RC positivity and Yau's rational connectedness conjecture

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Contents

(1) Background materials and Overview;

(2) A weak dual of Cartan-Serre-Grothendieck-Kodaira theorem;

(3) A solution to Yau's conjecture on rationally connected manifolds;

(4) RC-positivity, rigidity of harmonic maps and holomorphic maps.

- X: a compact complex manifold
- $(E,h) \to X$: a Hermitian holomorphic vector bundle
- There exists a unique connection (called Chern connection) compatible with the complex structure on (X,E) and also the Hermitian metric h
- In local coordinates $\{z^i\}$ on X and local frames $\{e_\alpha\}$ over E, the Chern curvature tensor $R^E \in \Gamma(X, \Lambda^{1,1}T_X^* \otimes \mathrm{End}(E))$ of (E,h,∇) is given by

$$R^E = \sqrt{-1} R_{i\bar{j}\alpha}^{\gamma} dz^i \wedge d\bar{z}^j \otimes e^\alpha \otimes e_\gamma, \tag{0.1}$$

where $R_{i\bar{j}\alpha}^{\gamma} = h^{\gamma\overline{\beta}} R_{i\bar{j}\alpha\overline{\beta}}$ and

$$R_{i\bar{j}\alpha\overline{\beta}} = -\frac{\partial^2 h_{\alpha\overline{\beta}}}{\partial z^i \partial \overline{z}^j} + h^{\gamma\overline{\delta}} \frac{\partial h_{\alpha\overline{\delta}}}{\partial z^i} \frac{\partial h_{\gamma\overline{\beta}}}{\partial \overline{z}^j}.$$

- Let (X, ω_g) be a compact Hermitian manifold with $\omega_g = \sqrt{-1} g_{i\bar{j}} dz^i \wedge d\bar{z}^j$. The curvature tensor of (T_X, ω_g) is

$$R_{i\bar{j}k\bar{\ell}} = -\frac{\partial^2 g_{k\bar{\ell}}}{\partial z^i \partial \bar{z}^j} + g^{p\bar{q}} \frac{\partial g_{k\bar{q}}}{\partial z^i} \frac{\partial g_{p\bar{\ell}}}{\partial \bar{z}^j}.$$

- (X, ω) has positive holomorphic bisectional curvature (HBSC)

$$R_{i\bar{j}k\bar{\ell}} \xi^i \bar{\xi}^j \eta^k \bar{\eta}^\ell > 0$$

for non-zero ξ, η.

- (X, ω) has positive holomorphic sectional curvature (HSC)

$$R_{i\bar{j}k\bar{\ell}} \xi^i \bar{\xi}^j \xi^k \bar{\xi}^\ell > 0$$

for non-zero ξ.

- (X, ω) has positive (Chern) Ricci curvature if

$$R_{i\bar{j}} = g^{k\bar{\ell}} R_{i\bar{j}k\bar{\ell}} = -\frac{\partial^2 \log \det(g)}{\partial z^i \partial \bar{z}^j}$$

  is positive definite. The Ricci curvature represents the first Chern class $c_1(X)$ of the complex manifold X.

- (X, ω) has positive (Chern) scalar curvature if the scalar curvature function

$$s = g^{i\bar{j}} R_{i\bar{j}} > 0.$$

Background materials: positivity of line bundles I

- $(L, h) \to X$: a Hermitian holomorphic line bundle. The curvature of (L, h)
$$R^L = -\sqrt{-1}\partial\overline{\partial}\log h$$

- L is called positive (or ample), if there exists a smooth Hermitian metric h such that
$$R^L = -\sqrt{-1}\partial\overline{\partial}\log h > 0$$
as a smooth $(1,1)$-form.

- L is called nef, if for any $\varepsilon > 0$ and metric ω, there exists a smooth metric h_ε such that $-\sqrt{-1}\partial\overline{\partial}\log h_\varepsilon \geq -\varepsilon\omega$.

- L is called pseudo-effective, if there exists a possibly singular metric h such that $-\sqrt{-1}\partial\overline{\partial}\log h \geq 0$ in the sense of current.

- Let X be a smooth projective variety. There are many equivalent definitions for ampleness of a line bundle L. The Nakai-Moishezon-Kleiman criterion asserts that: a line bundle L is ample if and only if

$$L^{\dim Y} \cdot Y > 0$$

for every positive-dimensional irreducible subvariety $Y \subset X$.

- Similarly, a line bundle L is nef if and only if

$$L \cdot C \geq 0$$

for every irreducible curve $C \subset X$.

- A line bundle L is said to be strictly nef, if for any irreducible curve C in X,

$$L \cdot C > 0.$$

- It is easy to see

$$\text{ample} \subsetneq \text{strictly nef} \subsetneq \text{nef}$$

- A vector bundle V is said to be ample (resp. nef, strictly nef) if the tautological line bundle $\mathcal{O}_V(1)$ of $\mathbb{P}(V) \to X$ is ample (resp. nef, strictly nef).

- differential geometry: positivity or negativity of
  (1) holomorphic bisectional curvature HBSC,
  (2) holomorphic sectional curvature HSC,
  (3) Ricci curvature,
  (4) scalar curvature.

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  (1) T_X or T_X^*;
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- more positivity in algebraic geometry:
  (1) uniruled, i.e. covered by rational curves;
  (2) rationally connected, i.e. any two points can be connected by some rational curve;
  (3) Fano, i.e. $c_1(X) > 0$.

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Question

Relations between geometric positivity and algebraic positivity?

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Theorem (Siu-Yau, Mori)

TX ample $\Longleftrightarrow$ X has a (Kähler) metric with $\mathrm{HBSC} > 0$.
Indeed, $X \cong \mathbb{P}^n$.

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- Siu-Yau's solution to the Frankel conjecture: (X, ω) Kähler with $\mathrm{HBSC} > 0 \Longrightarrow X \cong \mathbb{P}^n$. (analytical method.)

- Mori's solution to the Hartshorne conjecture: TX ample $\Longrightarrow X \cong \mathbb{P}^n$. (characteristic p.)

- Further uniformization and structure theorems: there are more than 100 mathematicians who contributes significantly along this line:
- N.Mok: uniformization theorem for Kähler manifolds with $HBSC \geq 0$;
- Chau, Chen, Fang, Feng, Gu, Liu, Ni, Tam, Zhang, Zhu and etc..
- Zheng ($n = 3$), Demailly-Peternell-Schneider(general): structure theorem for projective manifolds with T_X nef.
- There are also generalized structured theorems for K_X^{-1} nef due to Campana-Demailly-Peternell, Junyan Cao, A. Horing.

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- There are also generalized structured theorems for K_X^{-1} nef due to Campana-Demailly-Peternell, Junyan Cao, A. Horing.
- Recently, by using algebraic methods and analytical methods, we obtain a generalization of Mori's theorem:

Theorem (Li-Ou-Y., 2018)

Let X be a projective manifold. If T_X is strictly nef, then $X \cong \mathbb{P}^n$.

It is well-known that the Ricci curvature represents the first Chern class of the manifold, i.e.

$$[Ric] = 2\pi \cdot c_1(X) = 2\pi \cdot c_1(K_X^{-1}) = -2\pi \cdot c_1(K_X).$$

- Kodaira Embedding theorem: a line bundle L is ample $\Longleftrightarrow$ L has a smooth metric with positive curvature.

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- Theorem(Yau). K_X^{-1} is ample $\Longleftrightarrow$ X has a Kähler metric ω with $Ric(\omega) > 0$.
- Theorem(Yau). K_X is flat $\Longleftrightarrow$ X has a Kähler metric ω with $Ric(\omega) = 0$.

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K_X^{-1} is strictly nef $\iff$ K_X^{-1} is ample, i.e. X is Fano.

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- We have some progress

Theorem (Li-Ou-Y., 2018)

If $\Lambda^r T_X$ is strictly nef for some $1 \leq r \leq \dim X$, then X is rationally connected.

In particular, if K_X^{-1} is strictly nef, then X is rationally connected.

holomorphic sectional curvature (HSC) $\leftrightsquigarrow$ K_X or K_X^*?

Conjecture (Yau, 1970s)

Let X be a compact Kähler manifold. If X has a Kähler metric with $HSC < 0$, then X is a projective manifold and K_X is ample.

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- Breakthrough by Damin Wu and S.T. Yau:

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- Ideas of their proof:
  1. Schwarz Lemma: X has $HSC \leq 0 \Longrightarrow X$ has no rational curve;
  2. Mori: Projective manifold X + no rational curve $\Longrightarrow K_X$ nef.
  3. Yau: Monge-Ampere equation method $\Longrightarrow K_X$ is big, i.e. $c_1^n(K_X) > 0$;
  4. K_X *big* and X has no rational curve $\Longrightarrow K_X$ ample.

- Using Wu-Yau's idea and by passing Mori's theory:

Theorem (Tosatti-Y., JDG 2017)

If (X, ω) is a compact Kähler manifold with HSC ≤ 0, then K_X is nef.

Background materials: Yau's conjecture on negative HSC I

- Using Wu-Yau's idea and by passing Mori's theory:

Theorem (Tosatti-Y., JDG 2017)

If (X, ω) is a compact Kähler manifold with HSC ≤ 0, then K_X is nef.

- Note that Generic torus has $\mathrm{HSC} \leq 0$ but they are not projective manifold. Hence, Mori's theory can not apply in our situation.

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- Further works: Zheng-Y., Heier, Lu, Wong, Zheng, Nomura and etc..
- By using similar ideas

Theorem (Chen-Y., Math. Ann. 2018)

Let X be a compact Kähler manifold. If X is homotopic to a compact Riemannian manifold with negative sectional curvature,

$$\Longrightarrow X \text{ is a projective manifold and } K_X \text{ is ample.}$$

In his "Problem section", Yau proposed the following conjecture (Problem 47):

Conjecture (Yau, 1982)

Let X be a compact Kähler manifold. If X has a Kähler metric with $HSC > 0$, then X is a projective manifold and X is rationally connected, i.e. any two points can be connected by some rational curve.

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- Recently, I confirmed this conjecture by introducing a new concept called "RC-positivity":

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Let (X, ω) be a compact Kähler manifold with $HSC > 0$,

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$$\Longrightarrow X \text{ is a projective manifold and } X \text{ is rationally connected.}$$

- Note that the Hirzebruch surface $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(10))$ has $HSC > 0$, but the anticanonical bundle K_X^{-1} is not nef and so K_X^{-1} is not ample.

Let X be a projective manifold.

-

Theorem (Siu-Yau, 1980)

Let (X,ω) be a compact Kähler manifold with positive holomorphic bisectional curvature, then X contains a rational curve. Indeed, $X \cong \mathbb{P}^n$.

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- **Theorem (Siu-Yau, 1980)**

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- **Theorem (Mori, 1982)**

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- **Theorem (Mori, 1982)**

  If the canonical bundle K_X is not nef, then X contains a rational curve.

- The following fundamental result is the key in characterizing uniruled manifold.

Theorem (Boucksom-Demailly-Paun-Peternell, 2012)

X is uniruled if (and only if) K_X is not pseudo-effective

- A line bundle L is called RC-positive if it has a smooth metric h such that its Ricci curvature
$$\mathrm{Ric}(L,h) = -\sqrt{-1}\partial\overline{\partial}\log h$$
has at least one positive eigenvalue everywhere.

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- By using algebraic geometry and transcendental method (e.g. Calabi-Yau theorem), we obtain a differential geometric characterization of uniruled manifold (which was also a conjecture of Yau)

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Theorem (Y., Compositio. Math. 2018)

The following statements are equivalent for algebraic manifold X.

(1) *X is uniruled, i.e. covered by rational curves;*

(2) *$c_1(X)$ is RC-positive,*
i.e. there exists a Hermitian metric ω such that
$$\mathrm{Ric}(\omega) \text{ has at least one positive eigenvalue at each point.}$$

- RC-positive vector bundles.

Definition

A Hermitian holomorphic vector bundle (E, h) over a complex manifold X is called RC-positive, if at point $q \in X$, and for each nonzero vector $v \in E_q$, there exists some nonzero vector $u \in T_qX$ such that

$$R^E(u, \overline{u}, v, \overline{v}) > 0. \quad (0.2)$$

RC-positivity and rationally connected manifolds.

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- For a Hermitian line bundle (L, h),

$$\text{RC-positive} \iff \text{uniformly RC-positive}$$

- Examples of RC-positive tangent bundles.

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  - Fano manifolds;
  - manifolds with positive second Chern-Ricci curvature;
  - Hopf manifolds $\mathbb{S}^1 \times \mathbb{S}^{2n+1}$;
  - complex manifolds with $\mathrm{HSC} > 0$;
  - products of manifolds with $TM >_{\mathrm{RC}} 0$.

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- Examples of uniformly RC-positive tangent bundles.
  - Kähler manifolds with $\mathrm{HSC} > 0$;
  - Fano manifolds with $\mathrm{HSC} \geq 0$ (Mok);
  - Hopf manifolds $\mathbb{S}^1 \times \mathbb{S}^{2n+1}$;
  - products of manifolds with $TM >_{\mathrm{URC}} 0$.

- First result

Theorem (Y., 2018)

If a compact Kähler manifold X has uniformly RC-positive tangent bundle, then X is projective and rationally connected.

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- We also have a conjecture.

Conjecture

Let X be a projective manifold. We expect the equivalence:

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RC-positivity and rationally connected manifolds.

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Conjecture

Let X be a projective manifold. We expect the equivalence:

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Let X be a projective manifold. We expect the equivalence:

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Let X be a projective manifold. We expect the equivalence:

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(2) T_X is uniformly RC-positive.

- Here, we have proved (2) $\Longrightarrow$ (1).
- For (1) $\Longrightarrow$ (2), we have some progress

Theorem

Let X be a rationally connected manifold. Then the tautological line bundle $\mathcal{O}_{T_X^*}(-1)$ is uniformly RC-positive (e.g. there exists RC-positive Finsler metric on X).

Geometry of vector bundles, and using techniques in non-Kähler geometry/functional analysis/$\overline{\partial}$-estimate to investigate problems in algebraic geometry!

- Let's recall the Cartan-Serre-Grothendieck-Kodaira theorem

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- Ideas of the proof: using non-Kähler geometry/functional analysis/non-linear PDE/Siu's non-vanishing theorem.

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Conjecture (Problem 47 of Yau's "Problem Section", 1982)

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  Let X be a compact Kähler manifold. If X has a Kähler metric with $HSC > 0$, then X is a projective and rationally connected manifold.

- **Theorem**

  If a compact Kähler manifold (X, ω) has $HSC > 0$, then $\Lambda^p T_X$ is RC-positive and $H^{p,0}_{\overline{\partial}}(X) = 0$ for every $1 \leq p \leq \dim X$. In particular, X is projective and rationally connected.

- The proof is based on a "minimum prinicple":

Lemma

Let (X, ω) be a compact Kähler manifold. Let $e_1 \in T_qX$ be a unit vector which minimizes the holomorphic sectional curvature H of ω at point q, then

$$2R(e_1, \overline{e}_1, W, \overline{W}) \geq \left(1 + |\langle W, e_1\rangle|^2\right) R(e_1, \overline{e}_1, e_1, \overline{e}_1) \tag{0.4}$$

for every unit vector $W \in T_qX$.

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Corollary

Let X be a compact Kähler manifold. If there exist Hermitian metrics ω and h:

$$\mathrm{tr}_\omega R^{(T_X,h)} \in \Gamma(X, \mathrm{End}(T_X)) \text{ is positive definite,}$$

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The following statements are equivalent on a projective manifold X.

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- The following rigidity theorem is well-known

Theorem (Yau)

Let $f : M \to N$ be a holomorphic map between two complex manifolds. If

Then f is constant.

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- By using maximum principle

$$\mathrm{tr}_{\omega_h}\left(\sqrt{-1}\partial\overline{\partial}u\right) \geq \left(h^{\alpha\overline{\beta}}R_{\alpha\overline{\beta}\gamma\overline{\delta}}\right)h^{\mu\overline{\delta}}h^{\gamma\overline{\nu}}\left(g_{i\overline{j}}f^i_\mu\overline{f^j_\nu}\right)-R_{i\overline{j}k\overline{\ell}}\left(h^{\alpha\overline{\beta}}f^i_\alpha\overline{f^j_\beta}\right)\left(h^{\mu\overline{\nu}}f^k_\mu\overline{f^\ell_\nu}\right).$$

where

$$u = |\partial f|^2 = g_{i\overline{j}}h^{\alpha\overline{\beta}}f^i_\alpha\overline{f}^j_\beta$$

is the energy density of the map $f : (M, h) \to (N, g)$.

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- We introduce the generalized ∂-energy density on the projective bundle $\mathbb{P}(T_M) \to M$ is defined as

$$\mathscr{Y} = g_{i\bar{j}} f^i_\alpha \overline{f}^j_\beta \frac{W^\alpha \overline{W}^\beta}{\sum h_{\gamma\bar{\delta}} W^\gamma \overline{W}^\delta}.$$

By setting $\mathscr{H} = \sum h_{\gamma\bar{\delta}} W^\gamma \overline{W}^\delta$, we obtain the complex Hessian estimate:

$$\sqrt{-1}\partial\bar{\partial}\mathscr{Y} \geq \left(\sqrt{-1}\partial\bar{\partial}\log \mathscr{H}^{-1}\right)\cdot\mathscr{Y} - \frac{\sqrt{-1}R_{i\bar{j}k\bar{\ell}} f^i_\alpha \overline{f^j_\beta} f^k_\mu \overline{f^\ell_\nu} W^\mu \overline{W}^\nu dz^\alpha \wedge d\bar{z}^\beta}{\mathscr{H}}.$$

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- Recall that, there are many Kähler and non-Kähler complex manifolds with RC-positive tangent bundles
  - complex manifolds with $\mathrm{HSC} > 0$;
  - Fano manifolds;
  - manifolds with positive second Chern-Ricci curvature;
  - Hopf manifolds $\mathrm{S}^1 \times \mathrm{S}^{2n+1}$;
  - products of complex manifolds with RC-positive tangent bundles.

By using the Hessian estimate

$$\sqrt{-1}\partial\overline{\partial}\mathscr{Y} \geq \left(\sqrt{-1}\partial\overline{\partial}\log\mathscr{H}^{-1}\right)\cdot\mathscr{Y} - \frac{\sqrt{-1}R_{i\bar{j}k\bar{\ell}}f^i_\alpha\overline{f^j_\beta}f^k_\mu\overline{f^\ell_\nu}W^\mu\overline{W^\nu}dz^\alpha\wedge d\bar{z}^\beta}{\mathscr{H}},$$

we can also show

Theorem

Let $f:(M,h)\to(N,g)$ be a holomorphic map between two Hermitian manifolds. Suppose M is compact. If

(1) (M,h) *has* $\mathrm{HSC}>0$ *(resp.* $\mathrm{HSC}\geq 0$ *);*

(2) (N,g) *has* $\mathrm{HSC}\leq 0$ *(resp.* $\mathrm{HSC}<0$ *),*

then f is a constant map.

- Recall that: a smooth map f from a complex manifold M to a Kähler (or Riemannian) manifold N is pluri-harmonic if and only if for any holomorphic curve i: $C \to M$, the composition $f \circ i$ is harmonic.

Theorem

Let $f : M \to (N, g)$ be a pluri-harmonic map from a compact complex manifold M to a Riemannian manifold or a Kähler manifold (N, g)

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Let $f : (M, h) \to (N, g)$ be a harmonic map from a compact Kähler manifold (M, h) to a Riemannian manifold (N, g).

Then f is constant.

RC-positivity and rigidity of harmonic and pluri-harmonic maps.

- Recall that: a smooth map f from a complex manifold M to a Kähler (or Riemannian) manifold N is pluri-harmonic if and only if for any holomorphic curve i: $C \to M$, the composition $f \circ i$ is harmonic.

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Let $f : M \to N$ be a holomorphic map between two compact complex manifolds. If

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Question

Let $f : M \to N$ be a holomorphic map between two compact complex manifolds. If

(1) $\mathcal{O}_{T^*_M}(-1)$ is RC-positive;

(2) N is Kobayashi hyperbolic.

Then f is constant.

Thank you!