Conformal Invariance and Brownian loop measure

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$$\partial_t g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}, \qquad g_0(z) = z$$

Stochastic Loewner Equation—-Schramm-Loewner Equation

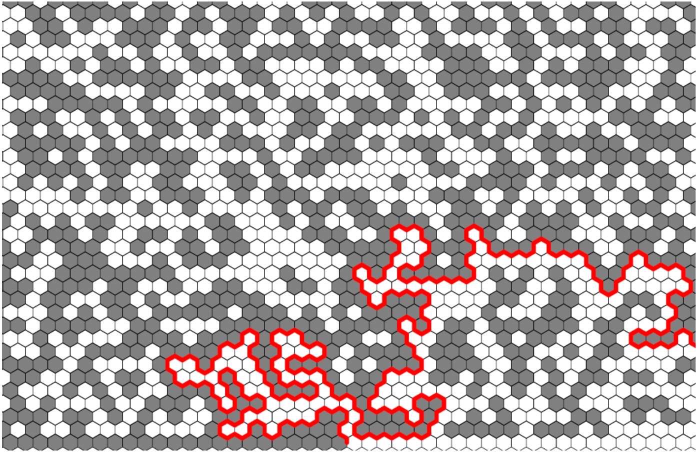
Motivation

- In statistical physics, many lattices models which generate random shapes.
- While the mesh of these models tends to zero, the random shapes converge in some sense to some random fractal shapes, —the scaling limits. And
- The distribution of the scaling limit is invariant under conformal maps.
- These phenomena have been observed by Statistical Physicists for a long time. But for most cases, the rigorous proof is missing, and little is known about the scaling limit.

Recent Progress

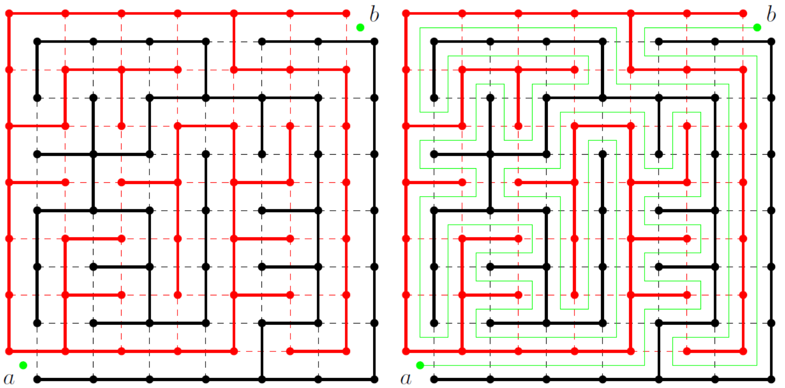
- The situation was changed since Oded Schramm introduced the Schramm-Loewner evolution (SLE) in 1999.
- SLE with different parameters have been identifed as the scaling limits of a number of lattice models.

Critical site percolation



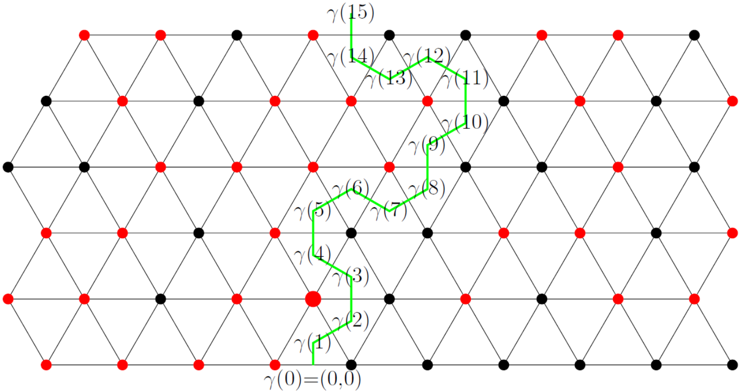
Critical site percolation on triangular lattice (by Schramm)

Uniform Spanning Tree



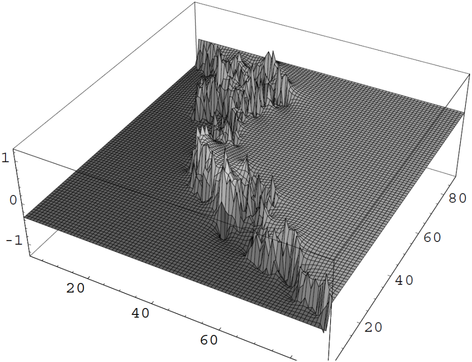
SLE(8)

Harmonic explorer



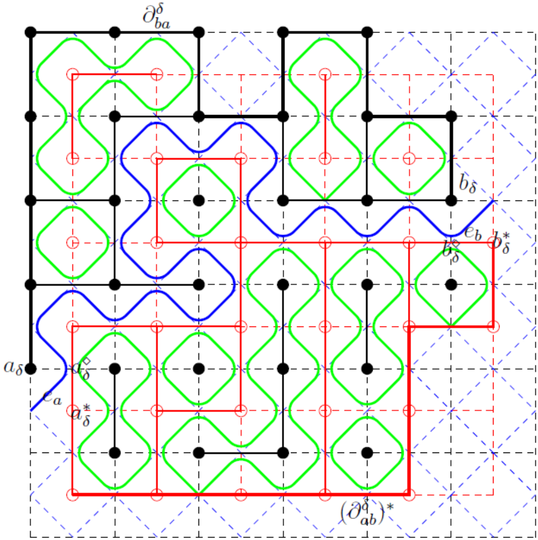
SLE(4)

Discrete Gaussian Free Field



SLE(4)

FK-Ising model



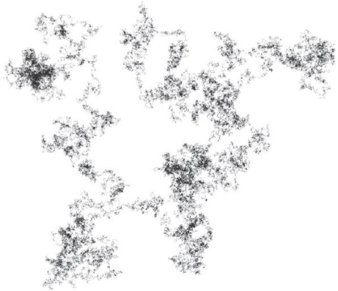
SLE($\frac{16}{3}$)

Random Walk

The following picture shows the image of a random walk on a square lattice.

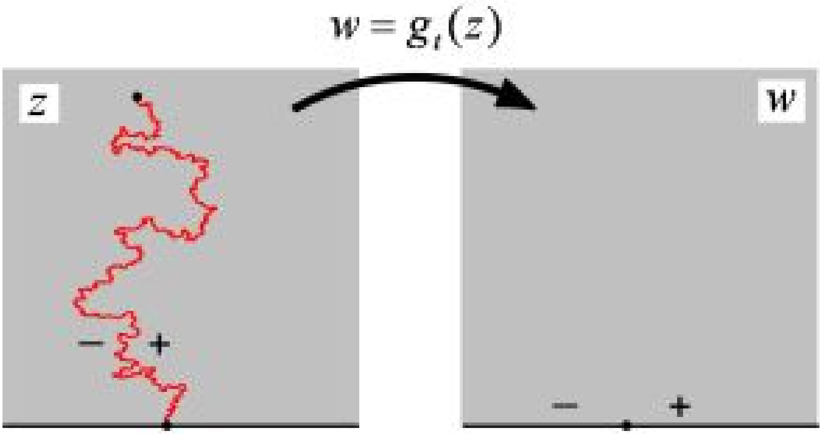
Brownian Motion

It is well known that random walk on regular lattices converge to planar Brownian motion when the mesh tends to 0.



Conformal Maps

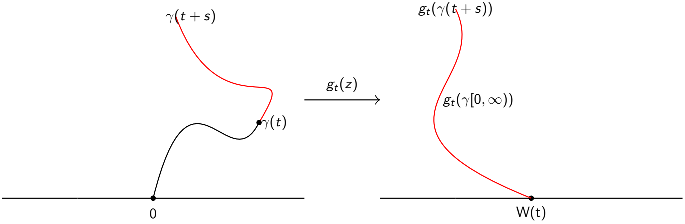
How to code the information of such a curve?



Loewner Equation

How to code the information of such a curve?

Answer: Loewner equation.



Loewner Equation

Theorem (Schramm 2001)

If the curve is parameterized such that $g_t(z) = z + \frac{2t}{z} + \ldots$ *at* ∞*, then*

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W(t)}$$

and $W(t)$ *is a continuous real valued function.*

The equation describes how the maps g_t evolve as γ_t grows.

Key tool:

$$f(z) = \int_{\partial D} f(w) H_D(z,w) d|w|$$

where $H_D(z,w)$ is the Poisson kernel.

Poisson Kernels

$$H_{\mathbb{D}}(z,w) = \frac{1}{2\pi}\Re\frac{w+z}{w-z}, w \in \partial\mathbb{D}, z \in \mathbb{D}.$$

$$H_{\mathbb{H}}(z,x) = -\frac{1}{\pi}\Im\frac{1}{z-x}, z \in \mathbb{H}, x \in \mathbb{R}.$$

$$H_{S_\pi}(z,x) = -\frac{1}{\pi}\Im\frac{e^x}{e^z-e^x}, z \in S_\pi, x \in \mathbb{R}.$$

Loewner equation

(1) Given a continuous function $W(t)$, one can get the conformal map from the ODE.

(2) Can one get a curve? Not necessarily a curve.

(3) $W(t)$: Holder-$\frac{1}{2}$ continuity.

$\Rightarrow \gamma_t$ is a simple curve (Rhode 2005).

Brownian Motion

1-dim standard Brownian Motion:

(1) $B_0 = 0$;

(2) For $0 \leq t_1 < t_2 < t_3 < ... < t_n$,
$B_{t_2} - B_{t_1}, B_{t_3} - B_{t_2}, ..., B_{t_n} - B_{t_{n-1}}$ are independent;

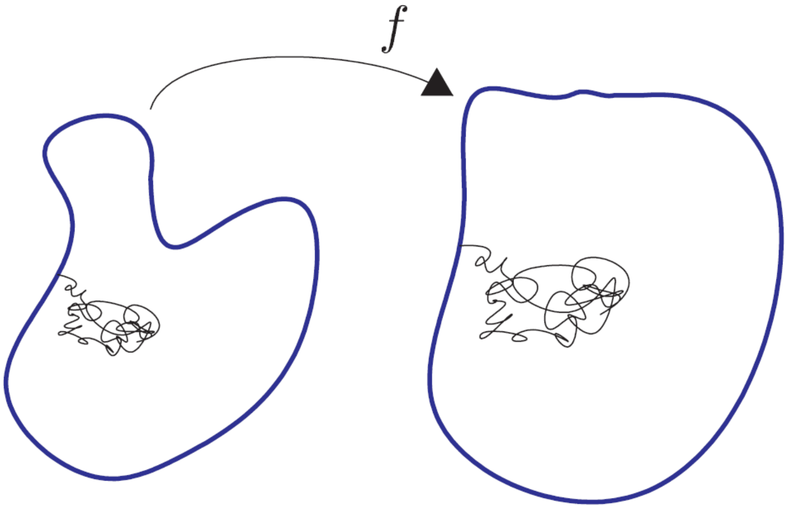
(3) For $0 \leq s \leq t$, $B_t - B_s \sim N(0, t-s)$;

(4) With probability 1, the sample path $t \rightarrow B_t$ is continuous.

$N(0, t-s) = \int (2\pi(t-s))^{-1/2} exp(-x^2/2(t-s))dx$,
with mean 0, and variance $t - s$.

Conformal invariance of planar Brownian motion

$$B(t) = B_1(t) + iB_2(t)$$



Brownian Motion

The planar Brownian motions are conformally invariant.

Theorem [Lévy, 1948]

Suppose that f maps D_1 conformally onto D_2, $z_1 \in D_1$, and $z_2 = f(z_1) \in D_2$. For $j = 1, 2$, let $B_j(t)$ be a planar Brownian motion started from z_j, and let T_j be the first t such that $B_j(t) \in \partial D_j$. Let $S_j = B_j[0, T_j]$, i.e., the image of the initial part of B_j before it leaves D_j. Then S_2 has the same distribution as $f(S_1)$.

The definition of SLE

Definition (Schramm 2001)

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}, \qquad g_0(z) = z$$

- Stochastic Loewner Equation—-Schramm-Loewner Equation
- SLE: driving function B_t.
- SLE Trace $\gamma_t = g_t^{-1}(\sqrt{\kappa} B_t)$,

  a continuous curve never crosses itself.
- The process $(g_t(z) : t \geq 0)$ is called the **Chordal** $SLE(\kappa)$.

Proposition (Schramm-Rohde-Lawler-Werner-Zhan)

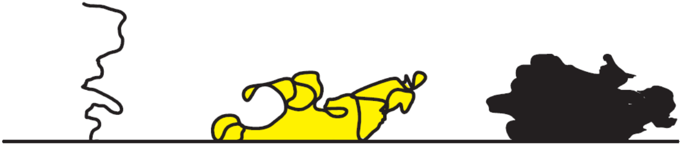
$\mathrm{SLE}(\kappa)$ *has the following propeties:*

- Generated by a curve;
- Scaling invariance;
- Phase dependence on κ;
- Locality and restriction;
- Reversibility and duality.

Main properties of SLE

With probability one, the following properties are satisfied:

- Schramm-Rohde (2001):



$\kappa \in [0, 4]$ $\kappa \in (4, 8)$ $\kappa \in [8, \infty)$

- If $\kappa > 0$, γ_κ is fractal; As $\kappa \uparrow$, γ_κ becomes more fractal; Phase transition at $\kappa = 4$ and $\kappa = 8$.

Main properties of SLE

· Vincent Beffara (2007): The Hausdorff dimension of the trace:

$$Dim_H \gamma_\kappa = \min\{1 + \frac{\kappa}{8}, 2\}$$

Other versions of SLE

Radial SLE:

$$\partial_t g_t(z) = g_t(z)\frac{e^{i\sqrt{\kappa}B_t} + g_t(z)}{e^{i\sqrt{\kappa}B_t} - g_t(z)}, g_0(z) = z \in \mathbb{D}$$

Whole plane SLE:

$$\partial_t g_t(z) = g_t(z)\frac{e^{i\sqrt{\kappa}B_t} + g_t(z)}{e^{i\sqrt{\kappa}B_t} - g_t(z)}, \lim_{t\to-\infty} e^t g_t(z) = z, \forall z \in \mathbb{C}\backslash\{0\}$$

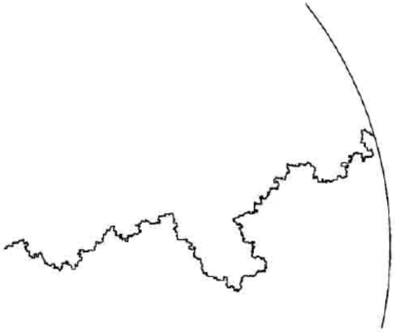
Strip SLE:

$$\partial_t g_t(z) = \coth\frac{g_t(z) - \sqrt{\kappa}B_t}{2}, g_0(z) = z \in S_\pi$$

$S_\pi := \{x + iy : x \in \mathbb{R}, 0 < y < \pi\}$

Loop-erased Random Walk

The picture below is a LERW inside a disc.

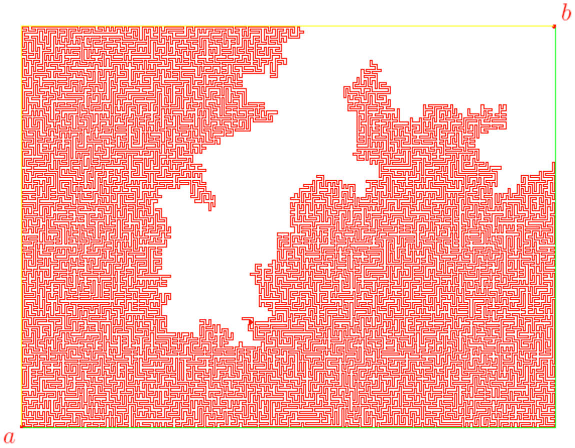


Loop-erased Random Walk

Theorem [Lawler-Schramm-Werner, 2002]

As the mesh tends to 0, the above LERW converges to a radial SLE(2) curve in D, which is a random simple curve connecting z_0 with ∂D, and satisfies conformal invariance.

Uniform Spanning Tree



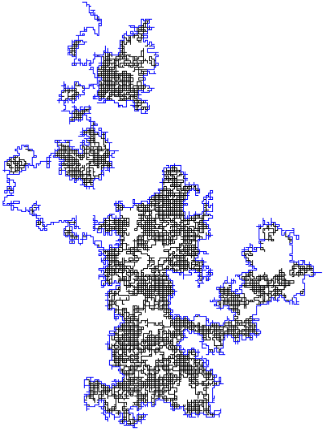
Uniform Spanning Tree

Theorem [Lawler-Schramm-Werner, 2002]

The Peano curve of the uniform spanning tree with the above boundary condition converges to the chordal SLE(8) curve as the mesh tends to 0. So the scaling limit of the uniform spanning tree Peanon curve is conformally invariant.

Brownian frontier

- **Mandelbrot conjecture（1987）.**



- (Schramm-Lawler-Werner 2001) $dim_H = \frac{4}{3}$

Brownian frontier

Key point:

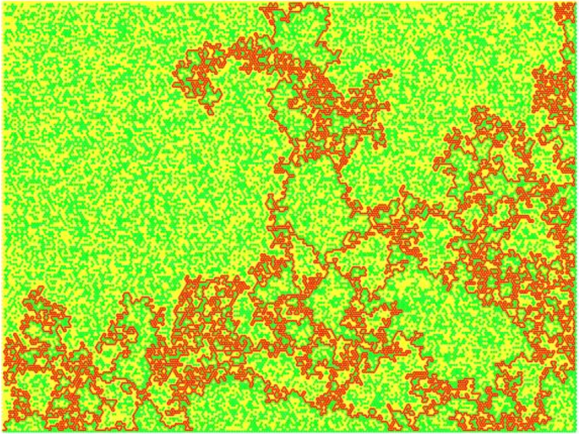
· SLE(8/3)$\Longrightarrow$ Brownian intersection exponent $\xi(2,0) = \frac{2}{3}$

· dim_H(frontier)$=2-\xi(2,0) \Longrightarrow dim_H=\frac{4}{3}$

· Dividing K BM into k groups, $E(R)$ is the event that the paths of any two different groups are non-intersecting.

$$\xi(p_1, p_2, ..., p_k) = \lim_{R\to\infty} \frac{\log \mathbb{P}[E(R)]}{R}$$

Percolation



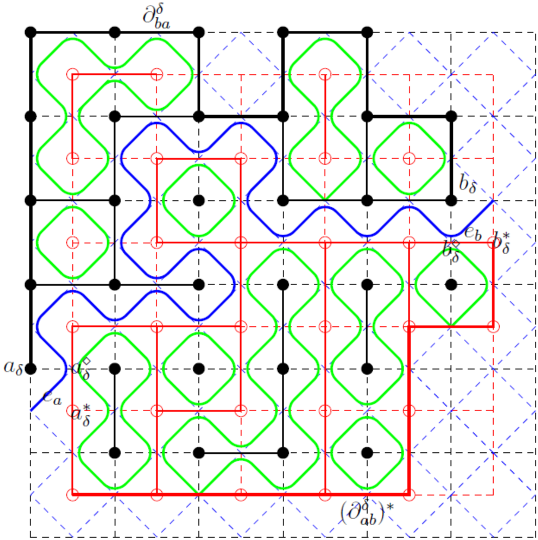
Converging to SLE

Theorem (Smirnov 2001)

The exploration path of the triangular percolation model in the upper half plane converges to $SLE(6)$ weakly.

Stanislav Smirnov:winner of the 2010 Fields Medal.

FK-Ising model



SLE($\frac{16}{3}$)

FK-Ising model

Theorem (Smirnov 2010)

If $q = 2, p = \frac{\sqrt{q}}{1+\sqrt{q}}$, then the FK-Ising exploration path γ_δ converging weakly to $SLE(\frac{16}{3})$.

Key Point: Discrete holomorphic function

Same method:Critical Ising Model $\rightarrow SLE(3)$

Brownian Loop Measure

Defnition(Lawler-Werner 2003)

$$\mu_{\mathbf{C}}^{\text{loop}} = \int_{\mathbf{C}} \int_0^{\infty} \frac{1}{t} \mu(z, z; t) \, dt \, dA(z)$$

the measure $\mu(z, z; t)$ can be consider as the measure induced by a Brownian motion starting from z and returning z at time t.

$$\mu(z, \mathrm{z}; t) = \lim_{\varepsilon \to 0} \frac{1}{\pi \varepsilon^2} \mathrm{P}^{\mathrm{z}} \big|_{B_t \in D(z, \varepsilon)}$$

a measure on the loop space with parameter w and

$$\mathrm{P}^{\mathrm{z}} \big|_{B_t \in \mathrm{D}(z, \varepsilon)} = \int_{B_t \in D(z, \varepsilon)} \frac{1}{2\pi t} e^{-\frac{1}{2t}(x^2 + y^2)} dx dy$$

Brownian Loop Measure（Wiener Measure）

The Brownian loop measure $\mu_{\mathbb{C}}^{\text{loop}}$ is a measure on the measurable space $(\Omega, \mathcal{F})$, where

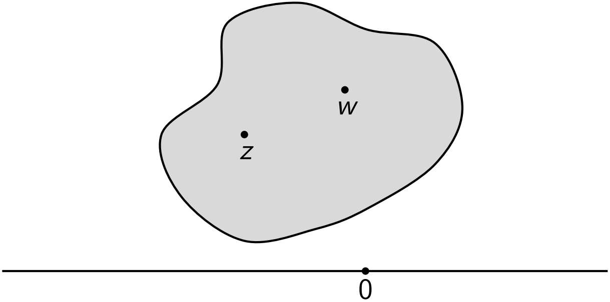
$$\Omega := \{\gamma : \gamma \text{ is unparameterized loops in the plane}\}.$$

$\mathcal{F}$ is the $\sigma-$algebra induced by the Hausdorff metric on closed sets of the plane.

Remark

By the property of planar Brownian motion, the outer boundary of a Brownian loop sample is a simple loop. And the outer boundary of a Brownian loop sample can be regarded as the scaling limit of the self-avoiding polygons.

Cardy-Gamsa's formula



$$\mu_{\mathbb{H}}^{\text{loop}}(\cdot) = ?$$

Cardy-Gamsa Formula

（predicted in 2006）

$$\tilde{\mu}_{\mathbb{H}}^{\text{loop}}[E(z,w)] = -\frac{1}{12\pi}\log(\eta(\eta-1)) - \frac{1}{12\pi}\eta\,{}_3F_2(1,\frac{4}{3},1;\frac{5}{3},2;\eta)$$

$$+\frac{\Gamma(\frac{2}{3})^2}{6\pi\Gamma(\frac{4}{3})}(\eta(\eta-1))^{\frac{1}{3}}\,{}_2F_1(1,\frac{2}{3};\frac{4}{3},\eta),$$

where

$$\eta = \eta(z,w) = -\frac{(x-u)^2+(y-v)^2}{4yv},$$

and ${}_3F_2$, ${}_2F_1$ are hypergeometric functions.

Han-W-Zinsmeister (2018)

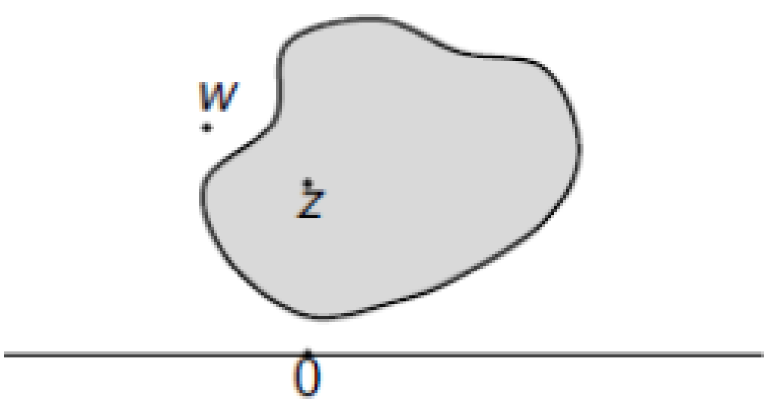
Theorem.

$$\mu_{\mathbb{H}}^{\mathrm{loop}}[E(z,w)] = -\frac{1}{10}\left[\log\sigma + (1-\sigma)\,{}_3F_2\!\left(1,\frac{4}{3},1;\frac{5}{3},2;1-\sigma\right)\right],$$

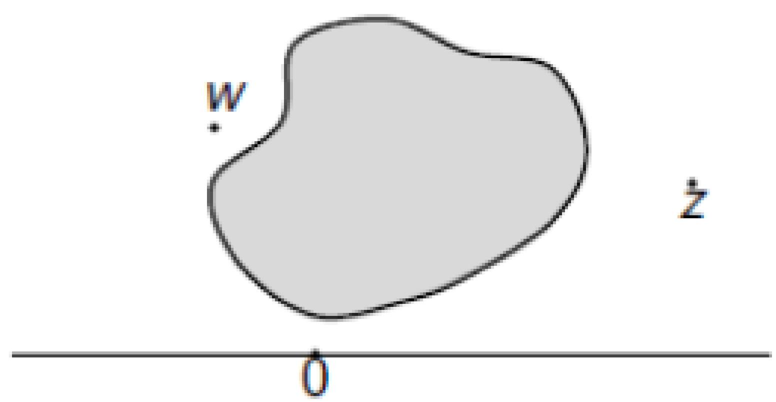
where

$$\sigma = \sigma(z,w) = \frac{|z-w|^2}{|z-\bar{w}|^2} = \frac{(x-u)^2+(y-v)^2}{(x-u)^2+(y+v)^2}.$$

The Other Cases



$\mu_{\mathbb{H}}^{\text{loop}}(\cdot) = \infty$



$\mu_{\mathbb{H}}^{\text{loop}}(\cdot) = \infty$

Discrete models converging to SLE

(1) **Uniform spanning Tree model $\rightarrow$SLE(8)** (Schramm-Lawler-Werner 2005);
(2) **Harmonic exploration model $\rightarrow$SLE(4)** (Schramm-Sheffield 2004);
(3) **Discrete Gaussian Free Field $\rightarrow$SLE(4)** (Schramm-Sheffield 2004);
(4) **Loop Erased Random Walk $\rightarrow$SLE(2)** (Schramm-Lawler-Werner 2005);
(5) **Ising Model $\rightarrow$SLE(3)**(Smirnov 2013);
(6) **FK-Ising Model (Random Cluster Model) $\rightarrow$SLE($\frac{16}{3}$)**(Smirnov 2013);
(7) **(Conjecture) Self-avoiding random walk $\rightarrow$SLE($\frac{8}{3}$)**.

Some Problems

(1) **University**. Critical percolation converging to SLE(6) on square lattice?

Change the underlying lattice of other models?

(2) **SAW** Self avoiding random walk $\to SLE(\frac{8}{3})$?

(3) A direct proof that $SLE(8)$ is generated by a curve?

(4) Schramm (ICM 2006):Conformally invariant scaling limits: an overview and a collection of problems.

(5) Scott Sheffield:Some problems with Gaussian Free Field.

Thank You

Brownian Loop Measure

$$u_t^z \leftarrow (B_t^z) \quad 0 \le t \le T,\ \text{a prob. measure;}$$

$$u_t(z,w) = \lim_{\varepsilon \to 0} \frac{1}{\pi\varepsilon^2} u^z(\cdot, B_t \in B(w,\varepsilon))$$

measure on the loop space;

$$\mathrm{P}^z|_{B_t \in B(w,\varepsilon)} = \int_{B_t \in B(w,\varepsilon)} \frac{1}{2\pi t} e^{-\frac{1}{2t}(x^2+y^2)} dxdy$$

$$u_t(z,w) = \lim_{\varepsilon \to 0} \frac{1}{\pi\varepsilon^2} \mathrm{P}^z|_{B_t \in B(w,\varepsilon)} \text{ a measure on}$$

the loop space with parameter w;

$$u_t^z = \int_{\mathbf{C}} u_t(z,w)\, dw$$

Total mass $|u_t(z,w)| = \frac{1}{2\pi t} e^{-\frac{|z|^2}{2t}}$.

$u_t(z,z)$ measure on the loop starting from z and returning z at time t.

$$u_{\mathbf{C}}^{loop}(E) = \int_{\mathbf{C}} \int_0^{\infty} \frac{1}{t} u_t(z,z)(E)\, dt\, dz$$

measure on the loop space.