A finite dimensional proof of Verlinde formula

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Motivation: Jacobian variety and theta functions

- Let C be a smooth projective curve of genus g,

$$J_C^d = \{\text{vector bundles } E \text{ of } \mathrm{rk}(E) = 1,\ \deg(E) = d \text{ on } C\}$$

- Let $\mathcal{E}$ be a universal line bundle on $C \times J_C^d \xrightarrow{\pi} J_C^d$, and

$$\Theta_{J_C^d} := \det R\pi(\mathcal{E})^{-k} \otimes \det(\mathcal{E}_y)^{k(d+1-g)}$$

- $H^0(J_C^d, \Theta_{J_C^d})$ is the so called space of **theta functions of order** k

$$\dim H^0(J_C^d, \Theta_{J_C^d}) = k^g$$

- A. Weil (1938) (Généralisation des fonctions abéliennes) suggested to **generalize the theory to higher rank** $r > 1$.

Motivation: Moduli spaces and generalized theta functions

- (Mumford, Narasimhan-Seshadri): There exist **moduli spaces**

$$\mathcal{U}_C = \{\text{s.s. bundles } E \text{ of } \mathrm{rk}(E) = r,\ \deg(E) = d \text{ on } C\}$$

and **theta line bundles** $\Theta_{\mathcal{U}_C}$ **on** $\mathcal{U}_C$.

- $H^0(\mathcal{U}_C, \Theta_{\mathcal{U}_C})$: space of **generalized theta functions of order** k

$$\dim H^0(\mathcal{U}_C, \Theta_{\mathcal{U}_C}) = ?$$

- A formula was predicted by **Conformal Field Theory**, when $r = 2$,

$$\dim H^0(\mathcal{U}_C, \Theta_{\mathcal{U}_C}) = \left(\frac{k}{2}\right)^g \left(\frac{k+2}{2}\right)^{g-1} \sum_{i=0}^{k} \frac{(-1)^{id}}{(\sin\frac{(i+1)\pi}{k+2})^{2g-2}}$$

The moduli spaces: $\mathcal{U}_{C,\omega} = \mathcal{U}_C(r, d, \omega)$

- C: projective curve of genus $g \geq 0$ with at most one node
- $\omega = (k, \{\vec{n}(x), \vec{a}(x)\}_{x \in I})$: a finite set $I \subset C$ of smooth points,

$$\vec{n}(x) := (n_1(x), n_2(x), \cdots, n_{l_x+1}(x))$$

$$\vec{a}(x) := (a_1(x), a_2(x), \cdots, a_{l_x+1}(x))$$

and an integer $k > 0$ such that

$$0 \leq a_1(x) < a_2(x) < \cdots < a_{l_x+1}(x) < k.$$

- $\mathcal{U}_{C,\omega}$: moduli space of semistable parabolic sheaves of rank r and degree d on C with parabolic structures determined by ω

The moduli spaces: Parabolic sheaves

- A torsion free sheaf E has a parabolic structure of type $\vec{n}(x)$ and weights $\vec{a}(x)$ at a smooth point $x \in C$, we mean a choice of

$$E_x = Q_{l_x+1}(E)_x \twoheadrightarrow \cdots\cdots \twoheadrightarrow Q_1(E)_x \twoheadrightarrow Q_0(E)_x = 0$$

of fibre E_x with $n_i(x) = \dim(ker\{Q_i(E)_x \twoheadrightarrow Q_{i-1}(E)_x\})$ and a sequence of integers

$$0 \le a_1(x) < a_2(x) < \cdots < a_{l_x+1}(x) < k.$$

- For any $F \subset E$, let $Q_i(E)_x^F \subset Q_i(E)_x$ be the image of F,

$$n_i^F = \dim(ker\{Q_i(E)_x^F \twoheadrightarrow Q_{i-1}(E)_x^F\})$$

$$\mathrm{par}\chi(F) := \chi(F) + \frac{1}{k}\sum_{x\in I}\sum_{i=1}^{l_x+1} a_i(x) n_i^F(x).$$

The moduli spaces: Semi-stability

- E is called **semistable** (resp., **stable**) for $\frac{\vec{a}}{k}$ if for any nontrivial subsheaf $F \subset E$ such that E/F is torsion free, one has

$$\mathrm{par}\chi(F) \leq \frac{\mathrm{par}\chi(E)}{r} \cdot r(F) \text{ (resp., } <).$$

- There exists a seminormal projective variety

$$\mathcal{U}_{C,\omega} = \mathcal{U}_C(r, d, \omega)$$

which is the coarse moduli space of s-equivalence classes of semistable parabolic sheaves E of rank r and $\deg(E) = d$ with parabolic structures of type $\{\vec{n}(x)\}_{x\in I}$ and weights $\{\vec{a}(x)\}_{x\in I}$ at points $\{x\}_{x\in I}$.

- If C is smooth, then it is normal, with only rational singularities.

Generalized theta functions on $\mathcal{U}_{C,\omega}$

- There is an algebraic family of ample line bundles $\Theta_{\mathcal{U}_{C,\omega}}$ on $\mathcal{U}_{C,\omega}$ (the so called Theta line bundles) when

$$\ell := \frac{k\chi - \sum_{x\in I}\sum_{i=1}^{l_x} d_i(x)r_i(x)}{r}$$

is an integer, where

$$d_i(x) = a_{i+1}(x) - a_i(x)$$

$$r_i(x) = n_1(x) + \cdots + n_i(x).$$

- $\mathrm{H}^0(\mathcal{U}_{C,\omega}, \Theta_{\mathcal{U}_{C,\omega}})$: The space of generalized theta functions. An explicit formula of

$$D_g(r,d,\omega) = \dim\, \mathrm{H}^0(\mathcal{U}_{C,\omega}, \Theta_{\mathcal{U}_{C,\omega}})$$

was predicted by **Conformal Field Theory**.

Verlinde formula: $D_g(r,d,\omega) = ?$

$$D_g(r,d,\omega) = (-1)^{d(r-1)}\left(\frac{k}{r}\right)^g (r(r+k)^{r-1})^{g-1}$$

$$\sum_{\vec{v}} \frac{\exp\left(2\pi i\left(\frac{d}{r}-\frac{|\omega|}{r(r+k)}\right)\sum_{i=1}^r v_i\right) S_\omega\left(\exp 2\pi i \frac{\vec{v}}{r+k}\right)}{\prod_{i<j}\left(2\sin \pi\frac{v_i-v_j}{r+k}\right)^{2(g-1)}}$$

where $\vec{v} = (v_1, v_2, \ldots, v_r)$ runs through the integers

$$0 = v_r < v_{r-1} < \cdots < v_2 < v_1 < r+k.$$

- For given $\omega = (k, \{\vec{n}(x), \vec{a}(x)\}_{x\in I})$, let $\lambda_i = k - a_i(x)$

$$\lambda_x = (\overbrace{\lambda_1,\ldots,\lambda_1}^{n_1(x)}, \overbrace{\lambda_2,\ldots,\lambda_2}^{n_2(x)}, \ldots, \overbrace{\lambda_{l_x+1},\ldots,\lambda_{l_x+1}}^{n_{l_x+1}(x)})$$

- Let $S_{\lambda_x}(z_1,\ldots,z_r)$ be Schur polynomial, $|\lambda_x| = \sum \lambda_i n_i(x)$,

$$S_\omega(z_1,\ldots,z_r) = \prod_{x\in I} S_{\lambda_x}(z_1,\ldots,z_r), \quad |\omega| = \sum_{x\in I}|\lambda_x|.$$

Rational Conformal Field Theory (RCFT)

- Let Λ be a finite set with an involution $\lambda \mapsto \lambda^*$, a **RCFT** is a functor:
$$(C, \overrightarrow{p}; \overrightarrow{\lambda}) \mapsto V_C(\overrightarrow{p}; \overrightarrow{\lambda})$$
where $\overrightarrow{p} = (p_1, \ldots, p_n)$, $p_i \in C$, $\overrightarrow{\lambda} = (\lambda_1, \ldots, \lambda_n)$, satisfies axioms:

- **A0:** $V_{\mathbb{P}^1}(\emptyset) = \mathbb{C}$, **A1:** $V_C(\overrightarrow{p}; \overrightarrow{\lambda}) \cong V_C(\overrightarrow{p}; \overrightarrow{\lambda}^*)$

- **A2:** Let $(C, \overrightarrow{p}; \overrightarrow{\lambda}) = (C', \overrightarrow{p}'; \overrightarrow{\lambda}') \sqcup (C'', \overrightarrow{p}''; \overrightarrow{\lambda}'')$. Then
$$V_C(\overrightarrow{p}; \overrightarrow{\lambda}) = V_{C'}(\overrightarrow{p}'; \overrightarrow{\lambda}') \otimes V_{C''}(\overrightarrow{p}''; \overrightarrow{\lambda}'')$$

- **A3:** For a family $\{C_t, \overrightarrow{p_t}; \overrightarrow{\lambda}\}_{t \in \triangle}$, there are canonical isomorphisms
$$V_{C_t}(\overrightarrow{p_t}; \overrightarrow{\lambda}) \cong V_{C_0}(\overrightarrow{p_0}; \overrightarrow{\lambda})$$

- **A4:** If C_0 has a node x, $\pi^{-1}(x) = \{x_1, x_2\}$, $\pi : \widetilde{C_0} \to C_0$. Then
$$V_{C_t}(\overrightarrow{p_t}; \overrightarrow{\lambda}) \cong \bigoplus_{\nu} V_{\widetilde{C_0}}(\overrightarrow{p_0}, x_1, x_2; \overrightarrow{\lambda}, \nu, \nu^*)$$

The fusion rules

- $\dim V_C(\overrightarrow{p}; \overrightarrow{\lambda})$ depends only on $g = g(C)$ and

$$(\lambda_1, ..., \lambda_n) := \lambda_1 + \cdots + \lambda_n.$$

- $\mathbb{N}^{(\Lambda)} := \{x = \lambda_1 + \cdots + \lambda_n \,|\, n \geq 0,\ \lambda_i \in \Lambda\}$,

$$N_g : \mathbb{N}^{(\Lambda)} \to \mathbb{N}, \qquad N_g(x) := \dim_{\mathbb{C}} V_C(\overrightarrow{p}; \overrightarrow{\lambda}).$$

- $N_g(x) = \sum_{\lambda \in \Lambda} N_{g-1}(x + \lambda + \lambda^*)$
- $N_0(0) = 1$
- $N_0(x) = N_0(x^*)$ $(\forall\ x \in \mathbb{N}^{(\Lambda)})$
- $N_0(x + y) = \sum_{\lambda \in \Lambda} N_0(x + \lambda) N_0(y + \lambda^*)$ $(\forall\ x,\ y \in \mathbb{N}^{(\Lambda)})$.

The fusion ring $\mathcal{F}$

- Let $\mathcal{F} = \mathbb{Z}^{(\Lambda)}$ be the free abelian group generated by Λ, define

$$\lambda \cdot \mu = \sum_{\nu \in \Lambda} N_0(\lambda + \mu + \nu^*) \cdot \nu.$$

- a bilinear form $(\cdot, \cdot) : \mathcal{F} \times \mathcal{F} \to \mathbb{Z}$ such that

$$(x \cdot z, y) = (x, z^* \cdot y)$$

- Λ is an orthonormal basis;

- $\mathcal{F}$ is called the **fusion ring** associated to the **RCFT**,

$$\mathcal{F}_{\mathbb{C}} := \mathcal{F} \otimes_{\mathbb{Z}} \mathbb{C}.$$

Formulation of Verlinde formula

- Let $\Sigma = \{\, \chi : \mathcal{F} \to \mathbb{C} \,\}$ be the set of characters of $\mathcal{F}$. Then

$$\dim V_C(\overrightarrow{p}; \overrightarrow{\lambda}) = \sum_{\chi \in \Sigma} \chi(\lambda_1) \cdots \chi(\lambda_n) \left(\sum_{\lambda \in \Lambda} |\chi(\lambda)|^2 \right)^{g-1}$$

- $\{\, m_x : \mathcal{F}_{\mathbb{C}} \to \mathcal{F}_{\mathbb{C}} \,|\, \forall\, x \in \mathcal{F}_{\mathbb{C}} \,\} \subset \mathrm{End}(\mathcal{F}_{\mathbb{C}})$ is a commutative subalgebra, let M_x be the matrix of linear operator m_x under the orthonormal basis Λ of $\mathcal{F}_{\mathbb{C}}$;

- There exists a unitary matrix $S = (S_{\lambda\mu})_{\lambda,\mu \in \Lambda}$ such that

$$S \cdot M_x \cdot S^{-1}$$

  is diagonal for all $x \in \mathcal{F}_{\mathbb{C}}$. Then

$$\dim V_C(\overrightarrow{p}; \overrightarrow{\lambda}) = \sum_{\nu \in \Lambda} \frac{S_{\nu\lambda_1} \cdots S_{\nu\lambda_n}}{S_{\nu 1}^{2g-2+n}}.$$

Verlinde conjecture

- Let $E_\tau = \mathbb{C}/\{1, \tau\}$ ($\tau \in \mathbb{H}$), $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Then

$$E_\tau \cong E_{\gamma\cdot\tau}.$$

- The axiom **A3** and axiom **A4** give a **unitary action** of $\mathrm{SL}_2(\mathbb{Z})$ on

$$V_E(\emptyset) \cong \bigoplus_{\lambda\in\Lambda} V_{\mathbb{P}^1}(p_1, p_2, \lambda, \lambda^*) \cong \mathcal{F}_\mathbb{C}$$

- The unitary action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{F}_\mathbb{C}$ can usually written explicitly;

- **Verlinde conjecture** (Nuclear Physics B300 (1988), 360–376): Let $S = (S_{\lambda\mu})_{\lambda,\mu\in\Lambda}$ be the matrix of modular transformation $\tau \mapsto -1/\tau$ (under orthonormal basis Λ of $\mathcal{F}_\mathbb{C}$). Then

$$S \cdot M_x \cdot S^{-1}$$

is diagonal for all $x \in \mathcal{F}_\mathbb{C}$.

Tsuchiya-Ueno-Yamada (1989): WZW model

- **Wess-Zumino-Witten (WZW) model** is associated to a simple complex Lie algebra $\mathfrak{g}$ and integer $k > 0$.

- Given a simple Lie algebra $\mathfrak{g}$ and integer $k > 0$, let P_k be the set of dominant weight of level $\leq k$, $V_{\vec{\lambda}} := V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$ $(\lambda_i \in P_k)$ and

$$V_C(\vec{p}; \vec{\lambda}) := \mathrm{Hom}_{\mathfrak{g} \otimes A_C}(\mathcal{H}_k, V_{\vec{\lambda}}), \quad A_C = \mathcal{O}_C(C - \{q\})$$

where $\mathcal{H}_k$ is the basic representation of level k of affine Lie algebra $\widehat{\mathfrak{g}}$, and $\mathfrak{g} \otimes A_C \hookrightarrow \mathfrak{g} \otimes \mathbb{C}((z)) \subset \widehat{\mathfrak{g}}$ is a Lie subalgebra of $\widehat{\mathfrak{g}}$.

- Tsuchiya-Ueno-Yamada (1989): Functor: $(C, \vec{p}; \vec{\lambda}) \mapsto V_C(\vec{p}; \vec{\lambda})$ satisfies the **axioms A0 to A4**.

- The dimension $N_g(\vec{\lambda}) = \dim V_C(\vec{p}; \vec{\lambda})$ of the "spaces of conformal blocks" is **computable**.

WZW model and generalized theta functions

- **Beauville- Laszlo** (1994): For $\mathfrak{g} = \mathfrak{sl}_r(\mathbb{C})$, we have

$$V_C(\emptyset) \cong H^0(\mathcal{SU}_C(r), \Theta^k_{\mathcal{U}_C})$$

- **Faltings** (1994): It is true for arbitrary simple Lie algebra $\mathfrak{g}$, if $\mathcal{SU}_C(r)$ is replaced by "moduli spaces of G-bundles on C", where G is the algebraic group with Lie algebra $\mathfrak{g}$.

- **Pauly** (1996): For $\mathfrak{g} = \mathfrak{sl}_r(\mathbb{C})$, $V_C(\overrightarrow{p}; \overrightarrow{\lambda}) \cong H^0(\mathcal{U}_C(r), \Theta^k_{\mathcal{U}_C})$, where $\mathcal{U}_C(r)$ is the moduli spaces of parabolic bundles on C.

- **Beauville**: As soon as the Verlinde formula became known to mathematicians, it became a **challenge** for them to give a rigorous proof, so a wealth of proofs have appeared.

Finite-dimensional proofs

- **Beauville**: The basic distinction between the proofs using standard algebraic geometry, **which up to now work only in the case** $r = 2$, and proofs that use infinite-dimensional algebraic geometry to **mimic the heuristic approach of the physicists-these work for all** r.

- Compute $\chi(\Theta^k_{\mathcal{U}_C})$: Bertram-Szenes, Zagier, Donaldson-Witten.

- Thaddeus (1994): Stable pairs, linear systems and Verlinde formula (Invent. Math. 117, 317-353).

- Narasimhan-Ramadas (1993): Factorization of generalized theta functions (Invent. Math. 114, 565-623).

- **Beauville**: ...which up to now work only in the case $r = 2$ and **an extension to higher rank seems out of reach**.

Degeneration method

- Degenerate $C_t \rightsquigarrow C_0 = X$ to a curve X with one node $x_0 \in X$.
- Need to prove: $\dim H^0(\mathcal{U}_{C_t,\omega_t}, \Theta_{\mathcal{U}_{C_t,\omega_t}}) = \dim H^0(\mathcal{U}_{X,\omega}, \Theta_{\mathcal{U}_{X,\omega}})$.
- Let $\pi : \widetilde{X} \to X$ be the normalization, $\pi^{-1}(x_0) = \{x_1, x_2\}$.

Theorem 1 (Sun, 2000-2003)

$$H^0(\mathcal{U}_{X,\omega}, \Theta_{\mathcal{U}_{X,\omega}}) \cong \bigoplus_{\mu} H^0(\mathcal{U}_{\widetilde{X},\omega^\mu}, \Theta_{\mathcal{U}_{\widetilde{X},\omega^\mu}})$$

$$H^0(\mathcal{U}_{X_1\cup X_2,\omega_1\cup\omega_2}, \Theta_{\mathcal{U}_{X_1\cup X_2,\omega_1\cup\omega_2}}) \cong \bigoplus_{\mu} H^0(\mathcal{U}_{X_1,\omega_1^\mu}, \Theta_{\mathcal{U}_{X_1,\omega_1^\mu}}) \otimes H^0(\mathcal{U}_{X_2,\omega_2^\mu}, \Theta_{\mathcal{U}_{X_2,\omega_2^\mu}})$$

where $\mu = (\mu_1, \cdots, \mu_r)$ *runs through* $0 \le \mu_r \le \cdots \le \mu_1 < k$.

Vanishing Theorem

Theorem 2 (Sun, 2000)

- *If* $g(C_t) \geq 2$*, then* $H^1(\mathcal{U}_{C_t,\omega_t}, \Theta_{\mathcal{U}_{C_t,\omega_t}}) = 0$.
- *If* $g(X) \geq 3$*, then* $H^1(\mathcal{U}_{X,\omega}, \Theta_{\mathcal{U}_{X,\omega}}) = 0$.

- $\mathcal{U}_{C,\omega} = \mathcal{R}^{ss}_{\omega}//G$, $\mathcal{R}^{ss}_{\omega} \subset \mathcal{R}$ is the set of GIT semistable points.
- $H^1(\mathcal{U}_{C,\omega}, \Theta_{\mathcal{U}_{C,\omega}}) = H^1(\mathcal{R}^{ss}_{\omega}, \Theta_{\omega})^{inv.} = H^1(\mathcal{R}, \Theta_{\omega})^{inv.}$
- $\Theta_{\omega} = \omega_{\mathcal{R}} \otimes \Theta_{\omega'}$, $H^1(\mathcal{R}, \Theta_{\omega})^{inv.} = H^1(\mathcal{R}^{ss}_{\omega'}, \omega_{\mathcal{R}} \otimes \Theta_{\omega'})^{inv.}$.
- $H^1(\mathcal{U}_{C,\omega}, \Theta_{\mathcal{U}_{C,\omega}}) = H^1(\mathcal{U}_{C,\omega'}, \Theta_{\mathcal{U}_{C,\omega'}} \otimes (\varphi_* \omega_{\mathcal{R}^{ss}_{\omega'}})^{inv.})$ where

$$\varphi : \mathcal{R}^{ss}_{\omega'} \to \mathcal{U}_{C,\omega'}$$

Where is the condition $g \geq 2$ needed ?

Proposition 1 (Sun, 2000)

For any $\omega = (r, d, I, \{\vec{n}(x), \vec{a}(x)\}_{x\in I}, k)$, *we have*

(1) $\mathrm{codim}(\mathcal{R}^{ss}_\omega \setminus \mathcal{R}^s_\omega) \geq (r-1)(g-1) + \frac{|\mathrm{I}|}{k}$,

(2) $\mathrm{codim}(\mathcal{R} \setminus \mathcal{R}^{ss}_\omega) > (r-1)(g-1) + \frac{|\mathrm{I}|}{k}$.

- $H^1(\mathcal{R}^{ss}_\omega, \Theta_\omega)^{inv.} = H^1(\mathcal{R}, \Theta_\omega)^{inv.}$ (It may hold unconditionally if $\mathcal{R}$ is projective with only rational singularity by C. Teleman).

- $H^1(\mathcal{R}, \omega_{\mathcal{R}} \otimes \Theta_{\omega'})^{inv.} = H^1(\mathcal{R}^{ss}_{\omega'}, \omega_{\mathcal{R}} \otimes \Theta_{\omega'})^{inv.}$ if

$$\mathrm{codim}(\mathcal{R} \setminus \mathcal{R}^{ss}_{\omega'}) > 2.$$

- (F. Knop): $\omega_{\mathcal{U}_{C,\omega'}} = (\varphi_* \omega_{\mathcal{R}^{ss}_{\omega'}})^{inv.}$ if $\mathrm{codim}(\mathcal{R}^{ss}_{\omega'} \setminus \mathcal{R}^s_{\omega'}) \geq 2$.

Recurrence relations of $D_g(r, d, \omega)$

Theorem 3 (Sun–Zhou, 2016)

For any positive integers c_1, c_2 *and partitions* $I = I_1 \cup I_2$, $g = g_1 + g_2$ *such that* $\ell_j = \frac{c_j \ell}{c_1 + c_2}$ $(j =, 2)$ *are integers, we have*

$$D_g(r, d, \omega) = \sum_{\mu} D_{g-1}(r, d, \omega^{\mu})$$

$$D_g(r, d, \omega) = \sum_{\mu} D_{g_1}(r, d_1^{\mu}, \omega_1^{\mu}) \cdot D_{g_2}(r, d_2^{\mu}, \omega_2^{\mu})$$

where $\mu = (\mu_1, \cdots, \mu_r)$ *runs through* $0 \leq \mu_r \leq \cdots \leq \mu_1 < k$ *and*

$$\omega^{\mu} = (k, \{\vec{n}(x), \vec{a}(x)\}_{x \in I \cup \{x_1, x_2\}}), \; \omega_j^{\mu} = (k, \{\vec{n}(x), \vec{a}(x)\}_{x \in I_j \cup \{x_j\}})$$

with $\vec{n}(x_j)$, $\vec{a}(x_j)$ *and* d_j^{μ} $(j = 1, 2)$ *determined by* μ.

Globally F-regular varieties

- Let M be a variety over a perfect field k of $char(k) = p > 0$,

$$F : M \to M$$

be the (absolute) Frobenius map and $F^e : M \to M$ be the e-th iterate of Frobenius map.

- When M is normal, for any (weil) divisor $D \in Div(M)$,

$$\mathcal{O}_M(D)(V) = \{\, f \in K(M) \,|\, div_V(f) + D|_V \geq 0 \,\}, \quad \forall\, V \subset M$$

is a reflexive subsheaf of constant sheaf $K = k(M)$

Definition 1

A normal variety M over a perfect field is called stably Frobenius D-split if

$$\mathcal{O}_M \to F^e_* \mathcal{O}_M(D)$$

is split for some $e > 0$.

Globally F-regular varieties: Projective case

Definition 2

A normal variety M over a perfect field is called globally F-regular if M is stably Frobenius D-split for any effective divisor D.

Proposition 2

Let M be a projective variety over a perfect field. Then the following statements are equivalent.

1. *M is normal and is stably Frobenius D-split for any effective D;*
2. *M is stably Frobenius D-split for any effective Cartier D;*
3. *For any ample line bundle $\mathcal{L}$, the section ring of M*

$$R(M, \mathcal{L}) = \bigoplus_{n=0}^{\infty} H^0(M, \mathcal{L}^n)$$

is strongly F-regular.

Globally F-regular varieties: Theory of Characteristic 0

Definition 3

*A variety M over a field of characteristic zero is said to be of globally F-regular type if its "**modulo p reduction of M**" are globally F-regular for a dense set of p.*

Proposition 3 (K. E. Smith)

Let M be a projective variety over a field of characteristic zero. If M is of globally F-regular type, then we have

(1) *M is normal, Cohen-Macaulay with rational singularities. If M is $\mathbb{Q}$-Gorenstein, then M has log terminal singularities.*

(2) *For any nef line bundle $\mathcal{L}$ on M, we have $H^i(M, \mathcal{L}) = 0$ when $i > 0$. In particular, $H^i(M, \mathcal{O}_M) = 0$ whenever $i > 0$.*

Moduli spaces: globally F-regular type

Definition 4

Let C be a smooth projective curve, $\omega = (k, \{\vec{n}(x), \vec{a}(x)\}_{x\in I})$ and

$$\det : \mathcal{U}_{C,\omega} \to J_C^d, \quad E \mapsto \det(E).$$

For any $L \in J_C^d$, the fiber $\mathcal{U}_{C,\omega}^L := \det^{-1}(L)$ is called **moduli spaces of semi-stable parabolic bundles with fixed determinant**, *which is normal with at most rational singularities.*

Theorem 4 (Sun-Zhou, 2016)

For any data ω, the moduli spaces $\mathcal{U}_{C,\omega}^L$ is **of globally F-regular type**.

Corollary 1

For any ample line bundle $\mathcal{L}$ on $\mathcal{U}_{C,\omega}$, we have $H^i(\mathcal{U}_{C,\omega}, \mathcal{L}) = 0, \ \forall\ i > 0$.

Vanishing Theorem for node curve X

Definition 5

Let $\pi : \widetilde{X} \to X$ *be the normalization of* X, $\pi^{-1}(x_0) = \{x_1, x_2\}$. *A generalized parabolic sheaf (GPS)* (E, Q) *consist:*

- *A parabolic sheaf* E *determined by* $\omega = (r, d, \{\vec{n}(x), \vec{a}(x)\}_{x \in I}, k)$,
- *A* r*-dimensional quotient* $E_{x_1} \oplus E_{x_2} \xrightarrow{q} Q \to 0$.
- (E, Q) *is semi-stable if* $\forall\, E' \subset E$, E/E' *torsion free outside* $\{x_1, x_2\}$

$$pardeg(E') - dim(Q^{E'}) \leq rk(E') \frac{pardeg(E) - dim(Q)}{r}$$

Normalization of $\mathcal{U}_{X,\omega}$: The moduli space $\mathcal{P}$

- $\mathcal{P} = \{$ semi-stable GPS $(E, Q) = (E, E_{x_1} \oplus E_{x_2} \to Q \to 0) \}$, which is called **moduli space of GPS** (*generalized parabolic sheaf*).

- $\phi : \mathcal{P} \to \mathcal{U}_{X,\omega}$ is defined by $\phi(E, Q) = F$, where F is given by
$$0 \to F \to \pi_* E \to_{x_0} Q \to 0$$

- $\phi : \mathcal{P} \to \mathcal{U}_{X,\omega}$ is the normalization of $\mathcal{U}_{X,\omega}$ such that
$$\phi^* : \mathrm{H}^1(\mathcal{U}_X, \Theta_{\mathcal{U}_X}) \hookrightarrow \mathrm{H}^1(\mathcal{P}, \Theta_{\mathcal{P}}).$$

- There exist a flat morphism $Det : \mathcal{P} \to J^d_{\widetilde{X}}$, let
$$\mathcal{P}^L = Det^{-1}(L).$$

Globally F-regular type of $\mathcal{P}^L$

Theorem 5 (Sun-Zhou, 2016)

The moduli space $\mathcal{P}^L$ of semi-stable generalized parabolic sheaves with fixed determinant L is of globally F-regular type.

Corollary 2

$H^i(\mathcal{P}^L, \mathcal{L}) = 0$ *for any* $i > 0$ *and nef line bundles* $\mathcal{L}$ *on* $\mathcal{P}^L$ *and*

$$H^i(\mathcal{P}, \Theta_{\mathcal{P}}) = 0 \quad \forall\ i > 0.$$

Corollary 3

Let X be a projective curve with at most one node and $\mathcal{U}_{X,\omega}$ be the moduli space of parabolic sheaves on X with any given data ω. Then

$$H^1(\mathcal{U}_{X,\omega}, \Theta_{\mathcal{U}_{X,\omega}}) = 0.$$

Sketch of Proof: When $|I|$ is large enough

- Recall $\widetilde{\mathcal{R}}'_I := \widetilde{\mathcal{R}}' = \mathrm{Grass}_r(\widetilde{\mathcal{F}}_{x_1} \oplus \widetilde{\mathcal{F}}_{x_2}) \to \widetilde{\mathcal{R}}_I = \times_{x\in I} Flag_{\vec{n}(x)}(\widetilde{\mathcal{F}}_x)$, $\mathcal{P}^L = \widetilde{\mathcal{R}}^{ss}_{I,\omega}//SL(V)$ is determined by $\omega = (r, d, \{\vec{n}(x), \vec{a}(x)\}_{x\in I}, k)$.

Proposition 4 (Sun, 2000)

There is ω_c such that $\mathcal{P}^L_{\omega_c} = \widetilde{\mathcal{R}}^{ss}_{I,\omega_c}//SL(V)$ is a Fano variety with only rational singularities (thus F-split type) if $(r-1)(g-1) + \frac{|I|}{2r} \geq 2$.

Proposition 5 (Sun, 2000)

For any $\omega = (r, d, I, \{\vec{n}(x), \vec{a}(x)\}_{x\in I}, k)$, we have

(1) $\mathrm{codim}(\widetilde{\mathcal{R}}'_I \setminus \widetilde{\mathcal{R}}'^{ss}_{I,\omega}) > (r-1)(g-1) + \frac{|\mathrm{I}|}{k}$,

(2) $\mathrm{codim}(\widetilde{\mathcal{R}}'^{ss}_{I,\omega} \setminus \{\mathcal{D}^f_1 \cup \mathcal{D}^f_2\}\widetilde{\mathcal{R}}'^{s}_{I,\omega}) \geq (r-1)(g-1) + \frac{|\mathrm{I}|}{k}$.

- Let $\widetilde{U} = \mathcal{R}'^{ss}_{I,\omega} \cap \widetilde{\mathcal{R}}'^{ss}_{I,\omega_c}$, then $codim(\widetilde{\mathcal{R}}'^{ss}_{I,\omega} \setminus \widetilde{U}) \geq 2$.

Sketch of Proof: Increase the number $|I|$

- Add extra parabolic points $x \in J \subset \widetilde{X}$, the projection

$$p_I : \widetilde{\mathcal{R}}'_{I\cup J} \to \widetilde{\mathcal{R}}'_I$$

is $\mathrm{SL}(V)$-invariant. Choose $|J|$ such that $(r-1)(g-1)+\frac{|I\cup J|}{k+2r}\geq 2$.

- Choose the canonical weight ω_c on $\widetilde{\mathcal{R}}'_{I\cup J}$, consider

$$p_I^{-1}(\widetilde{\mathcal{R}}'^{ss}_{I,\omega}) \supset \widetilde{U} = p_I^{-1}(\widetilde{\mathcal{R}}'^{ss}_{I,\omega}) \cap \widetilde{\mathcal{R}}'^{ss}_{I\cup J,\omega_c} \to \mathcal{P}^L_{\omega_c}.$$

Then $p_I^{-1}(\widetilde{\mathcal{R}}'^{ss}_{I,\omega}) \setminus \widetilde{U} = p_I^{-1}(\widetilde{\mathcal{R}}'^{ss}_{I,\omega}) \cap (\widetilde{\mathcal{R}}'_{I\cup J,\omega_c} \setminus \widetilde{\mathcal{R}}'^{ss}_{I\cup J,\omega_c})$ has codimension at least $(r-1)(g-1)+\frac{|I\cup J|}{2r}\geq 2$.

- Let $U \subset \mathcal{P}^L_{\omega_c}$ be the image of $\widetilde{U}$, then p_I induces a morphism $f : U \to \mathcal{P}^L$ such that $f_*(\mathcal{O}_U) = \mathcal{O}_{\mathcal{P}^L}$.

Problem and discussions

Definition 6

Let X be a scheme and $Y \subset X$ a closed sub-scheme. The pair $(X, Y))$ is called of **compatible Frobenius split type** *if*

- *X is of Frobenius split type*
- *For almost p, there is a F-split $\varphi : F_* \mathcal{O}_{X_p} \to \mathcal{O}_{X_p}$ such that*

$$\varphi(F_* \mathcal{I}_{Y_p}) \subset \mathcal{I}_{Y_p}.$$

Problem 1

Are the pairs $(\mathcal{P}, \mathcal{D}_j(a))$ ($j = 1,\ 2$), $(\mathcal{D}_1(a), \mathcal{D}_1(a) \cap \mathcal{D}_2 \cup \mathcal{D}_1(a-1))$ of **compatible Frobenius split type** *?*

- If the answer of above problem is Yes, then, for any ample line bundle $\mathcal{L}$ on $\mathcal{U}_X$,

$$H^i(\mathcal{U}_X, \mathcal{L}) = 0 \quad \forall\ i > 0.$$

Thanks !