

The ABCD of using history of mathematics in the (undergraduate) classroom

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“The history of science is science itself.”

Johann Wolfgang von Goethe, *Theory of Colour* (1808).

Introduction

Mathematics is a human endeavour which has spanned over four thousand years; it is part of our cultural heritage; it is a very useful, beautiful and prosperous subject. In his Presidential Address delivered to the British Association for the Advancement of Science in 1897 Andrew Russ Forsyth (1858 – 1942) said, “Mathematics is one of the oldest of sciences; it is also one of the most active; for its strength is the vigour of perpetual youth.” [1, Chapter VII] This quotation hints at a peculiar feature of mathematics, which other sciences do not seem to possess, or at least not to the same extent, viz. the past, the present and the future of the subject are intimately inter-related, making mathematics a cumulative science with its past forever assimilated in its present and future [2, 3]. No wonder in another Presidential Address to the British Association for the Advancement of Science in 1890, J.W.L. Glaisher (1848 – 1928) said that “no subject loses more than mathematics by any attempt to dissociate it from its history” [1, Chapter VI]. The great French mathematician Henri Poincaré (1854 – 1912) even said, “If we wish to forsee the future of mathematics, our proper course is to study the history and present condition of the science.”

For many years now various authors in different parts of the world have written on the important role played by history of mathematics in mathematics education. A good summary of some reasons for using history of mathematics in teaching

mathematics and of some ways in carrying it out can be found in [4, pp.4-5]. Perhaps it can be added that not only does the appropriate use of history of mathematics help in teaching the subject, but that in this age of “mathematics for all”, history of mathematics is all the more important as an integral part of the subject to afford perspective and to present a fuller picture of what mathematics is to the public community.

Be that as it may, enough has been said on a propagandistic level. Some enthusiasts have already channelled their effort into actual implementation, resulting in a corpus of interesting material published in recent years in the form of books or collections of papers [5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. With this in mind I wish to share with readers some of my experience in integrating history of mathematics into the day-to-day teaching in the (undergraduate) classroom. Do not be misled by the title into thinking that this article is a guide to the use of history of mathematics in the classroom! The letters *A*, *B*, *C*, *D* refer to four categories, or levels, of the use of history of mathematics in the classroom: *A* for anecdotes, *B* for broad outline, *C* for content, and *D* for development of mathematical ideas. Except for the last category, which describes a course by itself, the first three categories represent three aspects of the use of history of mathematics. Following a good practice in teaching, I shall illustrate each category with examples taken from actual classroom experience instead of just explaining what each category means in words. Even though such examples are admittedly piecemeal, I hope readers can still get an impression of how the four categories contribute to impart a sense of history in the study of mathematics in a varied and multifarious way.

***A* for Anecdotes**

Everybody agrees that anecdotes about mathematics and mathematicians can contribute to the teaching of the subject in various ways. In the preface to his book [15], Howard Eves sums it up beautifully, “These stories and anecdotes have proved very useful in the classroom — as little interest-rousing atoms, to add spice and a touch of entertainment, to introduce a human element, to inspire the student, to instill respect and admiration for the great creators, to yank back flagging interest,

to forge some links of cultural history, or to underline some concept or idea.” (For more anecdotes readers can consult two more books of a similar title by the same author [16, 17].)

When we make use of anecdotes we usually brush aside the problem of authenticity. It may be strange to watch mathematicians, who at other times pride themselves upon their insistence on preciseness, repeat without hesitation apocryphal anecdotes without bothering one bit about their authenticity. However, if we realize that these are to be regarded as anecdotes rather than as history, and if we pay more attention to their value as a catalyst, then it presents no more problem than when we make use of a heuristic argument to explain a theorem. Besides, though many anecdotes have been embroidered over the years, many of them are based on some kind of real occurrence. Of course, an ideal situation is an authentic as well as amusing or instructive anecdote. Failing that we still find it helpful to have a good anecdote which carries a message.

There are plenty of examples of anecdotes which serve to achieve the aims set out in Eves’ preface. I will give only two examples. The first example illustrates the function mentioned last in Eves’ list — to underline some concept or idea. The second example, besides introducing a human element, illustrates that mathematics is not an isolated intellectual activity.

The first example is an anecdote about the German mathematician Hermann Amandus Schwarz (1843 – 1921), reported by Hans Freudenthal [18]. Schwarz, who was noted for his preciseness, would start an oral examination at the University of Berlin as follows.

Schwarz: Tell me the general equation of fifth degree.

Student: $x^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$.

Schwarz: Wrong!

Student: ... where e is not the base of the natural logarithms.

Schwarz: Wrong!

Student: ... where e is *not necessarily* the base of the natural logarithms.

This anecdote, whether it is true, semi-true or even false, makes for a perfect appetizer to the main course of the general equation of degree m . It drives the

point home as to how *special* a general equation is! I have made use of this anecdote several times in a (second) course on abstract algebra, and each time students love it. After listening to it, they appreciate much better the definition of a general equation of degree m to be given subsequently.

The second example is a real historical document, a letter dated March 6, 1832 from Carl Friedrich Gauss (1777 – 1855) to his friend Farkas Bolyai (1775 – 1856), seven weeks after receipt of the amazing work on non-euclidean geometry by the latter’s son, János Bolyai (1802 – 1860). We can imagine the dismay (but not without a trace of delight!) of the proud father when he read the letter which said, “If I commenced by saying that I am unable to praise this work [by János], you would certainly be surprised for a moment. But I cannot say otherwise. To praise it, would be to praise myself. Indeed the whole contents of the work, the path taken by your son, the results to which he is led, coincide almost entirely with my meditations, which have occupied my mind partly for the last thirty or thirty-five years. ... of which up till now I have put little on paper, my intention was not to let it be published during my lifetime. ... On the other hand it was my idea to write down all this later so that at least it should not perish with me. It is therefore a pleasant surprise for me that I am spared this trouble, and I am very glad that it is just the son of my old friend, who takes the precedence of me in such a remarkable manner.” [19, p.100] From this passage we can unfold an interesting discussion on the interaction between philosophy and mathematics, and realize better how mathematics forms a “sub-culture” within a broader culture.

***B* for Broad Outline**

It is helpful to give an overview of a topic or even of the whole course at the beginning, or to give a review at the end. That can provide motivation and perspective so that students know what they are heading for or what they have covered, and how that relates to knowledge previously gained. In either case we can look for ideas in the history of the subject (even though in some cases the actual path taken in history was much too tortuous to be recounted to pedagogical advantage).

One good example which permeates different levels in the study of mathematics is the concept of a function. (See [20] for a discussion of an attempt to incorporate this mathematical-historical vein into the teaching of mathematics at various levels, from secondary school to university.) Let me give a more “localized” (to just one subject) example here, that of the differential geometry of surfaces.

With the invention of calculus came its application in the study of plane curves and later space curves. One crucial description is captured in the notion of curvature with its several different but equivalent definitions. If we view the curvature κ as the rate of change of a turning tangent, then it is not surprising that, as proved by Abraham Gotthelf Kästner (1719 – 1800) in 1761,

$$\oint_C \kappa ds = 2\pi$$

for a simple closed curve C on the plane. In the 18th century, knowledge about space curves allowed mathematicians to study a surface S in space, notably its curvature, through the investigation of intersecting curves on S by planes through the normal at a point. Leonhard Euler (1707 – 1783) introduced the notion of principal curvatures κ_1, κ_2 , which are the maximum and minimum values of the curvatures of sectional curves so obtained on a pair of mutually orthogonal planes. The product $\kappa = \kappa_1 \kappa_2$ turns out to be of significance and is known as the Gaussian curvature, which can also be described through the “Gauss map”, which measures how fast the surface bends away from the tangent plane by measuring the “dispersion of directions” of unit normal vectors at all points in a neighbourhood. Calculation of these quantities involves the use of coordinates, i.e. the surface S is regarded as something sitting in the 3-dimensional euclidean space. For this reason we say that such quantities are extrinsically determined. Mathematicians would like to talk about the intrinsic geometry of a surface, i.e. describe the surface as someone living on the surface without having to leave the surface and look at it from above or below! In a famous memoir of 1827 titled “Disquisitiones generales circa superficies curvas” (General investigation of curved surfaces) Gauss initiated this approach. (Remember that Gauss did a lot of survey work and mapmaking, and in those days one surveyed the terrain on the ground, not from the air!) The crucial notion is that of a geodesic, the line of shortest distance on the surface between two given points. Two surfaces which are applicable the one to the other,

by bending but without stretching so that the distance between two given points remains the same, will have the same geometry. For instance, the geometry on a cylindrical surface will be the same as that on a plane surface, but will be different from that on a spherical surface. However, both the plane surface and the spherical surface enjoy a common property, viz. a small piece cut on each will be applicable to any other part on that same surface; in other words, they are both surfaces of constant curvature. Indeed, Gauss proved in his memoir that the (Gaussian) curvature κ is an intrinsic property, a result so remarkable that he named it “Theorema Egregium”. He further showed that for a triangle Δ on S whose sides are geodesics,

$$\iint_{\Delta} \kappa dS = 2\pi - (\text{sum of exterior angles}) .$$

(More generally, for a simple closed curve C on a surface S , the analogue to Kästner’s result is

$$\oint_C \kappa_g ds = 2\pi - \iint_R \kappa dS ,$$

where κ_g is the so-called geodesic curvature and R is the region on the surface bounded by C . For a geodesic triangle Δ , the first integral becomes the sum of exterior angles.) This important result was later generalized by Pierre-Ossian Bonnet (1819 – 1892) in 1848 and by other mathematicians still later into the deep Gauss-Bonnet Theorem, which relates the topology of a surface to the integral of its curvature. (For the continued development initiated by the famous 1854 Habilitationsvortrag of Georg Friedrich Bernhard Riemann (1826 – 1866), readers can consult [21, Chapters 11-15].)

***C* for Content**

In [22] David Rowe points out that a major challenge facing the history of mathematics as a discipline will be to establish a constructive dialogue between the “cultural historians” (those who approach mathematics as historians of science, ideas, and institutions) and the “mathematical historians” (those who study the history of mathematics primarily from the standpoint of modern mathematicians).

In this connection one should also consult [23, 24, 25] to savour the different views held by some mathematicians and some historians of mathematics. I learn and benefit from both groups in my capacity as a teacher and student of mathematics, for I agree with what Charles Henry Edwards, Jr. says in the preface to his book [10], “Although the study of the history of mathematics has an intrinsic appeal of its own, its chief *raison d’être* is surely the illumination of mathematics itself to promote a more mature appreciation of [theories].” In this section I will give four examples borrowed from pages in the history of mathematics with an eye to the enhancement of understanding of the mathematics. This is a particularly pertinent function of history of mathematics for a mathematics teacher’s day-to-day work.

(1) The first example has appeared in [26], which is in turn gleaned from [27, Appendix I]. (It also appears as one example in [28].) In 1678 Gottfried Wilhelm Leibniz (1646 – 1716) announced a “law of continuity” which said that if a variable at all stages enjoyed a certain property, then its limit would enjoy the same property. Up to the early 19th century mathematicians still held this tenet so that Augustin-Louis Cauchy (1789 – 1857) might have been guided by it to arrive at the following result in 1821: If $\{f_n\}$ is a sequence of continuous functions with limit f , i.e. $\lim_{n \rightarrow \infty} f_n(x) = f(x)$, then f is a continuous function. Whenever I teach a calculus class I present Cauchy’s “proof” to the class as follows. For sufficiently large n , $|f_n(x) - f(x)| < \varepsilon$. For sufficiently large n , $|f_n(x+h) - f(x+h)| < \varepsilon$. Choose a specific n so that both inequalities hold, then

$$|f_n(x) - f(x)| + |f_n(x+h) - f(x+h)| < 2\varepsilon .$$

For this chosen f_n , we have $|f_n(x+h) - f_n(x)| < \varepsilon$ for sufficiently small $|h|$, since f_n is continuous at x . Hence, for sufficiently small $|h|$, we have

$$|f(x+h) - f(x)| \leq |f(x+h) - f_n(x+h)| + |f_n(x+h) - f_n(x)| + |f_n(x) - f(x)| < 3\varepsilon .$$

With ε being arbitrary to begin with, this says that f is continuous at x . (A picture will make the argument even more convincing!) While many students are still nodding their heads, I tell them that Jean Baptiste Joseph Fourier (1768 – 1830) at about the same time showed that certain very discontinuous functions could be represented as limits of trigonometric polynomials! (In hindsight we see

that the work of Fourier provided counter-examples to the “theorem” of Cauchy. But at the time it was not regarded in this light. Actually, when the Norwegian mathematician Niels Henrik Abel (1802 – 1829) offered in his memoir of 1826 the example

$$\sin \phi - \sin 2\phi/2 + \sin 3\phi/3 - \cdots ,$$

he remarked that “it seems to me that there are some exceptions to Cauchy’s theorem” and asked instead what “the safe domain of Cauchy’s theorem” should be [27, Appendix I]. Abel resolved the puzzle by restricting attention to the study of power series, but in so doing, missed an opportunity to investigate the way an infinite series (of functions) converge.) I ask the class to wrestle with the “proof” of Cauchy and see what is amiss. If they cannot spot it, I tell them not to feel bad since Cauchy could not spot it either, and it was left to Philipp Ludwig von Seidel (1821 – 1896) to find out the mistake twenty six years later! Rectification of the proof led later to the new notion of uniform convergence explicitly explained by Karl Theodor Wilhelm Weierstrass (1815 – 1897). With this historical overture we pass naturally on to a discussion of the mathematics of uniform convergence. (See [5, Chapter 5; 12, Chapter III.4] for an enlightening discussion of the mathematics.)

(2) The second example has been used several times in an advanced elective course on algebra. It started with an announced “proof” of Fermat’s Last Theorem by Gabriel Lamé (1795 – 1870) in the meeting of the Paris Academy on March 1, 1847. The key step lies in the factorization

$$z^p = x^p + y^p = (x + y)(x + y\zeta) \cdots (x + y\zeta^{p-1})$$

in the ring of cyclotomic integers $\mathbb{Z}[\zeta]$ (modern terminology), where ζ is a primitive p th root of unity. For an interesting account of the pursuit of this question in subsequent meetings of the Paris Academy, readers can consult [29, Chapter 4]. The account includes the deposit of “secret packets” with the Academy by Cauchy and Lamé — an institution of the Academy which allowed members to go on record as having been in possession of certain ideas at a certain time without revealing the content, in case a priority dispute developed later. The packets remained secret and the matter was put to rest when Joseph Liouville (1809 – 1882) read a letter from his friend Ernst Eduard Kummer (1810 – 1893) in the meeting of

the Paris Academy on May 24, 1847. In the letter Kummer pointed out that the “proof” broke down owing to failure of unique factorization in $\mathbb{Z}[\zeta]$ in general. He even included a copy of his memoir, published three years earlier, in which he demonstrated that unique factorization failed for $p = 23$. He went on to say that he could save unique factorization by introducing a new kind of complex number he christened “ideal complex numbers”. With suitably chosen illustrative examples to supplement the story, this is a natural point to launch into a detailed discussion on the unique factorization of ideals in a Dedekind domain.

(3) The third example is also on algebra. It concerns the basic result known as the Chinese Remainder Theorem. I will skip both the statement of the result in the language of abstract algebra and the origin of the result found in Problem 26 of Chapter 3 of *Sunzi Suanjing* (Master Sun’s Mathematical Manual, c. 4th century), which can be found in most textbooks, such as [30]. I will also skip the application of this type of problem, viz. $x \equiv a_i \pmod{m_i}$, $i \in \{1, 2, \dots, N\}$, in ancient Chinese calendrical reckoning. I will only highlight what I would do next after going through the two aforesaid issues with the class. I discuss with them an algorithmic method devised by Qin Jiushao (1202 – 1261), known as the “Dayan art of searching for unity” and explained in his book *Shushu Jiuzhang* (Mathematical Treatise in Nine Sections) of 1247. It is instructive to see how to find a set of “magic numbers” from which a general solution can be built by linear combination. It suffices to solve separately single linear congruence equations of the form $kb \equiv 1 \pmod{m}$, by putting $m = m_i$ and $b = (m_1 \cdots m_N)/m_i$. The key point in Qin’s method is to find a sequence of ordered pairs (k_i, r_i) such that $k_i b \equiv (-1)^i r_i \pmod{m}$ and the r_i ’s are strictly decreasing. At some point $r_s = 1$ but $r_{s-1} > 1$. If s is even, then $k = k_s$ will be a solution. If s is odd, then $k = (r_{s-1} - 1)k_s + k_{s-1}$ will be a solution. This sequence of ordered pairs can be found by using “reciprocal subtraction” (known as the Euclidean algorithm in the West), viz. $r_{i-1} = r_i q_{i+1} + r_{i+1}$ with $r_{i+1} < r_i$, and $k_{i+1} = k_i q_{i+1} + k_{i-1}$. If one looks into the calculation actually performed at the time, one will find that the method is even more streamlined and convenient. Consecutive pairs of numbers are put at the four corners of a counting board, starting with

$$\begin{array}{|c|c|} \hline 1 & b \\ \hline 0 & m \\ \hline \end{array}, \quad \text{ending in} \quad \begin{array}{|c|c|} \hline k & 1 \\ \hline * & * \\ \hline \end{array}.$$

The intermediate processes are shown below:

$$\begin{array}{|c|c|} \hline k_i & r_i \\ \hline k_{i-1} & r_{i-1} \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|} \hline k_i & r_i \\ \hline k_{i-1} & r_{i+1} \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|} \hline k_i & r_i \\ \hline k_{i+1} & r_{i+1} \\ \hline \end{array} \quad \text{if } i \text{ is even ,}$$

OR

$$\begin{array}{|c|c|} \hline k_{i-1} & r_{i-1} \\ \hline k_i & r_i \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|} \hline k_{i-1} & r_{i+1} \\ \hline k_i & r_i \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|} \hline k_{i+1} & r_{i+1} \\ \hline k_i & r_i \\ \hline \end{array} \quad \text{if } i \text{ is odd .}$$

The procedure is stopped when the upper right corner becomes a 1, hence the name “searching for unity”. Students will be amazed by noting how the procedure outlined in *Shushu Jiuzhang* can be phrased word for word as a computer program!

(4) The fourth example is about the Cayley-Hamilton theorem taught in a linear algebra course, viz. $\chi(A) = 0$ where $\chi(X)$ is the characteristic polynomial of the $n \times n$ matrix A . First I show students a letter dated November 19, 1857 from Arthur Cayley (1821 – 1895) to James Joseph Sylvester (1814 – 1897) [31, pp.213-214]. The letter illustrated the theorem by exhibiting a concrete 2×2 case. This is particularly pertinent for an average student, who at this stage may even be confused by the mere statement of the result, not to mention the explanation of why it is true. Hence I emphasize the point in class by repeating the words made by Cayley himself (in 1858), “The determinant, having for its matrix a given matrix less the same matrix *considered as a single quantity* (italics mine) involving the matrix unity, is equal to zero.” To drive the point home I continue to produce a “joke-proof”: set $X = A$ in the expression $\det(A - XI)$, hence $\det(A - AI) = \det 0 = 0$, which means A satisfies the characteristic polynomial $\chi(X) = \det(A - XI)$. Students are requested to find out why this is not a valid proof. I will give a valid proof in the next lecture, and for the more mathematically oriented students I may further explain how to turn the “joke-proof” into a rigorous proof by regarding both $\chi(X)$ and $A - XI$ as polynomials over the ring of $n \times n$ matrices (or more precisely, as polynomials over the *commutative* subring generated by A). My experience tells me that students are stimulated into discussion by the letter of Cayley, perhaps because they see from it that mathematicians do not work alone but talk shop with each other and engage in social interaction. Students are particularly “sympathetic” to the statement made by Cayley in his 1858 memoir after demonstrating the theorem for a 2×2 matrix: “I have not thought it necessary to undertake the labour of a formal proof of the theorem in the general case of a

matrix of any degree.” [32, p.624] As teachers we know how best to handle this sentiment!

D for Development of Mathematical Ideas

“Development of Mathematical Ideas” is the title of a course I have been teaching at my university since 1976. As an elective course for upper-level mathematics students (with an occasional few in other majors) with a moderate class size of around twenty, the course does not have a fixed syllabus nor a fixed format in teaching and assessment, thus allowing me to try out freely new approaches and new teaching material from year to year. Some past experience has been reported in [33, 34].

For the academic year 1995-96 I built the course around the anthology “Classics of Mathematics” edited by Ronald Calinger [35], which became more readily available as a textbook through its re-publication in 1995. The idea is to let students read some selected primary source material and to “learn from the masters”. The year-long course was roughly divided into five sections: (1) Euclid’s *Elements*, (2) Mathematical Thinking, (3) From Pythagoras, Eudoxus, ... (Incommensurable Magnitudes) to Dedekind, Cantor, ... (Real Numbers), (4) Non-euclidean Geometry, (5) Gödel’s Incompleteness Theorem. Passages in [35] were fitted into these five sections. Besides the primary source material, some of the general historical accounts (named “Introduction” of each chapter) make for useful assigned reading, to be supplemented by a general text such as [32]. Lectures were devoted to a more in-depth discussion, with more emphasis on the mathematics. I needed to add some extra source material from time to time, especially material on ancient Chinese mathematics. For instance, in the part on mathematical thinking I tried to let students experience, through the writings of mathematicians such as Liu Hui (c.250), Yang Hui (c.1250), Leonhard Euler (1707 – 1783), Julius Wilhelm Richard Dedekind (1831 – 1916), Henri Poincaré (1854 – 1912), and George Pólya (1887 – 1985), how working mathematicians go about their jobs. Students would learn that the logical and axiomatic approach exemplified in Euclid’s *Elements* is not the only way. The textbook by Calinger [35], with its extensive bibliography, also

provides useful support for the project work (in groups of two), which consists of an oral presentation and a written report on a topic of the students' choice. The course itself is in fact the presentation of my project work!

Conclusion

Using history of mathematics in the classroom does not necessarily make students obtain higher scores in the subject overnight, but it can make learning mathematics a meaningful and lively experience, so that (hopefully) learning will come easier and will go deeper. The awareness of this evolutionary aspect of mathematics can make a teacher more patient, less dogmatic, more humane, less pedantic. It will urge a teacher to become more reflective, more eager to learn and to teach with an intellectual commitment. I can attest to the benefits brought by the use of history of mathematics through my personal experience. The study of history of mathematics, though it does not make me a better mathematician, does make me a happier man who is ready to appreciate the multi-dimensional splendour of the discipline and its relationship to other cultural endeavours. It does enhance the joy derived from my job as a mathematics teacher when I try to share this kind of feeling with my class. I attempt to sow the seeds of appreciation of mathematics as a cultural endeavour in them. It is difficult to tell when these seeds will blossom forth, or whether they ever will. But the seeds are there, and I am content. I like the view proclaimed by the noted historian of science George Sarton (1884 – 1956), who said, “The study of the history of mathematics will not make better mathematicians but gentler ones; it will enrich their minds, mellow their hearts, and bring out their finer qualities.” [36, p.28]

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