p-adic period domains and the Fargues-Rapoport conjecture

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- **Non-archimedean counterparts/analogues ?**

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$$0 \to (\mathrm{Lie} H^\vee)_F^\vee \to \mathbb{D}(\overline{H})_F \to (\mathrm{Lie} H)_F \to 0,$$

and $\rho : \mathbb{D}(\overline{H})_F \simeq D \otimes_{\breve{\mathbb{Q}}_p} F$, $\dim(\mathrm{Lie} H)_F = d$,

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and $\rho : \mathbb{D}(\overline{H})_F \simeq D \otimes_{\breve{\mathbb{Q}}_p} F$, $\dim(\mathrm{Lie} H)_F = d$, $\rightsquigarrow$ **p-adic period map** :

$$\pi_{dR} : \mathcal{M}_{\mathbb{H}}(F) \to Gr(d, h).$$

- **Grothendieck (ICM 1970)** : describe the image $\mathcal{F}^a := \mathrm{Im}\pi_{dR} \subset Gr(d, h)$ as a subspace ?

p-adic period domains : Rapoport-Zink's construction

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Rapoport-Zink : $\mathcal{F}(G, \mu, b)^{wa} \neq \emptyset$, $\mathcal{F}(G, \mu, b)^{wa} = \mathcal{F} \setminus \bigcup_{i \in I} J_b(\mathbb{Q}_p) Z_i$, where I is finite, $\forall i \in I, Z_i$ is some closed Schubert variety.

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Example

1. $(D^{\times}_{\frac{1}{n}},[b],\{\mu\})$, $[b]$ basic, $\{\mu\} \longleftrightarrow (1,0,\cdots,0)$, $\mathcal{F}^{wa} = \Omega^n_{\breve{\mathbb{Q}}_p}$,
   $\Omega^n = \mathbb{P}^{n-1}_{\mathbb{Q}_p} \setminus \bigcup_{H \textit{ rational hyperlane}} H$.
2. $(\mathrm{GL}_n,[b],\{\mu\})$, $[b]$ basic, $\{\mu\} \longleftrightarrow (1,0,\cdots,0)$, $\mathcal{F}^{wa} = \mathbb{P}^{n-1,ad}_{\breve{\mathbb{Q}}_p}$.
3. $(\mathrm{GL}_2,[b],\{\mu\})$, $[b]$ ordinary, $\{\mu\} \longleftrightarrow (1,0)$, $\mathcal{F}^{wa} = \mathbb{A}^{1,ad}_{\breve{\mathbb{Q}}_p}$.

p-adic period domains : Rapoport-Zink conjecture

Conjecture (Rapoport-Zink)

$\exists$ open dense subspace $\mathcal{F}(G,\mu,b)^a \subset \mathcal{F}(G,\mu,b)^{wa}$, and a G- $\mathbb{Q}_p$-local system $\mathcal{L}$ over $\mathcal{F}(G,\mu,b)^a$, such that on all classical points $x \in \mathcal{F}(G,\mu,b)^a$, $\mathcal{L}_x$ gives the p-adic Galois rep. of $K(x)$ as above.

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- **non-archimedean $\Rightarrow$ archimedean** : similar characterization can be applied to the Borel embedding $X \subset \mathcal{F}(G,\mu)$, $\rightsquigarrow$ **Simpson's theory of twistor structures**...

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Recall the complex setting : $X^+\twoheadrightarrow\Gamma\setminus X^+$, **archimedean $\not\leftrightarrow$ non-archimedean !**

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  Recall the complex setting : $X^+ \twoheadrightarrow \Gamma \setminus X^+$, **archimedean $\not\leftrightarrow$ non-archimedean !**
- In general, $\mathcal{E}$ $\rightsquigarrow$ **local Shimura varieties** $(\mathcal{M}_K)_{K\subset G(\mathbb{Q}_p)}$ in the sense of Rapoport-Viehmann.

p-adic period domains : local Shimura varieties

- $\mathcal{F}(G,\mu,b)^a$: true p-adic analogues of Griffiths period domains. $G = \mathrm{GL}_h$, $\mathcal{F}(G,\mu,b)^a$ gives an answer of **the question of Grothendieck**.
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- In general, $\mathcal{E} \rightsquigarrow$ **local Shimura varieties** $(\mathcal{M}_K)_{K\subset G(\mathbb{Q}_p)}$ in the sense of Rapoport-Viehmann.
- For $[b]$ basic, **Kottwitz conjecture** : the supercuspidal part of $\varinjlim_K H_c^*(\mathcal{M}_{K,\mathbb{C}_p},\overline{\mathbb{Q}}_\ell)$ realizes the **local Langlands/Jacquet-Langlands correspondences** for $G(\mathbb{Q}_p)$ and $J_b(\mathbb{Q}_p)$.

p-adic period domains : $\mathcal{F}(G,\mu,b)^a$ vs. $\mathcal{F}(G,\mu,b)^{wa}$

$$(G,[b],\{\mu\}),\rightsquigarrow \mathcal{F}(G,\mu,b)^a \subset \mathcal{F}(G,\mu,b)^{wa} \subset \mathcal{F}(G,\mu)^{ad}.$$

- $\mathcal{F}(G,\mu,b)^{wa}$: easier to describe, but usually larger than what we need.
- $\mathcal{F}(G,\mu,b)^{a}$: exactly what we want, but usually hard to explicitly describe.

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- **Example (Hartl)** : $(\mathrm{GL}_5,[b],\{\mu\}), \{\mu\} \longleftrightarrow (1,1,1,0,0)$, $[b]$ basic, then $\mathcal{F}(G,\mu,b)^a \subsetneq \mathcal{F}(G,\mu,b)^{wa} = \mathcal{F}(G,\mu)^{ad} = Gr(3,5)^{ad}$.

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- However, $\mathcal{F}(G,\mu,b)^a = \mathcal{F}(G,\mu,b)^{wa}$ in the previous examples :
  1. **Drinfeld** : $\mathcal{M}_{Dr} \twoheadrightarrow \Omega^n_{\breve{\mathbb{Q}}_p}$
  2. **Lubin-Tate, Gross-Hopkins** : $\mathcal{M}_{E[p^\infty]} \twoheadrightarrow \mathbb{P}^{1,ad}_{\breve{\mathbb{Q}}_p}$, $E/\overline{\mathbb{F}}_p$ **supersingular** elliptic curve
  3. **Katz, Serre-Tate** : $\mathcal{M}_{E[p^\infty]} \twoheadrightarrow \mathbb{A}^{1,ad}_{\breve{\mathbb{Q}}_p}$, $E/\overline{\mathbb{F}}_p$ **ordinary** elliptic curve

The Fargues-Rapoport conjecture

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Conjecture (Fargues-Rapoport)

Assume $[b] = [b_0]$ ***basic****. Then* $\mathcal{F}(G, \mu, b)^a = \mathcal{F}(G, \mu, b)^{wa}$ *if and only if* $(G, \{\mu\})$ *is* ***fully Hodge-Newton decomposable****.*

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Example

The following pairs are fully Hodge-Newton decomposable :

1. $\mathrm{GL}_h, \{\mu\} \longleftrightarrow (1, 0, \ldots, 0)$
2. $GU(V, \langle, \rangle), \{\mu\} \longleftrightarrow ((1, \cdots, 1, 0), 0)$
3. $GSp_4, \{\mu\} \longleftrightarrow (1, 1, 0, 0)$
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In fact, **Görtz-He-Nie** give a complete classification of such fully Hodge-Newton decomposable $(G, \{\mu\})$. ($\Rightarrow G$ classical group)

Some key ideas in the proof

- Weakly admissibility in terms of the Fargues-Fontaine curve.
- Harder-Narasimhan stratification of the p-adic flag variety $\mathcal{F} = \coprod_{b' \in B(G,0,\nu_b\mu^{-1})} \mathcal{F}^{b'}$.
- Explicit description of the generalized Kottwitz set $B(G, 0, \nu_b\mu^{-1})$.
- Explicit description of each stratum $\mathcal{F}^{b'}$ using "dual" local Shimura varieties and the Hodge-Tate period map π_{HT}.
- ...

Thank you !