# Asymptotic growth of the cohomology of Bianchi groups

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# Contents

Let $k \geq 2$, $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ congruence, $\mathcal{M}_k(\Gamma)$ (resp. $\mathcal{S}_k(\Gamma)$) space of (cusp) modular forms of weight $k$ and level $\Gamma$. We can compute dimensions of $\mathcal{M}_k(\Gamma)$ and $\mathcal{S}_k(\Gamma)$ using Riemann-Roch theorem.

Example.

$$\dim_{\mathbb{C}} \mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z})) = \begin{cases} \lfloor \frac{k}{12} \rfloor & k \equiv 2 \mod 12 \\ \lfloor \frac{k}{12} \rfloor + 1 & \text{else.} \end{cases}$$

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In particular, when $k$ tends to $\infty$,

$$\dim_{\mathbb{C}} \mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z})) \sim c \cdot k.$$

True for general $\Gamma$, i.e. as $k \to \infty$, $\dim_{\mathbb{C}} \mathcal{M}_k(\Gamma) \sim k$ (meaning: $c_1 k \leq f(k) \leq c_2 k$).

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Similar for Hilbert modular forms (with growth $\sim k^d$, where $d$ is the degree of the totally real field).

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- $F = \mathbb{Q}(\sqrt{-d})$, $d > 0$ square-free, $\mathcal{O}_F$ = ring of integers,
- $\mathrm{SL}_2(\mathcal{O}_F)$ acts on the hyperbolic 3-space $\mathbb{H}^3 := \mathbb{C} \times \mathbb{R}^+$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z, r) := \frac{((az+b)\overline{(cz+d)} + a\bar{c}r^2, r)}{|cz+d|^2 + |c|^2 r^2}.$$

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### Definition

*A Bianchi group* $\Gamma$ *is a congruence subgroup of* $\mathrm{SL}_2(\mathcal{O}_F)$.

- e.g. $\mathfrak{p}|p$ prime ideal,

$$\Gamma(\mathfrak{p}^n) := \left\{ \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) | a-1, d-1, b, c \equiv 0 \mod \mathfrak{p}^n \right\}$$

$$\Gamma_1(\mathfrak{p}^n) := \left\{ \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) | c \equiv 0 \mod \mathfrak{p}^n \right\}.$$

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**Theorem (Harder)**

$$\mathcal{S}_{k+2}(\Gamma) \cong H^1_{cusp}(\Gamma, V_k(\mathbb{C}))$$

where

- $V_k(\mathbb{C}) = \mathrm{Sym}^k\mathbb{C}^2 \otimes \overline{\mathrm{Sym}^k\mathbb{C}^2}$, where
- $\mathrm{Sym}^k\mathbb{C}^2 =$ standard representation of $\mathrm{SL}_2$
- $H^1(\Gamma, V_k)$: group cohomology; if $X_\Gamma =$ compatification of $\Gamma\backslash\mathbb{H}^3$,

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**Remark**: (Borel-Wallach) $H^1_{cusp}(\Gamma, \mathrm{Sym}^k \otimes \overline{\mathrm{Sym}^l}) = 0$, if $k \neq l$.

**Question:** dimensions of $\mathcal{M}_k(\Gamma)$, $\mathcal{S}_k(\Gamma)$ ?

Unknown: no precise formula is known so far. $\Gamma\backslash\mathbb{H}^3$ has no complex structure (3-dim over $\mathbb{R}$).

Rohlfs (1984): $\dim_{\mathbb{C}} \mathcal{S}_2(\mathrm{SL}_2(\mathcal{O}_F)) \geq \frac{1}{24}\varphi(d) - \frac{1}{4} - \frac{h(d)}{2}$, $h(d)$ =class number.

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**Goal**: study the asymptotic behavior of the dimension of $H^1(\Gamma, V_k)$.

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**Trivial bound**:

1. $k$ fixed $\Rightarrow \dim_{\mathbb{C}} H^1 \ll [\mathrm{SL}_2(\mathcal{O}_F) : \Gamma(\mathfrak{p}^n)]$;
   (use Shapiro's lemma to reduce to the case $\Gamma_1 = 1$ is trivial)
2. $\Gamma$ fixed $\Rightarrow \dim_{\mathbb{C}} H^1 \ll (k+1)^2 \sim k^2$,
   (because $\Gamma$ is finitely generated.)

**Theorem (Calegari-Emerton, 2009)**

*If $k$ is fixed and $\mathfrak{p}|p$, then*

$$\dim_{\mathbb{C}} H^1(\Gamma(\mathfrak{p}^n), V_k) \ll_k \begin{cases} p^{2n} & \text{if } f(\mathfrak{p}|p) = 1 \\ p^{5n} & \text{if } f(\mathfrak{p}|p) = 2. \end{cases}$$

**Remark:**

- Trivial bound: $|\mathrm{SL}_2(\mathcal{O}_F/\mathfrak{p})| \sim p^{3n}$, $p^{6n}$ (resp.)
- The bound is *sharp* if $f(\mathfrak{p}|p) = 1$ (Kionke-Schwermer, 2015)!
- Proof: Emerton's completed cohomology + (non-comm.) Iwasawa algebra.
- Can't replace $\Gamma(\mathfrak{p}^n)$ by $\Gamma_0(\mathfrak{p}^n)$ or $\Gamma_1(\mathfrak{p}^n)$, since need $\bigcap_{n\geq 1} \Gamma(\mathfrak{p}^n) = \{1\}$.

If $\Gamma$ is fixed:

**Theorem (Finis-Grunewald-Tirao, 2010)**

$$k \ll_\Gamma \dim_{\mathbb{C}} H^1(\Gamma, V_k) \ll_\Gamma \frac{k^2}{\ln k}.$$

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**Conjecture**

*The actual growth is $\sim k$.*

More generally, if $[F : \mathbb{Q}] = d$ with $d = r_1 + 2r_2$, then conjecturally

$$\dim_{\mathbb{C}} \mathcal{S}_{(k,\ldots,k)}(\Gamma) \sim k^{r_1+r_2}.$$

**Theorem (Marshall, 2012)**

$$\dim_{\mathbb{C}} H^1(\Gamma, V_k) \ll_{\Gamma,\epsilon} k^{\frac{5}{3}+\epsilon}, \quad \forall \epsilon > 0.$$

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**Mod $p$ method:** if $\mathcal{V}_k \subset V_k(\mathbb{Q}_p)$ is an $\mathbb{Z}_p$-lattice,

$$\dim_{\mathbb{C}} H^1(\Gamma, V_k(\mathbb{C})) = \dim_{\mathbb{Q}_p} H^1(\Gamma, V_k(\mathbb{Q}_p)) \leq \dim_{\mathbb{F}_p} H^1(\Gamma, \mathcal{V}_k \otimes \mathbb{F}_p).$$

For $K$ compact $p$-adic Lie group (e.g. $\mathbb{Z}_p$, $\mathrm{SL}_2(\mathbb{Z}_p)$), define Iwasawa algebra:

$$\mathbb{Z}_p[[K]] = \varprojlim_{K_n} \mathbb{Z}_p[K/K_n].$$

**Example**: $K = \mathrm{Ker}(\mathrm{SL}_2(\mathbb{Z}_p) \to \mathrm{SL}_2(\mathbb{F}_p))$, then

$$g_1 = \begin{pmatrix} 1+p & 0 \\ 0 & (1+p)^{-1} \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}$$

top. generate $K$, so $\mathbb{Z}_p[[K]]$ top. generated by $X_i := g_i - 1$; non-commutative.

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**Fundamental properties** (Lazard, 1965):

- $\mathbb{Z}_p[[K]]$ is noetherian.
- $K$ uniform pro-$p$ $\Rightarrow$ $\mathbb{Z}_p[[K]]$ integral domain.
- gradedly $\mathbb{Z}_p[[K]]$ is commutative, looks like a polynomial ring. (e.g. $\mathrm{Gr}(\mathbb{Z}_p[[K]]) = \mathbb{F}_p[\bar{p}, \bar{X}_1, \bar{X}_1, \bar{X}_3]$.)

**Notation**:

$$\Lambda = \mathbb{F}_p[[K]], \quad \tilde{\Lambda}_{\mathbb{Q}_p} = \mathbb{Z}_p[[K]][\frac{1}{p}],$$

**Theorem (Harris, 1979)**

*Let $M$ be a finitely generated module over $\Lambda$ (or $\tilde{\Lambda}_{\mathbb{Q}_p}$) and $d = \dim K$.*

1. *There exists $c \leq d$, $\lambda \neq 0$ such that*

$$\dim H_0(K_n, M) = \lambda p^{cn} + O(p^{(c-1)n}), \quad n \to \infty;$$

2. *$c < d \iff M$ is torsion (i.e. $M \otimes_\Lambda \mathrm{Frac}(\Lambda) = 0$);*
3. *$H_i(K_n, M) = O(p^{(d-1)n})$, $\forall i \geq 1$.*

**Remark**: $c$ is called the Gelfand-Kirillov dimension of $M$.

Fix prime $p$, let $\Gamma_n = \Gamma \cap \Gamma(p^n)$, $\hat{\Gamma} = \varprojlim_n \Gamma/\Gamma_n$. Choose $\mathcal{V}_k \subset V_k(\mathbb{Q}_p)$: $\mathbb{Z}_p$-lattice.

**Definition (Emerton, Completed cohomology)**

$$\hat{H}^j(\mathcal{V}_k) = \varprojlim_m \varinjlim_n H^j(\Gamma_n, \mathcal{V}_k/p^m)$$

$$\hat{H}^j(V_k) = \hat{H}^j(\mathcal{V}_k)[\frac{1}{p}].$$

1. $\hat{H}^j(\mathcal{V}_k)$ admissible (i.e. the dual of $\hat{H}^j(\mathcal{V}_k)$ is f.g. $\mathbb{Z}_p[[\hat{\Gamma}]]$-module).
2. Spectral sequence

$$E_2^{ij} := H^i(\hat{\Gamma}, \hat{H}^j(\mathcal{V}_k)) \Rightarrow H^{i+j}(\Gamma, \mathcal{V}_k),$$

   ($\Rightarrow$ reduces to bound $H^i(\hat{\Gamma}, \hat{H}^j(\mathcal{V}_k))$ for $i + j = 1$.)
3. $H^j(\mathcal{V}_k) \cong \hat{H}^j(\mathbb{Z}_p) \otimes \mathcal{V}_k$.
4. $\mathrm{SL}_2(F_p)$ acts on $\hat{H}^j(\mathcal{V}_k)$; $F_p :=$ completion of $F$ at $p$.

The key fact is:

**Proposition**

*The dual of $\hat{H}^j(\mathbb{Z}_p)$ is a torsion $\mathbb{Z}_p[[\hat{\Gamma}]][\frac{1}{p}]$-module.*

**Proof** Key point: $\mathrm{SL}_2(\mathbb{C})$ does not admit discrete series.) $(\mathrm{GL}_2(\mathbb{C})$ does not contain compact Cartan subgroups) □

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Prop. $\Longrightarrow$ Thm. of Calegari-Emerton.

Indeed, $\hat{\Gamma}$ has dimension $d = 3$ if $f(\mathfrak{p}|p) = 1$ and $d = 6$ if $f(\mathfrak{p}|p) = 2$, then Harris' theorem implies

$$\dim H^i(\hat{\Gamma}(\mathfrak{p}^n), \hat{H}^j(V_k)) \ll p^{(d-1)n}.$$

**Proof of Marshall's theorem**:

- By mod $p$ method, it suffices to bound by mod $p$ method:

$$\dim_{\mathbb{F}_p} H^i(\hat{\Gamma}, \overline{T} \otimes \overline{\mathcal{V}_k})$$

where $T \subset \hat{H}^i(\mathbb{Q}_p)$, and $\mathcal{V}_k \subset V_k$ are $\mathbb{Z}_p$-lattices. This holds for any $p$; we choose $p$ which splits in $F$, i.e. $p\mathcal{O}_F = \mathfrak{p}_1\mathfrak{p}_2$, so $F_{\mathfrak{p}_1} \cong F_{\mathfrak{p}_2} \cong \mathbb{Q}_p$ and

$$\hat{\Gamma} \subset \mathrm{SL}_2(\mathbb{Z}_p) \times \mathrm{SL}_2(\mathbb{Z}_p).$$

Since $\Gamma$ is fixed, may assume $\hat{\Gamma} = \mathrm{SL}_2(\mathbb{Z}_p) \times \mathrm{SL}_2(\mathbb{Z}_p)$ for simplicity.

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- Observation (Serre) For char. $p$ modular forms, there is a bijection

$$\mathcal{S}_k(\mathrm{SL}_2(\mathbb{Z})) \longleftrightarrow \mathcal{S}_2(\Gamma_0(p)).$$

Marshall generalized to get: if $k \sim p^n$,

$$\text{wt } k \text{ of level } \hat{\Gamma} \quad \sim \text{ wt } 2 \text{ of level } \widehat{\Gamma_0(\mathfrak{p}^n)}.$$

## Proposition (Marshall)

*If (the dual of)* $\overline{T}$ *is a f.g.* *torsion* *module over* $\mathbb{F}_p[[\hat{\Gamma}]]$*, then*

$$\dim_{\mathbb{F}_p} H^i\left(\widehat{\Gamma_0(\mathfrak{p}^n)}, \overline{T}\right) \ll p^{(\frac{5}{3}+\epsilon)n}, \quad \forall \epsilon > 0.$$

**Remark**. $\bigcap_{n\geq 1} \widehat{\Gamma_0(\mathfrak{p}^n)} \neq \{1\}$, so no analog of Harris' theorem.

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**Proof**. Use mod $p$ Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$, precisely, mod $p$ representation theory of $\mathrm{GL}_2(\mathbb{Q}_p)$ (with central character):

- Barthel-Livné (1993), Breuil (2001): classify irreducible objects $\pi$.
- Paškūnas (2012): deformation theory of $\pi$.