using non-Kähler geometry/functional analysis/non-linear PDE/Siu's non-vanishing theorem.

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Conjecture (Problem 47 of Yau's "Problem Section", 1982)

Let X be a compact Kähler manifold. If X has a Kähler metric with HSC > 0, then X is a projective and rationally connected manifold.

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Theorem

If a compact Kähler manifold (X, ω) has HSC > 0, then $\Lambda^p T_X$ is RC-positive and $H^{p,0}_{\overline{\partial}}(X) = 0$ for every $1 \le p \le \dim X$. In particular, X is projective and rationally connected.

A proof of Yau's conjecture for HSC > 0

• The proof is based on a "minimum prinicple":

Lemma

Let (X, ω) be a compact Kähler manifold. Let $e_1 \in T_q X$ be a unit vector which minimizes the holomorphic sectional curvature H of ω at point q, then

$$2R(e_1,\overline{e}_1,W,\overline{W}) \ge \left(1 + |\langle W,e_1\rangle|^2\right)R(e_1,\overline{e}_1,e_1,\overline{e}_1) \tag{0.4}$$

for every unit vector $W \in T_q X$.

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- (6) A criterion of Campana-Demailly-Peternal implies X is rationally connected.

Corollary

Let X be a compact Kähler manifold. If there exist Hermitian metrics ω and h: $\operatorname{tr}_{\omega} R^{(T_X,h)} \in \Gamma(X, \operatorname{End}(T_X))$ is positive definite,

then X is projective and rationally connected.

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The following statements are equivalent on a projective manifold X.

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(2) there exists a Hermitian metric ω with $\operatorname{Ric}^{(2)}(\omega) > 0$.

Theorem (Yau)

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• By using maximum principle

$$\operatorname{tr}_{\omega_{h}}\left(\sqrt{-1}\partial\overline{\partial}u\right) \geq \left(h^{\alpha\overline{\beta}}R_{\alpha\overline{\beta}\gamma\overline{\delta}}\right)h^{\mu\overline{\delta}}h^{\gamma\overline{\nu}}\left(g_{i\overline{j}}f_{\mu}^{i}\overline{f_{\nu}^{j}}\right) - R_{i\overline{j}k\overline{\ell}}\left(h^{\alpha\overline{\beta}}f_{\alpha}^{i}\overline{f_{\beta}^{j}}\right)\left(h^{\mu\overline{\nu}}f_{\mu}^{k}\overline{f_{\nu}^{\ell}}\right)$$

where

$$u = \left|\partial f\right|^2 = g_{i\bar{j}} h^{\alpha \overline{\beta}} f_{\alpha}^i \overline{f}_{\beta}^j$$

is the energy density of the map $f: (M, h) \rightarrow (N, g)$.

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• We introduce the generalized ∂ -energy density on the projective bundle $\mathbb{P}(\mathcal{T}_M) \to M$ is defined as

$$\mathscr{Y} = \mathbf{g}_{i\overline{j}} \mathbf{f}_{\alpha}^{i} \overline{\mathbf{f}}_{\beta}^{j} \frac{W^{\alpha} \overline{W}^{\beta}}{\sum \mathbf{h}_{\gamma \overline{\delta}} W^{\gamma} \overline{W}^{\delta}}$$

By setting $\mathscr{H} = \sum h_{\gamma \overline{\delta}} W^{\gamma} \overline{W}^{\delta}$, we obtain the complex Hessian estimate:

$$\sqrt{-1}\partial\overline{\partial}\mathscr{Y} \geq \left(\sqrt{-1}\partial\overline{\partial}\log \mathscr{H}^{-1}
ight)\cdot \mathscr{Y} - rac{\sqrt{-1} \mathcal{R}_{i\overline{j}k\overline{\ell}}f^i_lpha \overline{f^j_eta} f^j_lpha \overline{f^j_
u} \mathcal{W}^\mu \overline{W}^
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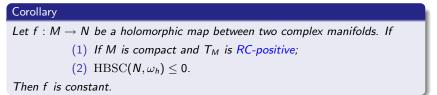
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- Recall that, there are many Kähler and non-Kähler complex manifolds with RC-positive tangent bundles
 - complex manifolds with HSC > 0;
 - Fano manifolds;
 - manifolds with positive second Chern-Ricci curvature;
 - Hopf manifolds $S^1 \times S^{2n+1}$;
 - products of complex manifolds with RC-positive tangent bundles.

By using the Hessian estimate

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we can also show

Theorem Let $f : (M, h) \rightarrow (N, g)$ be a holomorphic map between two Hermitian manifolds. Suppose M is compact. If (1) (M, h) has HSC > 0 (resp. $HSC \ge 0$); (2) (N, g) has $HSC \le 0$ (resp. HSC < 0),

then f is a constant map.

 Recall that: a smooth map f from a complex manifold M to a Kähler (or Riemannian) manifold N is pluri-harmonic if and only if for any holomorphic curve i: C → M, the composition f ∘ i is harmonic.

Theorem

Let $f : M \to (N, g)$ be a pluri-harmonic map from a compact complex manifold M to a Riemannian manifold or a Kähler manifold (N, g)

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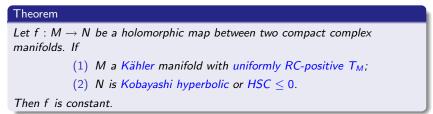
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Question

Let $f: M \to N$ be a holomorphic map between two compact complex manifolds. If

(1)
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Thank you!