Learning and teaching of analysis in the mid 20th century : A semi-personal observation

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A (somewhat personal) prologue

When I was approached by the organizers of the Symposium on One Hundred Years of *L'Enseignement Mathématique* as a prospective speaker, I certainly felt honoured but at the same time surprised and timorous, because I am neither a mathematics educator nor a historian of mathematics by training or by profession. I am just a plain mathematics teacher who believes in the value of the cultural and the historical dimensions of the discipline in general education. The topic assigned to me does not help to alleviate my nervousness either, because my research interest in mathematics lies more in algebra and discrete mathematics. (I wrote a thesis in algebraic *K*-theory under the supervision of Hyman Bass, who is the present President of the ICMI, and I later turned to work in combinatorial designs.) But of course as a student in mathematics we all study analysis; as a teacher in mathematics we all have taught some courses in calculus or analysis. As a school pupil and undergraduate I studied analysis in the mid 1960s. I suppose the organizers chose me as a speaker mainly for my experience as a learner of the subject of analysis during the middle part of the 20th century.

The excellent opportunity of meeting the many colleagues in the community of mathematics education on that memorable occasion, not to mention using it as an excuse to visit the beautiful city of Genève, is too tempting for me not to accept the invitation. Hence I set to work by proceeding in four directions:

(a) I read up on relevant papers that appeared in the decade 1955-1965 in the journal *L'Enseignement Mathématique.*

(b) I consulted related sources, of which the following are particularly pertinent :

** *Tendances nouvelles de l'enseignement des mathématiques*, Vol.I (1967), Vol.II (1967), Vol.III (1973), Vol.IV (1979), ICMI/UNESCO.

** *Studies in Mathematics Education, Vol. 4: The Education of Secondary School Teachers of Mathematics*, edited by R. Morris, UNESCO, 1985.

** M. Artigue, Analysis, in *Advanced Mathematical Thinking*, edited by D. Tall, Kluwer Academic Publ., 1991, 167-198.

** M. Artigue, Réformes et contre-réformes dans l'enseignement de l'analyse au lycée (1902-1994), in *Les sciences au lycée, un siècle de réformes de l'enseignement des mathématiques et de la physique en France et à l'étranger*, edited by B. Belhoste, H. Gispert, N. Hulin, Vuibert-INRP, 1996, 192-217.

** Proceedings of ICM or ICME held between 1950 to 1980.

(c) I consulted some textbooks in common use in the mid 20th century. This is certainly not meant to be a comprehensive survey, but is to a large extent dependent on my personal acquaintance with the books.

(d) I corresponded or talked with some mathematicians (of different nationalities) who were students or young teachers in the 1950s and early 1960s in order to get to know about their own learning or teaching experience in analysis. It must be admitted that this part of the project is not carried out systematically nor scientifically and tends to be anecdotal. But it may supplement (a) and (b) in that we can get a non-Western perspective through talking with several mathematicians in China. Papers in *L'Enseignement Mathématique* usually recount learning and teaching in European countries, sometimes also in North American countries, but very seldom in Asian countries as well. As for my own self, I began undergraduate study in 1963. When I was a school pupil, the tidal wave of the "new math" movement had not yet swept over Hong Kong, so I was brought up in a more traditional curriculum, modelled after the British system, sometimes with a time lag, as Hong Kong was in those days a British colony. It was in the university that I first felt the impact of modern mathematics.

Learning and teaching of analysis in the 1950s and 1960s

What was school mathematics like in the 1950s? There is no dearth of reports on this question even if we confine our search to *L'Enseignement Mathématique*. Instead of going over them one by one let me just quote a passage from the report by Howard F. Fehr [Fehr, 1959]:

"This, then is the picture of what the pupil has been taught. What does he really know? This is hard to tell, but it can be said that the 15 year old in all countries, who has continued his study of mathematics through the first 9 or 10 years of school can compute in a mature manner with the positive rational numbers, in a decimal system of notation, even though he cannot rationalize what he does; he has a fairly useful and practical knowledge of geometry with respect to mensuration and common relationships; and he can manipulate algebraic expressions and solve equations and problems in a structureless system of algebra. He can make simple deductions, but his entire concept of proof, if any, is limited to that of theorems in geometry. He really does not know what mathematics is, or how it is applied, but he has a large body of information, upon which, if he is inclined or interested, a study of mathematics can be built in the ages 16 years to 21 years."

(As an aside, from what I observe of my own students today, most of them still do not know what mathematics is, but they have a far more meagre body of information mastered so that they experience difficulty in continuing their study of mathematics.)

Fehr continues with the aims of mathematical instruction labelled as :

"(1) mathematics for the better life, i.e. for its intrinsic value, or for its own sake and

(2) mathematics for a better living, i.e. for its application to science, technology, and social problems that will result in more efficient practical day by day living."

This should ring equally true forty years after it was written, but today people seem to pay more and more attention to (2) than to (1), at most paying some lip-service to the latter.

It was with the academic background depicted above that students in the upper secondary school embarked on the study of analysis, which was usually just called calculus in most places (as it still is today). Calculus had been introduced into the upper secondary school curriculum in the early part of the 20^{th} century. It met with some success, for instance in the reform of 1902 in France [Artigue, 1996]. By the 1950s the syllabus was more or less stabilized to include: differentiation and integration; simple applications such as rate of change, maxima and minima, area and volume, centre of mass, moment of inertia; the trigonometric, logarithmic and exponential functions; Taylor series expansion of functions. The instruction was in large part intuitive and informal, emphasizing calculation rather than conceptual understanding. For instance, the idea of limit was explained through an intuitive sense that the dependent variable approaches a certain value as the independent variable approaches a given value. The derivative of a function $f(x)$ at $x = a$ is described geometrically as the slope of the tangent line to the curve $y - f(x) = 0$ at the point $(x, f(a))$ of the euclidean plane $\mathbb{R}^2$. From this description students were led to the calculation of the limit of a Newton quotient $[f(a+h)-f(a)]/h$ as h approaches 0, from which point on they were drilled in the calculation of the derivatives of many different functions. The subject was meant to be preparatory for those who would go on to university and need it as a tool or prerequisite for further mathematical pursuits. Let me illustrate with my own learning experience. I came into contact with calculus before formal upper secondary schooling when in the preceding summer vacation my mathematics teacher kindly offered to give me extra lessons out of the popular American textbook *Elements of the Differential and Integral Calculus* by William Anthony Granville (revised by Percy F. Smith and William Raymond Longley in 1929; originally published in 1904). One would read definitions like:

> "the variable v is said to approach the constant l as a limit when the successive values of v are such that the numerical value of the difference $v - l$ ultimately becomes and remains less than any preassigned positive number, however small."

and recipes like :

> "GENERAL RULE FOR DIFFERENTIATION
>
> FIRST STEP. In the function replace x by $x + \Delta x$, and calculate the new value of the function, $y + \Delta y$.
>
> SECOND STEP. Subtract the given value of the function from the new value and thus find Δy (the increment of the function).

THIRD STEP. Divide the remainder Δy (the increment of the function) by Δx (the increment of the independent variable).

FOURTH STEP. Find the limit of this quotient when Δx (the increment of the independent variable) varies and approaches zero as a limit. This is the derivative required."

Integration is introduced as the inverse operation to differentiation, viz. "to find a function $f(x)$ whose derivative $f'(x) = \phi(x)$ is given". The definite integral follows next, again with a recipe :

"FIRST STEP. Find the indefinite integral of the given differential expression.

SECOND STEP. Substitute in this indefinite integral first the upper limit and then the lower limit for the variable, and subtract the last result from the first."

In my school days I enjoyed doing all these but I would not claim that I really understood what was going on. I still did not really understand what was going on even after studying later upper secondary school textbooks such as *Techniques of Mathematical Analysis* by C.J. Tranter (The English Universities Press, 1957). I solved quite a number of exercises in that book, most of them of a rather technical nature, such as:

- If a, b are positive and $a + b = 1$, show that $ab \le ¼$ and deduce that

$$\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2 \ge \frac{25}{2}.$$

- Prove that, if $y = x^2 \cos x$, then

$$x^2\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+(x^2+6)y=0.$$

Deduce that, when $x = 0$,

$$(n-2)(n-3)\frac{d^n y}{dx^n} + n(n-1)\frac{d^{n-2} y}{dx^{n-2}} = 0.$$

- If $\pi y = \int_0^{\pi} \cos(x\sin\theta)d\theta$, show that

$$\frac{d^2 y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0 .$$

Occasionally I ran into some theoretical discussion, such as:

- If $f(a) = f(b)$, there is one point in the open interval (a, b) at which $f'(x) = 0$.
- If $f'(x) > 0$ for every x in the open interval (a, b), $f(x)$ is strictly increasing throughout the interval.

Somewhere in the book there is a passage on the Fundamental Theorem of Calculus. But I was not aware of the significance of this beautiful result at the time. In fact, without grasping the concept of a definite integral as the limit of a certain summation, I would have had difficulty in applying integration at will to solve problems other than the stereotyped ones. If I had studied with all my might the book *A Course of Pure Mathematics* by Godfrey Harold Hardy (tenth edition 1952; originally published in 1908), I could have understood better. But the book was clearly beyond my comprehension at the time — I bought the book because of its misleading title, since the subject I registered for the university entrance examination was called “Pure Mathematics” (to be distinguished from another subject called “Applied Mathematics”)! However, one thing I remember vividly about reading that book is its appendix on two proofs of the Fundamental Theorem of Algebra with the beginning remark that “it belongs more properly to analysis”— in the mind of a 16-year-old, it is strange to learn that the root of an algebraic equation has to do with infinitesimal analysis.

Despite this shadowy understanding (but with reasonably adequate technical competence in computation) I was thrilled with the subject of calculus. During the summer vacation when my teacher gave me the extra tutoring, I could solve a problem, albeit in a rather formal symbol-pushing manner without knowing what a differential equation is, that asked for the escape speed of a rocket. A rocket (of mass m) is to be launched straight up from the surface of earth (of mass M) having a radius R. The student is asked to show that the minimum speed v_0 at which the rocket must be launched to reach a distance r_0 from the centre of earth is given by $v_0 = \sqrt{2GM\left(\frac{1}{R} - \frac{1}{r_0}\right)}$ and to deduce that the escape speed is $\sqrt{\frac{2GM}{R}}$. I started the calculation with the formula $-m\frac{dv}{dt} = \frac{GMm}{r^2}$ (this much is physics) followed by a formal manipulation

$$\frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt} = \frac{dv}{dr}v$$

that yielded $\quad -mv\frac{dv}{dr} = \frac{GMm}{r^2}$, or $\quad -v\,dv = \frac{GM}{r^2}dr$ so that

$$-\int_{v_0}^{0} v\,dv = \int_{R}^{r_0} \frac{GM}{r^2}\,dr\,,$$

hence $\quad -\left.\frac{v^2}{2}\right|_{v_0}^{0} = -\left.\frac{GM}{r}\right|_{R}^{r_0}$,

or $\quad \frac{1}{2}v_0^2 = -GM\left(\frac{1}{r_0} - \frac{1}{R}\right)$, $\qquad$ i.e. $\quad v_0 = \sqrt{2GM\left(\frac{1}{R} - \frac{1}{r_0}\right)}$.

(Just to indicate how shadowy my understanding was at that stage, I should confess that I added the minus sign in the starting formula *after* I found out that I arrived at an expression $\frac{1}{2}v_0^2 = GM\left(\frac{1}{r_0} - \frac{1}{R}\right) < 0$, which is blatantly wrong!) Finally, by putting $r_0 = \infty$, I obtained the escape speed $v_0 = \sqrt{\frac{2GM}{R}}$. This little feat on my part was particularly thrilling at the time when a little over three years previously the first Soviet sputnik was launched into orbit, heralding the age of space travel!

The "new math" movement and analysis

The word “sputnik” mentioned in a paper on mathematics education reminds one of the “new math” movement. There is no need to go into the history of the new math movement nor the dispute on the pros and cons during and after the movement, although this is eventful, instructive and worth the discussion. There is a large body of literature on that. For an overview in the UK and the USA one can consult [Thwaites, 1972; CBMS, 1975]. Two thought-provoking papers published in *L’Enseignement Mathématique* discussed the issues in the early phase of the movement [Freudenthal, 1963; Wittenburg, 1965]. The introduction of untraditional subject matter and the emphasis on abstraction and structure in the new math movement did not seem to affect basically the learning and teaching of analysis as a school subject, in the sense that what was a difficult pedagogical problem before the movement remained a difficult pedagogical problem during and after the movement. It is true that in the new math movement, the subject of analysis, at least on what concerns notions and topics closely related to it (such as function, absolute value, error estimation, sequence and series, manipulation of inequalities, metrics, etc.), was taught in even lower grades for even younger pupils, and that the language of sets and mappings was adopted for the pursuit of preciseness, and that there was a trend towards abstraction. But even without the language of sets and mappings, even without the strengthened dose of abstraction, students found analysis (calculus) difficult, not really because they could not do the (routine) calculations but because they did not understand and could not make good *sense* of the subject. The transition from school to university posed (as it still does today) even more serious problems. Several papers which appeared in *L’Enseignement Mathématique* in the mid 1950s were devoted to that issue [Behnke, 1957; Delessert, 1965; Freudenthal, 1956; Maxwell, 1956]. Kay Piene also discussed this issue in an address in the ICM54 at Amsterdam [Piene, 1956]. Again, let me illustrate with my own learning experience. Besides studying textbooks such as *Differential and Integral*

Calculus, Vol. I and II by Richard Courant (1937), *Mathematical Analysis: A Modern Approach To Advanced Calculus* by Tom Apostol (1957), *Principles of Mathematical Analysis* by Walter Rudin (1953; 2nd edition 1964), I could also have access to Chinese textbooks which were modelled after the Soviet tradition of textbooks written by authors like Alekandr Iakovlevich Khinchin or Grigorii Mikhailovich Fichtenholz that were popular in the USSR and in China in the mid 20th century. The course would start with a detailed discussion of the completeness of the real number system in its various formulations and the basic theorems about a continuous function defined on a closed interval. (A particularly lucid account, still worth studying today, is the little book *Eight Lectures On Mathematical Analysis* by A.I. Khinchin, first published in Russian in 1943.) Thus I had the opportunity to sample the very rigorous treatment of analysis by the Soviet school, which was hard work but good solid training. In Europe, budding mathematicians in their first year at university went through a similar rigorous diet of textbooks by authors like Édouard Goursat, Georges Valiron, or Jean Bass. The consequence of going through such a rigorous diet is that one either makes it or one gets irretrievably lost. In the mid 20th century, when tertiary education in most places still catered for élitism, this state of affair was allowed to go on. With the opening up of tertiary education in later decades, the problem is getting more and more noticeable and has to be faced and resolved.

Two questions on transition from school to university

There are two questions I wish to raise concerning the transition from school to university pertaining to the learning and teaching of analysis.

(a) By observing students I have taught from the mid 1970s till now, I find that I was quite fortunate to have experienced a smoother transition. To what extent has this to do with my "classical" education in geometry? As Richard Courant puts it in an introductory remark to his famous textbook on calculus [Courant, 1937], "Its intimate association with geometrical ideas and its stress on individual niceties give the older mathematics a charm of its own." I was brought up with a large dose of synthetic geometry replete with lots of proofs and construction problems. Not only was I accustomed to the notion of proof and logic before starting undergraduate study, but in school geometry I tasted the joy of discovery and the joy of succeeding in understanding something which is tangible (you can at least draw some pictures even if you do not know why it has to be like that at first) but not obvious (you do not know why it is like that at first). Geometry is a subject in which one can exercise logical discipline and free imagination *at the same time*. Developing a liking for geometry also enabled me to look at problems in other subjects from a geometric viewpoint. This helped in particular in the study of analysis. After all, calculus is the process of linear approximation, and linear problems fall within the purview of linear algebra, which is in itself akin to geometry. Of course, it does not work all the time and different persons are accustomed to different ways of thinking; nevertheless it offers an alternative. Many students today are not accustomed to this flexibility in framework in their study of mathematics.

(b) How extensive is the influence of "Bourbakism" felt on the teaching of analysis in the university? In school mathematics I do not think there is much influence at all. I first learnt about the work of Bourbaki from my teachers at the university. I cannot say much more further on this issue as I have not found out enough, and besides it would be somewhat remote from school mathematics, which is what we are more focused on in our discussion.

What can we learn from looking at the past?

What can we learn from looking at those bygone days as far as the learning and teaching of analysis in the school or university classroom of the 21^{st} century are concerned? Back in those days, the all-purpose electronic computer was just making its début; the hand-held calculator was still a luxury in the classroom; the Internet was not thought of even in works of science fiction; the mathematics curriculum was not as broad and as diversified; the application of analysis was mainly confined to physics and engineering. As a result, contemporary issues in the learning and teaching of analysis, such as the use of modern technology in the classroom, the relevance of the subject in daily life or the relationship of the subject to information technology, were not yet on the agenda. However, the difficulties a student encountered in those days are still the difficulties a student encounters today. In this sense, many of these contemporary issues, though they may play a significant role in improving the learning and teaching of the subject, are secondary since they may well breathe new life into the subject but they are not at the root of the difficulty. What *is* at the root is the perennial controversy between computational skill and understanding, between concreteness and abstraction. This controversy, which can sometimes unfortunately develop into an unnecessary false dichotomy, is a theme which reverberated throughout the last century from the introduction of calculus into the school curriculum to the calculus reform of the 1980s and 1990s. In this connection let me quote two relevant passages from two great teachers:

"The point of view of school mathematics tempts one to linger over details and to lose one's grasp of general relationships and systematic methods. On the other hand, in the "higher" point of view there lurks the opposite danger of getting out of touch with concrete details, so that one is left helpless when faced with the simplest cases of individual difficulty, because in the world of general ideas one has forgotten how to come to grips with the concrete. The reader must find his own way of meeting this dilemma. In this he can only succeed by repeatedly thinking out particular cases for himself and acquiring a firm grasp of the application of general principles in particular cases; herein lies the chief task of anyone who wishes to pursue the study of Science." [Courant, 1937]

"Abstraction was not a hormone which can be imposed from outside, but one that the patient must generate for himself in response to appropriate stimulation." [Quadling, 1985]

Finally, let me inject a historical remark on the term "analysis". In the ancient Greek usage of this word it means, in contrast to "synthesis", the process of working backward from what is sought until something already known is arrived at. Towards the beginning of the 17th century François Viète used the term "analysis" to denote algebra, since he did not favour the word "algebra" (coming from the Arabic word "al-jabr") which has no meaning in any European language. In his book *In artem analyticam isagoge,* much of the algebra developed is motivated by the intention to solve geometric problems. In this sense it is related to the ancient usage of the word in describing the process. This was brought even more into focus by the work of René Descartes as illustrated by the famous appendix *La géométrie.* When calculus came on stage in the era of Isaac Newton and Gottfried Leibniz, they regarded the subject as an extension of the algebra of the infinite in which lots of functions were expressed as power series that behave like polynomials, just longer! This was again emphasized in the work of Joseph Louis Lagrange with the title *Théorie des fonctions analytiques.*

Leonhard Euler titled his book on calculus *Introductio in analysin infinitorum*. By and by the term "analysis" was used to denote the study of calculus and its extension. In the preface to an unpublished book on analysis, Henri Lebesgue discusses the relationship and distinction between arithmetic, algebra and analysis (published posthumously as [Lebesgue, 1956]). Gustave Choquet emphasizes the inseparable relationship between algebra, geometry and analysis throughout his paper [Choquet, 1962]. Thus, the subject of analysis is closely tied in with the subjects of arithmetic, algebra and geometry. Maybe this historical episode is a reminder of the integrated unity of the four basic subjects in the mathematics curriculum, which we wish our students to realize and to appreciate.

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