A dispersive regularization for modified Camassa-Holm equation

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Outline

The modified Camassa-Holm (mCH) equation

The mCH equation

$$m_t + [(u^2 - u_x^2)m]_x = 0, \quad m = u - u_{xx}, \; x \in \mathbb{R}, \; t > 0.$$

(Fokas, 1995; Olver & Rosenau, 1996; Qiao, 2006)

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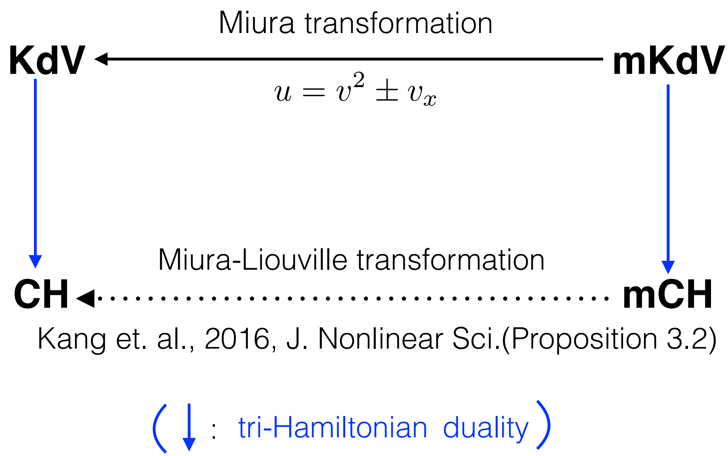

Figure 1: Relations with KdV, mKdV and CH equations

Local solutions

For the mCH equation:

- local existence and uniqueness of strong solutions:
  Fu, Gui, Liu, Qu, 2013, JDE (Besov space)
  Gui, Liu, Olver, Qu, 2013, CMP ($H^s(\mathbb{R}), \quad s > 1/2$)
  Himonas, Mantzavinos, 2014, J. Nonlinear Sci. (Hölder space)
  ...
- finite time blow-up behaviors:
  Gui, Liu, Olver, Qu, 2013, CMP
  Liu, Olver, Qu, 2014, Anal. Appl.
  Chen, Liu, Qu, Zhang 2015, Adv. Math.
  ...

Question: How to extend weak solutions globally?

Definition of weak solutions

mCH equation is equivalent to

$$
\begin{aligned}
&(1-\partial_{xx})u_t + [(u^2-u_x^2)(u-u_{xx})]_x \\
&= (1-\partial_{xx})u_t + (u^3+uu_x^2)_x - \frac{1}{3}(u^3)_{xxx} + \frac{1}{3}(u_x^3)_{xx} = 0.
\end{aligned}
$$

Multiply the equation by a test function $\phi \in C_c^\infty(\mathbb{R}\times[0,\infty))$ and integrate by parts:

$$
\begin{aligned}
\mathcal{L}(u,\phi) &= \int_0^\infty \int_{\mathbb{R}} u(x,t)[\phi_t(x,t)-\phi_{txx}(x,t)]\,\mathrm{d}x\,\mathrm{d}t \\
&- \frac{1}{3}\int_0^\infty \int_{\mathbb{R}} u_x^3(x,t)\phi_{xx}(x,t)\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{3}\int_0^\infty \int_{\mathbb{R}} u^3(x,t)\phi_{xxx}(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_0^\infty \int_{\mathbb{R}} (u^3+uu_x^2)\phi_x(x,t)\,\mathrm{d}x\,\mathrm{d}t = -\int_{\mathbb{R}} \phi(x,0)m_0(x)\,\mathrm{d}x.
\end{aligned}
$$

Properties of G and G'

Fundamental solution of Helmholtz operator $1 - \partial_{xx}$: $(m = u - u_{xx})$

$$G(x) = \frac{1}{2}e^{-|x|} \qquad \Rightarrow \qquad u = G * m.$$

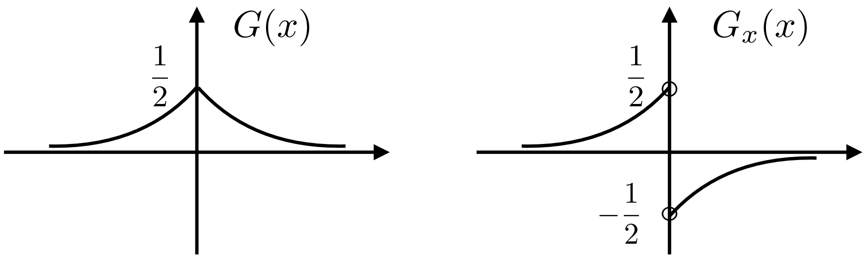

Figure 2: G and G'.

- $G - G_{xx} = \delta; \quad V(G) = 1, \quad V(G') = 2.$
- $\|G\|_{L^\infty} = \frac{1}{2}, \quad \|G'\|_{L^\infty} = \frac{1}{2}; \quad \|G\|_{L^1} = 1, \quad \|G'\|_{L^1} = 1;$

N-peakon weak solutions

N-peakon weak solutions:

$$u^N(x,t)=\sum_{i=1}^{N}p_i(t)G(x-x_i(t)),\quad m^N=\sum_{i=1}^{N}p_i(t)\delta(x-x_i(t)).$$

When $x_1(t)<x_2(t)<\cdots<x_N(t)$, one has (Gui et.al. 2013):

$$\begin{cases}\frac{\mathrm{d}}{\mathrm{d}t}p_i=0,\\ \frac{\mathrm{d}}{\mathrm{d}t}x_i=\frac{1}{6}p_i^2+\frac{1}{2}\sum_{j<i}p_ip_je^{x_j-x_i}+\frac{1}{2}\sum_{j>i}p_ip_je^{x_i-x_j}+\sum_{1\le m<i<n\le N}p_mp_ne^{x_m-x_n}.\end{cases}$$

$N=1$: solitary wave solutions (one peakon):

$$u(x,t)=pG(x-x(t)),\qquad x'(t)=\frac{p^2}{6}.$$

Finite time collision

Consider initial date

$$m_0 = p_1\delta(x-c_1) + p_2\delta(x-c_2), \quad c_1 < c_2, \quad p_1^2 > p_2^2.$$

The two peakons ODE:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}x_1(t) = \frac{1}{6}p_1^2 + \frac{1}{2}p_1p_2e^{x_1(t)-x_2(t)}, \\ \frac{\mathrm{d}}{\mathrm{d}t}x_2(t) = \frac{1}{6}p_2^2 + \frac{1}{2}p_1p_2e^{x_1(t)-x_2(t)}. \end{cases}$$

$\Longrightarrow$

$$\frac{\mathrm{d}}{\mathrm{d}t}(x_1 - x_2) = \frac{1}{6}(p_1^2 - p_2^2) > 0.$$

This two peakons will collide with each other at

$$t^* := \frac{6(c_2 - c_1)}{p_1^2 - p_2^2}.$$

Global trajectories $\not\Longrightarrow$ global peakon solutions

Finite time collide between peakons

Question: How to extend trajectories globally and obtain global N-peakon weak solutions?

Problem: non-Lipschitz vector field

Characteristic equation for mCH:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = u^2(x(t),t) - u_x^2(x(t),t).$$

Consider $u^N(x,t) = \sum_{i=1}^{N} p_i G(x - x_i(t))$:

$$\frac{\mathrm{d}}{\mathrm{d}t}x_i(t) = \left[u^N(x_i(t),t)\right]^2 - \left[u_x^N(x_i(t),t)\right]^2, \quad i = 1, \cdots, N.$$

Jump discontinuous, non-Lipschitz vector field.

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Jump discontinuous, non-Lipschitz vector field.

How about mollify the vector field one time?
(Vortex blob method for 2D Euler equation)

Speed for one peakon $u(x,t) = pG(x - x(t))$

Fact: $x'(t) = \frac{p^2}{6}$.

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From characteristics equation: $x'(t) = (u^2 - u_x^2)(x(t), t)$:

$$\begin{aligned} x'(t) &= p^2 G^2(x(t) - x(t)) - p^2 G_x^2(x(t) - x(t)) \\ &= p^2 G^2(0) - p^2 G_x^2(0) = \frac{p^2}{6}. \end{aligned}$$

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$$= p^2 G^2(0) - p^2 G_x^2(0) = \frac{p^2}{6}.$$

$\Rightarrow$ the **correct** definition for $G_x^2(0)$:

$$G_x^2(0) = G^2(0) - \frac{1}{6} = \frac{1}{12}.$$

One peakon: one time mollification

Let $\rho(x)$ be an even function with $\int \rho \, \mathrm{d}x = 1$, $\varepsilon > 0$.

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon}\rho\left(\frac{x}{\varepsilon}\right), \quad G^\epsilon(x) = (\rho_\epsilon * G)(x).$$

Then, one time mollification gives wrong travelling speed:

$$(\rho_\varepsilon * G_x)^2(0) \to 0, \quad [\rho_\varepsilon * (G_x^2)](0) \to \frac{1}{4}, \quad \varepsilon \to 0.$$

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Double mollification gives the right one (independent of ρ)

$$\lim_{\epsilon \to 0}[\rho_\epsilon * (G_x^\epsilon)^2](0) = \frac{1}{12}. \qquad (*)$$

Double mollification

For N-peakon:

$$u^{N,\epsilon}(x;\{x_k\}) := \sum_{i=1}^{N} p_i G^{\epsilon}(x-x_i),\ U_{\epsilon}^{N}(x;\{x_k\}) := \left[u^{N,\epsilon}\right]^2 - \left[u_x^{N,\epsilon}\right]^2.$$

Double mollification (particle blob method):

$$\frac{\mathrm{d}}{\mathrm{d}t} x_i^{\epsilon}(t) = (\rho_{\epsilon} * U_{\epsilon}^{N})(x_i^{\epsilon}(t);\{x_k^{\epsilon}(t)\}), \quad i = 1, \cdots, N.$$

$\Rightarrow$ Global approximated trajectories:

$$x_i^{\varepsilon}(t): \quad t \in [0,\infty), \quad i = 1, 2, \cdots, N.$$

Collision avoidance

Initial data $x_i^\epsilon(0) = c_i$:

$$m_0^N = \sum_{i=1}^N p_i \delta(x - c_i), \quad c_1 < c_2 < \cdots < c_N, \quad \sum_{i=1}^N |p_i| \leq M_0.$$

Theorem (Theorem 3.2, Gao, Li & Liu 2018)

Let $\{x_i^\epsilon(t)\}_{i=1}^N$ be trajectories obtained by approximated ODEs with initial data $x_i^\epsilon(0) = c_i$. Then, for any $t > 0$, we have

$$x_1^\epsilon(t) < x_2^\epsilon(t) < \cdots < x_N^\epsilon(t), \quad \forall t > 0.$$

Proof: only need the following estimate:

$$-C_\epsilon(x_{k+1}^\epsilon - x_k^\epsilon) \leq \frac{\mathrm{d}}{\mathrm{d}t}(x_{k+1}^\epsilon - x_k^\epsilon) \leq C_\epsilon(x_{k+1}^\epsilon - x_k^\epsilon).$$

Weak consistence

Approximated N-peakon solutions:

$$u^{N,\epsilon}(x,t) = \sum_{i=1}^{N} p_i G^{\epsilon}(x - x_i^{\epsilon}(t)).$$

"Distance" to weak solutions:

$$E_{N,\epsilon} = \mathcal{L}(u^{N,\epsilon}, \phi) + \int_{\mathbb{R}} \phi(x,0) m_0^N(\mathrm{d}x).$$

We have

Proposition (Proposition 3.1, Gao, Li & Liu 2018)

There exists a constant C independent of N, ϵ such that $E_{N,\epsilon}$ satisfies

$$|E_{N,\epsilon}| \le C\epsilon.$$

Limiting trajectories

Trajectories $x_i^\epsilon(t)$ are globally Lipschitz:

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}x_i^\epsilon(t)\right| \leq \frac{1}{2}M_0^2.$$

$\Rightarrow \{x_i^\epsilon(t)\}_{\epsilon>0}$ is uniformly bounded and equicontinuous in $[0,T]$.

Arzelà-Ascoli Theorem $\Rightarrow \{x_i(t)\}_{i=1}^N \subset C([0,+\infty))$ satisfying

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}x_i(t)\right| \leq \frac{1}{2}M_0^2, \quad i=1,\cdots,N$$

$x_i(t)$ never cross with each other.

Global N-peakon weak solutions

Approximated N-peakon weak solutions:

$$u^{N,\epsilon}(x,t) = \sum_{i=1}^{N} p_i G^{\epsilon}(x - x_i^{\epsilon}(t))$$

converges a.e. (x,t) to

$$u^{N}(x,t) = \sum_{i=1}^{N} p_i G(x - x_i(t)).$$

Lebesgue dominated convergence theorem implies

$$u^{N,\epsilon} \to u^N, \quad u_x^{N,\epsilon} \to u_x^N,\ \epsilon \to 0, \quad L^1_{loc}(\mathbb{R} \times [0,+\infty))$$

$+$ weak consistency $\Rightarrow$

$u^N(x,t)$ is an N-peakon weak solution to the mCH equation.

Gloabl existence of weak solutions in $\mathcal{M}(\mathbb{R})$

Theorem (Theorem 4.1, Gao, Li & Liu 2018)

For any initial Radon measure $m_0 \in \mathcal{M}(\mathbb{R})$, there is a global weak solution u to the mCH equation, which satisfies that

$$u \in C([0,\infty); H^1(\mathbb{R})) \cap L^\infty(0,\infty; W^{1,\infty}(\mathbb{R})) \cap W^{1,\infty}(0,\infty; L^\infty(\mathbb{R})),$$

and for any $T > 0$,

$$u, \quad u_x \in BV(\mathbb{R} \times [0,T)).$$

Moreover, we have

$$m = (1 - \partial_{xx})u \in \mathcal{M}(\mathbb{R} \times [0,T))$$

and there exists a subsequence of m^N (also labeled as m^N) such that

$$m^N \overset{*}{\rightharpoonup} m \text{ in } \mathcal{M}(\mathbb{R} \times [0,T)) \quad (\text{as } N \to \infty).$$

Equations with multi-peakon solutions

fg-family of equations for multi-peakon solutions (Anco & Recio 2019):

$$m_t + f(u,u_x)m + [g(u,u_x)m]_x = 0, \quad x \in \mathbb{R},\ t > 0.$$

(i) $f = u_x$, $g = u$: CH equation;

(ii) $f = 0$, $g = (u^2 - u_x^2)^n$ $(n \in \mathbb{N}_+)$: generalized modified Camassa-Holm (gmCH) equation.

$$m_t + [(u^2 - u_x^2)^n m]_x = 0, \quad m = u - u_{xx}, \quad x \in \mathbb{R},\ t > 0.$$

$n = 1$: modified Camassa-Holm equation.

Double mollification for the gmCH

The gmCH equation ($n \in \mathbb{Z}_+$):

$$m_t + [(u^2 - u_x^2)^n m]_x = 0, \quad m = u - u_{xx}, \;\; x \in \mathbb{R}, \;\; t > 0.$$

Double mollification:

$$u^{N,\epsilon}(x; \{x_k\}_{k=1}^N) := \sum_{k=1}^N p_k G^\epsilon(x - x_k),$$

$$U_\epsilon^N(x; \{x_k\}_{k=1}^N) := \left[(u^{N,\epsilon})^2 - (\partial_x u^{N,\epsilon})^2\right]^n (x; \{x_k\}_{k=1}^N),$$

and

$$U^{N,\epsilon}(x; \{x_k\}_{k=1}^N) := (\rho_\epsilon * U_\epsilon^N)(x; \{x_k\}_{k=1}^N).$$

The regularized ODEs:

$$\frac{\mathrm{d}}{\mathrm{d}t} x_i^\epsilon(t) = U^{N,\epsilon}(x_i^\epsilon(t); \{x_k^\epsilon(t)\}_{k=1}^N), \quad i = 1, \cdots, N,$$

Traveling speed of single peakon solutions:

Consider the case for single peakon weak solutions. When $N = 1$,

$$\frac{\mathrm{d}}{\mathrm{d}t} x^{\varepsilon}(t) = U^{1,\varepsilon}(x^{\varepsilon}(t), t) = p^{2n} \left(\rho_{\epsilon} * \left[(G^{\epsilon})^2 - (G^{\epsilon}_x)^2 \right]^n \right)(0).$$

We have the following theorem:

Theorem (Theorem 3.2, Y. Gao & H. Liu 2021)

The following identity holds

$$\lim_{\epsilon \to 0} \left(\rho_{\epsilon} * \left[(G^{\epsilon})^2 - (G^{\epsilon}_x)^2 \right]^n \right)(0) = \frac{1}{2^{2n}} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{2k+1}.$$

For any amplitude $p \neq 0$ and $n \in \mathbb{N}$, the single peakon weak solutions to the gmCH equation is given by

$$u(x,t) = pG(x - c_n t), \quad c_n = \left(\frac{p}{2} \right)^{2n} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{2k+1}.$$

Thank you!