EQUATIONS IN CHINA: TWO MILLENNIA OF INNOVATION, TRANSMISSION AND RE-TRANSMISSION

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ABSTRACT

In a rather broad sense, mathematics deals with the solving of equations, different sorts of equations which help us to handle problems and to understand the world around us. It is therefore natural that the art of solving equations was developed in very early time, from the days of ancient Egypt, Babylonia, India and China. This paper tells (part of) the story of the art of solving equations in China as recorded in mathematical texts from ancient to medieval times, reaching a "golden age" in the thirteenth century, followed by a period of stagnation or even becoming a lost art until the study was revived during the eighteenth/nineteenth centuries when European mathematics was transmitted into China, first through the Jesuits since the beginning of the seventeenth century and later through the Protestant missionaries in the nineteenth century. It is hoped that this story may enrich the pedagogical aspect of learning the topic of equations in a modern day classroom.

1 Introduction

In a video made on the occasion of the award of the Abel Prize in 2018 the recipient, Robert Langlands, said, "What is called, not by me, "Langlands Program", is about various things. Among other things it is about the solution of equations."

In a rather broad sense, mathematics deals with the solving of equations, different sorts of equations which help us to handle problems and to understand the world around us. It is therefore natural that the art of solving equations was developed in very early time, from the days of ancient Egypt, Babylonia, India and China. In those early days such problems invariably appeared as word problems for a natural reason (Swetz, 2012). Symbolic language and manipulation in dealing with equations took a few thousand years more to materialize.

In China, the art of solving different types of equations appeared in various mathematical texts from ancient to medieval times, reaching a "golden age" in the thirteenth century, followed by a period of stagnation or even becoming a lost art until the study was revived during the eighteenth/nineteenth centuries when European mathematics was transmitted into China, first through the Jesuits since the beginning of the seventeenth century and later through the Protestant missionaries in the nineteenth century.

Before the era of transmission of European mathematics into China, another kind of transmission of learning took place through the famous Silk Road. As the main trade route in Central Asia that established links between a cross-cultural mix of religions, civilizations and people of many different regions, it also naturally enabled exchanges of learning and cultures of people of different races. Transmission of mathematical knowledge, either directly or indirectly, between China and regions in Central Asia and the Middle East, India, the Islamic Empire and even Europe further to the West went on for many centuries from the Han Dynasty to the Yuan Dynasty

throughout fifteen centuries. A well-known example often referred to is the Method of Double False Position, originated from the method of *ying buzu* [盈不足 excess and deficit] explained in Chapter 7 of *Jiuzhang Suanshu* [九章算術 The Nine Chapters on the Mathematical Art], the prototype Chinese mathematical classics that is believed to have been compiled between the second century BCE and the first century CE (Siu, 2016).

2 A general account on solving equations in ancient and medieval China

The art of solving equations has a long history in China. Despite what the title may have suggested this paper will not give a detailed account of this history, which has been so far covered in a large collection of books and papers, some in the nature of scholarly research in history of mathematics and some in the nature of general exposition for a wider readership. Thus, there is no dearth of reading material on this topic. Primary sources are not hard to locate either, for example, see (Guo, 1993). Instead of offering a long list of references, which definitely cannot cover even a fraction of the relevant literature, I will just give some references which I frequently consult, and among that only those written in a Western language, leaving out a rich repertoire of the literature written in the Chinese language (Chemla, Guo, 2004; Hoe, 1977; Lam, 1977, Lam, & Ang, 1992/2004; Libbrecht, 1973; Shen, Crossley, & Lun, 1999). Readers who can read Chinese can begin with the encyclopaedic series on the historical development of mathematics in China in a span of four millennia from the very early time to the Qing Dynasty (Wu, 1982-2004).

If this topic has been so extensively studied, what is it that this paper wishes to contribute? As will become apparent in the next three sections we have the pedagogical aspect in mind. However, it is helpful to still give a brief account of the general background.

The art of solving equations was recorded and explained (in later commentaries) throughout the different chapters of *Jiuzhang Suanshu*. This book was woven around the main notion of ratio and proportion which already appeared in Chapter 1 and was much expanded and elaborated in the next five chapters, evolving into the method of excess and deficit of Chapter 7 and systems of linear equations of Chapter 8, then on to geometric interpretation and applications in the method of *gou-gu* [勾股 right triangles] of the final Chapter 9. Problems in ratio and proportion involve linear equations, but problems in area and volume of geometric figures and problems in right triangles invariably involve simple quadratic and cubic equations of the type $x^2 = A$ or $x^3 = V$, for which the methods of extracting square roots and cube roots were developed. In Chinese mathematics, the method of extracting square roots, called *kaifangshu* [開方術] and explained in Chapter 4 of *Jiuzhang Suanshu*, is a crucial discovery that led to highly elaborated methods in stages in subsequent dynasties, reaching a high point in the thirteenth century when it developed into an algorithmic procedure for solving polynomial equations of any degree. By that time the art of solving equations of any degree was known as the *tianyuanshu* [天元術 celestial source procedure] and was extended to solving a system of polynomial equations in four unknowns known as the *siyuanshu* [四元術 four sources procedure] propounded

by the mathematician ZHU Shi-jie [朱世傑 1249-1314] in his treatise *Siyuan Yujian* [四元玉鑑 Precious Mirror of the Four Sources] of 1303.

In another direction the art of solving a system of linear congruence equations was developed originating from the famous Problem 26 of Book 3 of *Sunzi Suanjing* [孫子算經 Master Sun's Mathematical Manual] of the third century: "There are an unknown number of things. Counting by threes we leave two; counting by fives we leave three; counting by sevens we leave two. Find the number of things." This was further elaborated into a general method known as the *dayan qiuyi shu* [大衍求一術 great extension art of searching for unity] propounded by the mathematician Qin Jiu-shao [秦九韶 *ca*. 1202-1261] in his treatise *Shushu Jiuzhang* [數書九章 Mathematical Treatise in Nine Sections] of 1247. This significant achievement gains recognition in the naming of an important result in abstract algebra of the modern era that is called the "Chinese Remainder Theorem".

In yet another direction the art of solving indeterminate equations was developed in the text *Zhangqiujian Suanjing* [張丘建算經 Mathematical Manual of Zhang Qiu-jian] of the fifth century with the famous "problem of the hundred coins and hundred fowls" which seems to have transmitted through the Silk Road to become part of the folklore of different mathematical cultures in the Islamic world and later in the Western world. This problem was re-transmitted into China during the seventeenth to eighteenth centuries, a period of transition between traditional Chinese mathematics and modern mathematics learnt from the Westerners. This particular type of problem aroused the interest of a group of Qing mathematicians who thereby contributed to the investigation of solving indeterminate equations.

In 1613 LI Zhi-zao [李之藻 1565-1630], a scholar-official in the Ming Court, collaborated with the Italian Jesuit Matteo Ricci [利瑪竇 1552-1610] to compile the treatise *Tongwen Suanzhi* [同文算指, literally meaning "rules of arithmetic common to cultures"] based on the 1583 European text *Epitome Arithmeticae Practicae* (literally meaning "abridgement of arithmetic in practice") of Christopher Clavius (1538-1612) and the 1592 Chinese mathematical classic *Suanfa Tongzong* [算法統宗, literally meaning "unified source of computational methods"] of CHENG Da-wei [程大位 1533-1606]. Besides being an attempt of LI Zhi-zao to synthesize European mathematics with traditional Chinese mathematics this is also the first book which transmitted into China in a systematic and comprehensive way the art of written calculation that had been in common practice in Europe since the sixteenth century (Siu, 2015). With this the road was well paved for the ensuing transmission of the art of solving equations from the Western world.

3 Solving problems not by solving equations

Many problems in ancient Chinese mathematical texts involve solving equations, such as "Chasing after the visitor to return his coat" (Problem 16 in Chapter 6 of *Jiuzhang Suanshu*), "Broken bamboo"(Problem 13 in Chapter 9 of *Jiuzhang Suanshu*), "Two mice drilling a wall" (Problem 12 of Chapter 7 of *Jiuzhang Suanshu*), "Inscribed circle in a right triangle" (Problem 16 in Chapter 9 of *Jiuzhang Suanshu*), "Pheasants and rabbits in a cage" (Problem 31 in Book 3 of *Sunzi Suanjing*). For each such problem a school pupil of today will no doubt set up an equation and solve it. But in

those days methods for solving equations (other than that of solving a system of linear equations using a systematic algorithm employing counting rods on the board that is equivalent to the Gaussian Elimination Method, which is explained in detail in Chapter 8 of *Jiuzhang Suanshu*) were not yet developed either in the East or in the West, so these problems were solved by other clever means, sometimes even relying on geometric explanation.

In the same manner, similar problems come up in primary school classrooms that are amenable to mathematical reasoning without relying on the knowledge of solving equations, and they serve a pedagogical purpose. This leads one to ask, "What advantage do we gain by learning to solve equations, if we can do without it but by some other means? Why do we need to learn the topic of solving equations?" I shall get back to this point at the end of this section. To keep this paper within the preferred length, among the batch of five problems mentioned in the beginning of this section let us only look at two examples, taken from ancient Chinese mathematical texts, in more detail.

(1) In Book 3 of *Sunzi Suanjing* we find the famous problem about pheasants and rabbits: "Now there are pheasants and rabbits in the same cage. The top [of the cage] has 35 heads and the bottom has 94 legs. Find the number of pheasants and rabbits." (Lam, & Ang, 1992/2004, p. 180).

A school pupil of today would probably see it as a problem in solving a pair of simultaneous linear equations by putting r as the number of rabbits and p as the number of pheasants. If H and L are respectively the number of heads and legs, then $r + p = H$, $4r + 2p = L$. Hence, we eliminate p to obtain

$$r = (L - 2H) / 2 = L/2 - H,$$

that is, the number of rabbits r is equal to half the number of legs minus that of heads. In the text of *Sunzi Suanjing* the explanation is given exactly as what the last sentence says. Even though ancient Chinese mathematicians several centuries before were already well versed in solving a system of linear equations as explained in Chapter 8 of *Jiuzhang Suanshu*, apparently that is not how the problem was solved in *Sunzi Suanjing*.

From the text of *Sunzi Suanjing* it appears that the author was thinking along the line of tying up the legs two by two so that each pheasant has one head and one pair of legs, while each rabbit has one head and two pairs of legs. Among these $L/2$ pairs of legs we can take off H pairs, one from a pheasant and one from a rabbit, leaving behind $L/2 - H$ pairs, one for each rabbit. Thus, $r = L/2 - H$.

As a pedagogical means one can let students make up more funny and imaginative stories to explain the same formula. For instance, suppose we ask the pheasants and rabbits to each raise two legs. After doing that all the pheasants fall flat on the ground, while all the rabbits can each still stand on both legs. Count the heads (only of the rabbits) still held up, which is half of the $L - 2H$ legs standing on the ground, so $r = L/2 - H$. Or, for a more gruesome story, first cut off all legs, then distribute back to each pheasant and each rabbit a pair of legs. A pheasant can go away with a pair of legs, while a poor rabbit cannot move with only two legs. So, $L - 2H$ legs stay put, two for each rabbit, hence $r = (L - 2H) / 2 = L/2 - H$. Instead of eliminating p we can eliminate r to arrive at

$$p = (4H - L) / 2 = 2H - L/2.$$

Again, as a pedagogical means one can let students make up a story to explain this formula.

(2) Problem 13 in Chapter 9 of *Jiuzhang Suanshu* has a rich heritage, for it also appeared (with different data) in *Lilavati* written by the Indian mathematician Bhāskara II (1114-1185) in the twelfth century as well as in a European text written by the Italian mathematician Filippo Calandri in the fifteenth century. The problem asks: "Now given a bamboo 1 *zhang* high, which is broken so that its tip touches the ground 3 *chi* away from the base. Tell: What is the height of the break?" (Shen, Crossley, & Lun, 1999, p.485). In modern day mathematical language it means that we want to calculate the side *b* of a right triangle given *a* and *c* + *b*, where *c* is the hypotenuse. The answer given in *Jiuzhang Suanshu*, expressed in modern day mathematical language, is given by the formula

$$b = \frac{1}{2}\left[(c + b) - \frac{a^2}{(c + b)} \right]$$

A school pupil of today would probably arrive at this answer by invoking Pythagoras' Theorem and solving a certain equation. However, how was it done more than two thousand years ago, when the Chinese did not yet have the facility afforded by symbolic manipulation at their disposal?

Both LIU Hui [劉徽 *ca*. 225-295] and YANG Hui [楊輝 1238-1298], who were noted Chinese mathematicians more than a thousand years apart, explained in their commentaries how the answer is obtained. Their ingenious method is an excellent example to illustrate how shapes and numbers, or geometry and arithmetic/algebra, go hand in hand in traditional Chinese mathematics. By using Pythagoras' Theorem (known as *gou-gu* [勾股] in the Chinese context) we see that the gnomon formed by the larger square of side *c* minus the smaller square of side *b* has area a^2. Fold down the gnomon to form a rectangle of side *c* + *b* = *L* and *c* − *b*, and complete this rectangle to a square of side *L* (See Figure 1a, 1b, 1c). Subtracting this rectangle from the square of side *L* we obtain a rectangle of area $L^2 - a^2$, which is also equal to the product of the two sides *L* and *L* − (*c* − *b*) = 2*b*, that is, 2*bL*. Therefore, $2bL = L^2 - a^2$, thence the answer for *b* (See Figure 1d). For an animated demonstration of this process (credited to assistance from Anthony C. M. OR) readers are invited to go to a *Geogebra* applet at the link https://ggbm.at/2772025.
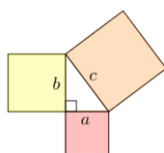

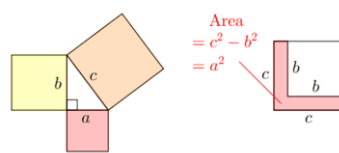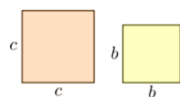Figure 1a: Broken Bamboo Problem   Figure 1b: Broken Bamboo Problem
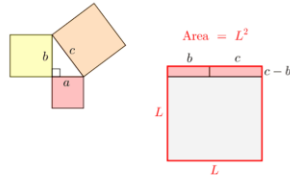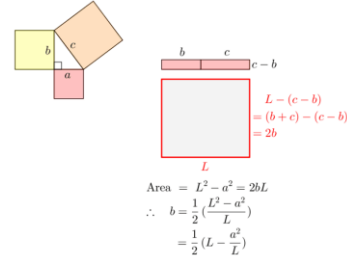
Figure 1c: Broken Bamboo Problem

Figure 1d: Broken Bamboo Problem

Both examples convey the point I wish to make, namely, it may be possible to solve a problem without setting up an equation then solve it, but it may require to some extent a clever idea. Not everybody can be a clever craftsman full of bright ideas as such, but most people can become skilful workers once they have learnt the methods of setting up and solving equations reasonably well enough. In history these methods were developed since the sixteenth century and matured during the next three centuries. Today a school pupil learns it in class and can thereby solve various problems by setting up and solving equations, be they linear or quadratic or of higher degree, while in the old days only masters could accomplish the same.

4   The *Tianyuan* procedure

In the book *Yigu Yanduan*[益古演段 Development of Pieces of Area Augmenting the Ancient Knowledge] of 1259 written by LI Ye [李冶 1192-1279] Problem 8 asks: "A square field has a circular pond in the middle. The area of land off the pond is 13 *mu* and 7 plus a half *fen*. We only know that the sum of the perimeters of the square and the circle is 300 *bu*. What are the perimeter of the square and that of the circle?" The method says: "Set *tianyuan yi* as the diameter [立天元一為圓徑]**.**" In a language familiar to a school pupil of today, this means "let x be the diameter". The solution follows (if expressed in today's language): The circumference is $3x$ (the value of π is taken to be 3), so the perimeter of the square is $300 - 3x$. Sixteen times the area of the square is equal to $9x^2 - 1800x + 90000$, while sixteen times the area of the circle is equal to $12x^2$. Hence, sixteen times the area of the square minus sixteen times the area of the circle is equal to $-3x^2 - 1800x + 90000$, which, according to the given data (knowing the conversion of units in measurement), comes out to be 52800. Thus, we have set up the equation $-3x^2 - 1800x + 37200 = 0$. Solving it, we obtain $x = 20$, and so the circumference of the circle is 60 *bu* and the perimeter of the square is 240 *bu*. To solve such an equation the Chinese mathematicians developed a method called *zengcheng kaifangfa* [增乘開方法 method of extraction of roots by successive addition and multiplication], which was the same as what is known in the Western world as Horner's method, devised by William George Horner (1788-1837) in 1819.

This method of setting up and solving a polynomial equation was a significant achievement in Chinese mathematics of the thirteenth century. As early as in the Qin-Han period (third century BCE to second century CE) the famous mathematical classics *Jiuzhang Suanshu* explained the method called *daicong kaifangshu* [帶從開方術 procedure of extraction of square root with accompanying number], which, as an extension of *kaifangshu*, calculated a root of a quadratic equation to any degree of

accuracy. This method was extended by later mathematicians of the twelfth/thirteenth centuries during the Song-Yuan period to solve a polynomial equation of higher degree, known by the name of *tianyuanshu* [天元術 celestial source procedure]. For instance, in the mathematical classics *Shushu Jiuzhang of* 1247 QIN Jiu-shao treated an equation of degree ten in Problem 2 (measuring a circular castle from the distance) of Chapter 8 as an illustration on how one could solve a polynomial equation of any degree by finding a root to any degree of accuracy (Siu,1995; Siu, 2009).

Ironically, this high-point of Chinese mathematics was also the beginning of its stagnation! One reason is that there was no need at the time to solve an equation of such high degree in practice. When the technical capability far exceeded the demand imposed by practical matters, motivation otherwise arising from an inner curiosity would not attract those with mainly a pragmatic attitude. For instance, the pragmatic attitude would not induce Chinese mathematicians in those days to think about existence of a root of an equation, not to mention solvability by radical, because they were sufficiently satisfied with an algorithm that could obtain a root to any degree of accuracy in terms of decimal places (Siu, 1995; Siu, 2009). One mathematician in the early Qing period, namely, WANG Lai [汪萊 1768-1813], was a rare exception. He was not satisfied with the work of mathematicians in the Song-Yuan period and wished to surpass it by posing questions such as deciding whether a quadratic equation has a real root or an imaginary root, thus touching on the theory of equations. However, he was so much ahead of his time that he was regarded at the time as a "maverick"!

Therefore, this *tianyuan* procedure was unfortunately almost lost, at best vaguely known but not well understood, after the thirteenth century owing to the turbulent era which lasted until the Ming Dynasty was established by the mid-fourteenth century. In the next section we will see the *tianyuan* procedure mentioned by Chinese mathematicians of the eighteenth century but in a different light.

5 Emperor Kangxi's study on solving equations

The story starts with the first period of transmission of European learning into China during the reign of Emperor Kangxi [康熙帝 1654-1722], which spanned the 1660s and 1670s. For a much more detailed account and discussion, interested readers are invited to read (Jami, 2011).

Following the practice of the Ming Court the Qing Court employed the service of foreign missionaries as official astronomers to make an accurate calendar, which was held by many as a major tool of legitimization of the Imperial rule in traditional Chinese thinking. The German Jesuit Johann Adam Schall von Bell [湯若望 1591-1666] convinced Emperor Shunzhi [順治帝 1638-1661] to adopt a new calendar, which was actually the fruition of a huge programme accomplished by XU Guang-qi [徐光啟 1562-1633] and his team in the Ming Dynasty but could not be implemented because of the collapse of the Ming Dynasty in 1644. When Emperor Kangxi succeeded to the throne in 1661 as a boy not yet seven-year-old, Schall lost the support of the late emperor and was accused of treason by the conservative ministers backed up by the powerful regents of the boy-emperor. Along with some other Chinese Catholic converts he was sentenced to death and would have met his tragic ending were it not for a strong earthquake that shook the capital for several days in

1665. People took this to be a warning from heaven that told them it was wrong to accuse Schall of treason. Furthermore, the Grand Empress Dowager (grandmother of Emperor Kangxi) intervened so that Schall was spared the death sentence but was expelled to Macao instead, where he died there soon afterwards in 1666. This event, known as the "calendar case", was rehabilitated in 1669 at the initiation of another foreign missionary, the Belgian Jesuit Ferdinand Verbiest [南懷仁 1623-1688].

Verbiest, who spent two years at the famous University of Coimbra before he was sent to China for missionary work, succeeded Adam Schall von Bell as the Head of the Imperial Astronomical Bureau, serving from 1669 to 1688. The way Verbiest managed to turn the tide was to challenge his opponent, the conservative minister YANG Guang-xian [楊光先 1597-1669], to compete in measuring the shadow of the sun on one December day of 1668. This event left a deep impression on the young Kangxi, as he later recounted the story:

> You all know that I am good at mathematics but do not know the
> reason why I study it. When I was very young there were frequent
> disputes between Han officials and Westerners in the Imperial
> Astronomical Bureau. They accused each other so badly that
> somebody might get beheaded! Yang Guangxian and Tang
> Ruowang [Adam Schall von Bell — apparently Kangxi's
> memory did not serve him well here!] competed in predicting
> the sun shadow at the Wu Gate in the presence of the nine chief
> ministers. However, none of them knew what the astronomers
> were doing. In my opinion, how can a person who lacks
> knowledge judge who is right or wrong? Hence I determined
> to study with all my might and main the subject of mathematics.
> Now that the methods were compiled and explained clearly in
> books, learners find it easy. Heaven knows how difficult it
> was for me to learn it in those days! (Wang, 1978, p.69.)

Verbiest was very much respected as the teacher of Emperor Kangxi in mathematics, astronomy and science. Realizing that he was himself getting old, Verbiest wrote and asked the Society of Jesus to send more younger missionaries learned in mathematics and astronomy to China. His call was answered by King Louis XIV of France (1638-1715), who sent a group of French Jesuits to China in 1685. For a political reason of not causing problems with the Portuguese authority this group of French Jesuits was sent under a sort of disguise as the "*King's Mathematicians*". By the time the five French Jesuits settled in the Imperial Court in 1688, Verbiest had passed away, so Emperor Kangxi continued to learn assiduously Western mathematics and astronomy from these "*King's Mathematicians*" in the second period.

Two of the French Jesuits, Joachim Bouvet [白晉 1656-1730] and Jean-François Gerbillon [張誠 1654-1707], left us with their diaries which gave a detailed account of their days spent in the Imperial Court. Bouvet wrote in his diary, published in France in 1697 and soon translated into English in 1699 [*Portrait histoirique de*

l'empereur de la Chine: presentée au Roy (The history of Cang-hy, the present emperour of China: presented[sic] to the most Christian King)] (Bouvet, 1697):

His natural genius is such as can be parallel'd but by few, being
endow'd with a quick and piercing Wit, a vast memory,
and great Understanding; His constancy is never to be shaken
by any sinister Event, which makes him the fittest Person in the
World, not only to undertake, but also to accomplish Great
designs.[…] But, what may seem most surprising, is, that so
great a Monarch, who bears upon his shoulders the weight
of so vast an Empire, should apply himself with a great deal
of Assiduity to, and have a true relish of all Sorts of useful Arts
and Sciences.
[…] so there is not any Science in Europe that ever came to his
Knowledge, but he showed a great Inclination to be instructed
in it. The first Occasion which had a more than ordinary
Influence upon his Mind, happened (as he was pleased to tell us
himself) upon a Difference arisen betwixt Yang quansien [Yang
Guang-xian], the Famous Author of the last Persecution in
China, and father Ferdinand Verbiest, of the Society of Iseus.
[…] As this Tryal of Skill in the Mathematiks was the first
Occasion that introduced the Father Missionaries into the
Emperor's acquaintance; so from that time, he always shew'd
a great inclination to be instructed in the Mathematical
Sciences, which in effect, are in great Esteem among the Chineses.
During the space of two Years, Father Verbiest instructed him
in the Usefulness of the best of the Mathematical Instruments, and
in what else was most Curious in Geometry, the Statique, and
Astronomy; for which purpose he wrote several Treatises.[…]
He did the Honour to us four Iesuits, Missionaries then at Peking,
to receive our Instructions, sometimes in the Chinese, sometimes
in the Tartarian Language; […] Much about the same time,
Father Anthony Thomas, did give him further Instruction concerning
the Use of the best Mathematical Instruments, in the Chinese
language, and the Practical part of Geometry and Arithmatik, the
principles of which he had formerly been taught by Father Verbiest.
He would also have us explain him the Elements of Euclid in the
Tartarian Language, being desirous to be well instructed in them,
as looking upon them to be the Foundation, upon which to build
the rest.
After he was sufficiently instructed in the Elements of geometry,
he ordered us to compile a whole System of both the Theorick
and Practick of Geometry, in the Tartarian Language, which we
afterwards explain'd to him in the same manner as we had done
with the Elements of Euclid. At the same time, Father Thomas
made a Collection of all the Calculations of geometry and

Arithmaticks (in the Chinese language) containing most of the
Curious Problems extant, both in the European and Chinese
Books, that treat this matter. He was so much delighted in
the pursuit of these Sciences, that besides betwixt two and three
Hours, which were set aside every day on purpose to be spent
in our Company, he bestowed most of his leisure time, both in
the day and at night in his Studies. (Bouvet, 1699, p.2, 3, 47,
50, 51, 52, 55.)

The third period of transmission began with the establishment of the Office of Mathematics in *Mengyangzhai* [蒙養齋 Studio for the Cultivation of the Youth], a place situated in the garden inside the Imperial Palace. Prince Yinzhi [胤祉 1677-1732], third son of Emperor Kangxi, was made the Head of Office of Mathematics. Besides serving as a school for taking lessons in mathematics, astronomy and science, the main task of this Office was to compile the monumental treatise *Lüli Yuanyuan* [律曆淵源 Origin of Mathematical Harmonics and Astronomy], comprising three parts: *Lixiang kaocheng* [曆象考成 Compendium of Observational Computational Astronomy] in forty-two volumes, *Shuli Jingyun* [數理精蘊 Collected Basic Principles of Mathematics] in fifty-three volumes, and *Lülü Zhengyi* [律呂正義 Exact Meaning of Pitchpipes] in five volumes. The compilers were all Chinese officials and scholars, but the book mentioned the contribution of foreign missionaries at the beginning of *Shuli Jingyun*. The treatise *Shuli Jingyun* includes both traditional Chinese mathematics, the part that was still in extant and was understood at the time, as well as Western mathematics, highly likely from the "lecture notes" prepared by the missionaries for Emperor Kangxi during his first period of ardent study in the 1670s.

6 Method of Borrowed Root and Powers

Another teacher who taught Emperor Kangxi the art of solving equations was the Belgian Jesuit Antoine Thomas [安多 1644-1709], who studied at University of Coimbra from 1678 to 1680 and was ordered to go to Peking in 1685. By the time Thomas came to China he had compiled *Synopsis mathematica*, which was based on the book *De numerosa potestatum ad exegesim resolution* (On the numerical resolution of powers by exegetics) written by François Viète (1540-1603) in 1600. Thomas later revised it as *Suanfa Zuanyao Zonggang* [算法纂要總綱 Outline of the Essential Calculations] and *Jiegenfang Suanfa* [借根方算法 Method of Borrowed Root and Powers], to be used as lecture notes for the mathematics lessons in the Imperial Court and later incorporated into Books 31-36 of *Shuli Jingyun*.

For the purpose of illustration let us look at an example in the book of Viète, namely, to solve the equation "x squared plus A times x equals B". Let x_1 be a first approximation of a root, which is $x_1 + x_2$. Substitute into the equation and neglect the comparatively much smaller term x_2 squared. We obtain x_2 in terms of x_1, A and B. So we have a better approximation $x_1 + x_2$. Keep reiterating the process to obtain a better and better approximation. Let us look at a similar problem in Book 33 of *Shuli Jingyun*: "If the cube [of root] and eight roots are equal to 1824 *che*, how much is one root?" In modern day mathematical language we want to solve $x^3 + 8x = 1824$, which

is accomplished by a method called "extraction of cube root with accompanying number". The basic idea is the same as that in the previous example. If the answer is not exact, the process will give a better and better approximation to any number of decimal places. Emperor Kangxi not only studied the method in earnest but even did homework assignments which can still be read in the archive today!

The Chinese mathematician MEI Jue-cheng [梅穀成 1681-1763] told the story on how he learnt this new method in his book *Chishui Yizhen* [赤水遺珍 Pearls Remaining in the Red River] of 1761:

> While I was serving in the Imperial Court of the late Emperor
> canonized as Shengzu Humane [Kangxi], I was instructed
> on the method of *jiegenfang* [借根方 borrowed root and powers]
> by the late Emperor. He issued an edict to say that the Westerners
> called the book *aerrebala* [阿爾熱巴拉 algebra] that means
> "Method from the East". I respectfully learnt the method, which is
> really marvellous and is the guide to mathematics.
> I suspected that the method resembles that of *tianyuan* [天元
> celestial source], so I took up the book *Shoushi Licao* [授時曆草
> Calculation Draft of the *Shoushi* Calendar] and studied it, thereby
> clarifying the matter. Though the terminologies are different the
> two methods are the same, not just a mere resemblance.
> Scholars in the Yuan period wrote on calendar reckoning using
> this method, which became a lost art for some unknown reason.
> It was fortunate that those [foreigners] who resided afar admired
> our culture and sent it back so that we retrieved it. From their
> naming it "Method from the East" we see that they did not forget
> from whom they learnt this method. (Mei, 1761/1877, p.7; Han,
> 2010, p.473.)

This is an indication of how the slogan of the time, "Western learning has its origin in Chinese learning", got promulgated. In making his subjects believe that Western learning originated in older Chinese learning, Emperor Kangxi knew that the Chinese would be more than willing to learn it and would not regard it as something opposing traditional value. Or, maybe he really thought that the method originated in older Chinese learning. Indeed, similar methods were explained in mathematical classics of earlier days, most of which became less known by the Ming and early Qing period. However, it is grossly wrong to say that *aerrebala* (algebra) means "Method from the East". For the Europeans the method was transmitted to them from the Islamic world — indeed "east" to them, but not from China! The word "algebra" itself does not connote anything about "Method from the East" but comes from the Arabic word *al-jabr*, meaning "restoration", which is to be understood together with another Arabic word *al-muqābala*, meaning "reduction". These two words appear in the title of a famous Arabic treatise of the ninth century from the pen of Muhammad ibn Mūsā al-Khwārizmī (*ca*. 780-850). The two words together describe the method in solving an algebraic equation by transposing terms and simplifying the expression, something a school pupil of today would be quite familiar with.

Catherine Jami once made a very pertinent remark:

> […] the cross-cultural transmission of scientific learning cannot
> be read in a single way, as the transmission of immutable objects
> between two monolithic cultural entities. Quite the contrary: the
> stakes in this transmission, and the continuous reshaping of what
> was transmitted, can be brought to light only by situating the
> actors within the society in which they lived, by retrieving their
> motivations, strategies, and rationales within this context.
> (Jami, 1999, p.430.)

In the process of transmission, sometimes between several different cultures thousands of miles apart in a span of hundreds of years, the reshaping would be intricate and hard to trace. Nevertheless, what finally evolved is the result of collective wisdom that became part of human heritage for which it is actually quite unnecessary to ascertain who the originator was. I subscribe to what Joseph Needham says in the preface of his monumental treatise:

> The citizen of the world has to live with his fellow-citizens […].
> He can only give them the understanding and appreciation which
> they deserve if he knows the achievements of the sages and
> precursors of their culture as well as of his own. […] Certain it is
> that no people or group of peoples has had a monopoly in contributing
> to the development of Science. Their achievements should be mutually
> recognized and freely celebrated with the joined hands of universal
> brotherhood." (Needham, 1954, p.9.)

7 New Method of *aerrebala*

Viète wrote another book in 1591 with more important influence, namely, *In Artem Analyticem Isagoge* [Introduction to the Analytical Art]. In his book he introduced what he called "*logistica numerosa*" and "*logistica speciosa*", that is, numerical calculation and symbolic calculation. Viète was so pleased with his idea that he concluded his book by the exclamation: "*Quod est, nullum non problema solver* [There is no problem that cannot be solved]"! It led to subsequent work of René Descartes (1596-1650), *La géométrie* [Geometry] of 1637, and that of Isaac Newton (1642-1726), *Arithmetica Universalis* [Universal Arithmetic] of 1707 (with the work actually done about forty years earlier), by which time mathematicians in Europe were familiar with the use of symbolic calculation. Again, a school pupil of today would be quite familiar with that too, but when Viète first introduced it in his book, it was a very novel idea. Let us see how Emperor Kangxi reacted to it when another French Jesuit, Jean-François Foucquet [傅聖澤 1665-1741] taught him this new method, which Foucquet called the "new method of *aerrebala*".

In an Imperial Edict issued by Kangxi Emperor between 1712 and 1713 he said:

Every day soon after getting up I study with the Princes the method of *aerrebala* [algebra] and find it most difficult. He [J.-F. Foucquet] says that it is easier than the old method, but it looks more difficult than the old method and has more errors as well as many awkward features. […] Copy this Imperial Edict and issue the book in the capital to the Westerners for them to study it in details, and to delete those parts that do not make sense. It says something like *Jia* multiplies *Jia*, and *Yi* multiplies *Yi*, without any concrete number appearing. One never knows what the result of the multiplication is. It seems that this man [J.-F. Foucquet] is only mediocre in mathematical skill! (Zhongguo diyi lishi dangan guan, 2003, p.52; Han, 2010, p.468.)

In defence Foucquet said in his book *Aerrebala Xinfa* [阿爾熱巴拉新法 New Method of *Aerrebala*], the only existing copy now kept at the Biblioteca Apostolica Vaticana as document *Borgia Cinese* 319:

The old method uses numerical values, while the new method uses symbols that are accommodating [通融記號 *tongrong jihao*] […] Using this accommodating notation, it is easy to perform calculation, and it enables one to see the situation clearly so that one can focus on the method and understand the underlying rationale of the calculation. The use of numerical value works only for a particular value, while the use of accommodating notation encompasses all values in general. (Han, 1991, p.24-25; Han, 2010, p.467.)

The frustration expressed by Emperor Kangxi reminds us of the joke in the classroom in which a teacher announces routinely, "Let x be the age of the boy, then…" only to receive a matter-of-fact protest from a pupil, "What happens if it is not?" The explanation given by Foucquet reminds us of an important underlying message, which is however seldom made apparent to school pupils, namely, that we treat numerical quantities as general objects, and manipulate such general objects as if they were numerical quantities. Although we do not know (prior to solving the equation) what they are, we know that they stand for certain numbers and, as such, obey general rules. For instance we do not know what A and B are, but we know that $A \times B = B \times A$. We can therefore apply these general rules systematically to solve problems which can be formulated in terms of equations. This is in fact the underlying essence of modern day abstract algebra in which the study and vista broaden from the familiar number systems to various algebraic structures of interest.

For a teacher this phenomenon should not be unfamiliar. Many students who face some difficulty in learning a new subject, instead of putting in more effort, choose the easier way out by blaming the teacher for poor teaching and telling everybody that the teacher is incapable! In some sense, the episode between Emperor Kangxi and Father

Foucquet was like that! It is important for a teacher to try to convince the pupil why symbolic manipulation is needed, why the effort to understand it is worthwhile, and to explain the meaning behind those symbols.

Of the several methods of solving algebraic equations, the "new method" gradually replaced the other methods by the eighteenth century and was further developed. However, because of this personal dislike of the subject by Emperor Kangxi, probably resulting from reluctance to admit his own inadequacy in learning it coupled with arrogance, the transmission into China of the powerful method of symbolic calculation was delayed for nearly one-and-half century! China had to wait until in 1859 LI Shan-lan [李善蘭 1811-1882] and the British missionary Alexander Wylie [偉烈亞力 1815-1887] collaborated to translate the treatise *Elements of Algebra* written by Augustus De Morgan (1806-1871) in 1835, which was given the title *Daishuxue* [代數學 The Study of *Daishu*] (Chan, Siu, 2012). Thirteen years later HUA Heng-fang [華蘅芳 1833-1902] and another British missionary John Fryer [傅蘭雅 1839-1928] translated the book *Algebra* William Wallace (1768-1843) wrote for the *Encyclopaedia Britannica* between 1801 and 1810, which was given the title *Daishushu* [代數術 The Method of *Daishu*], in 1872. In both translated text, the term *daishu* literally means "the study of numbers represented by a character", which is no doubt motivated by what De Morgan wrote in his book:

> A letter denotes a number, which may be, according to
> circumstances, as will hereafter appear, either any number we
> please; or some particular number which is not known, and
> which, therefore, has a sign to represent it till it is known.
> (Demorgan, 1835/1837, p. vi.)

8 Conclusion

When we were in school we all learnt to solve equations. Did it ever occur to you at the time how miraculous the phrase "let x be …" is? Without knowing what this magic x stands for *a priori*, somehow at the end after certain symbolic manipulations the value for x falls out as an answer! Or, at the other extreme, did you find the routine working of solving an equation step by step boring at the time ? Now that we have become teachers ourselves do we realize the difficulty most students have in understanding what the phrase "let x be …" means ? Indeed, what is this x ? Is it an unknown (in an equation) ? Is it a variable (in a function) ? Or, is it an indeterminate (in a polynomial) ?

Although the Chinese had built up a rather refined kind of machinery in solving different types of equations from the ancient to medieval times, what is learnt in school today basically follows what was developed in the Islamic world since the eighth century and subsequently in Europe from the sixteenth century to the nineteenth century. This was transmitted into China in the Ming Dynasty and the Qing Dynasty. In some sense, a school pupil of today, when first encountered the topic of solving equations, might feel the same as what a Chinese in those days first encountered Western ways of solving equations. In this paper we attempted to look at

the pedagogical issue from this viewpoint and hope that learners can benefit from the discussion.

REFERENCES

Bouvet, J. (1697/1699). *Portrait histoirique de l'empereur de la Chine*: *presentée au Roy*. Paris: Etienne Michallet.

Bouvet, J. (1699) The history of Cang-Hy, the present emperor of China presented to the most Christian king (English translation of Bouvet's *Portrait histoirique de l'empereur de la Chine*: *presentée au Roy*). London: F. Coggan.

Chan, Y. C. [陳葉祥], & Siu, M. K. [蕭文強] (2012). On Alexander Wylie's *Jottings on the Science of the Chinese Arithmetic*. In *Proceedings of HPM-2012, Daejeon* (pp. 229-237). Daejeon:HPM-2012.

Chemla, K., & Guo, S. C. [郭書春] (2004). *Les Neuf Chapitres: Le classique mathématique de la Chine ancienne et ses commentaires*. Paris: Dunod.

De Morgan, A. (1835/1837). *Elements of algebra*. London: Taylor and Walton.

Guo, S. C. [郭書春] (Ed.) (1993). *Zhongguo kexue jishu dianji tonghui (shuxue juan)* [中國科學技術典籍通彙(數學卷) Collection of Chinese classics in science and technology (Mathematics), Volumes 1-5]. Zhengzhou: Henan Educational Press.

Han, Q.[韓琦] (1991) Kangxi shidai chuanru de xifang shuxeu ji qi dui zhongguo shuxue de yingxiang [康熙時代傳入的西方數學及其對中國數學的影響 Transmission of Western mathematics in the reign of Kangxi and its influence on Chinese mathematics], PhD dissertation. Beijing: Institute of the History of Natural Sciences.

Han, Q.[韓琦] (2010) Xifang shuxue de chuanru he qianjia shiqi gusuan de fuxing — yi *jiegenfang* de chuanru he *tianyuanshu* yanjiu de guanxi wei li [西方数学的传入和乾嘉时期古算的复兴 — 以借根方的传入和天元术研究的关系为例 Transmission of Western mathematics and the revival of traditional mathematics in the reign of Qian Long and Jia Qing — relationship between the transmission of *jiegenfang* and the study of *tianyuanshu* as a case study]. In P. Y. Chu [祝平一] (Ed.) *Zhongguoshi xinlun keji yu zhongguo shehui fence* [中國史新論：科技與中國社會分冊 New perspective on Chinese history : volume on science and technology and Chinese society] (pp.459-486). Taipei: Academia Sinica/Linking Publishing.

Hoe, J. (1977). *Les systèmes d'équations polynômes dans le Siyuan Yujian (1303)*. Paris: Collège de France, Institut des hautes études chinoises.

Jami, C. (1999). "European science in China" or "Western learning"? Representations of cross-cultural transmission, 1600-1800, *Science in Context*, *12* (3), 413-434.

Jami, C. (2011). *The Emperor's new mathematics: Western learning and imperial authority during the Kangxi reign (1662-1722)*. Oxford: Oxford University Press.

Lam, L. Y [藍麗蓉] (1977). *A critical study of the Yang Hui Suan Fa: A thirteenth-century Chinese mathematical treatise*. Singapore: University of Singapore Press.

Lam, L.Y. [藍麗蓉], & Ang, T.S. [洪天賜] (1992/2004). *Fleeting footsteps: Tracing the conception of arithmetic and algebra in ancient China*. Singapore: World Scientific.

Libbrecht, U. (1973). *Chinese mathematics in the thirteenth century: The Shu-shu Chiu-chang of Ch'in Chiu-shao*. Cambridge: MIT Press.

Mei, J.C. [梅穀成] (1761/1877) *Chishui yizhen* [赤水遺珍 Pearls remaining in the red river].Suzhou: Jiangsu Publishing House.

Needham, J. (1954) *Science and civilization in China, Volume* 1. Cambridge: Cambridge University Press.

Shen, K. S.[沈康身], Crossley, J. N., & Lun, A. W. C. [倫華祥] (1999). *The nine chapters on the mathematical art: Companion and commentar*y. Oxford: Oxford University Press.

Siu, M. K. [蕭文強] (1995). Mathematics education in ancient China: What lesson do we learn from it ?, *Historia Scientiarum*, *4*(3), 223-232.

Siu, M. K. [蕭文強] (2009). Mathematics education in East Asia from antiquity to modern times, In K. Bjarnadottir, F. Furinghetti, & G. Schubring (Eds.), *Dig where you stand: Proceedings of a conference on on-going research in the history of mathematics education, Garðabær, June 20-24, 2009* (pp. 197-208). Reykjavik: University of Iceland.

Siu, M. K. [蕭文強] (2015). *Tongwen Suanzhi* (同文算指) and transmission of *bisuan* (筆算 written calculation) in China: from an HPM (History and Pedagogy of Mathematics) viewpoint, *Journal for History of Mathematics*, *28*(6), 311-320.

Siu, M. K. [蕭文強] (2016). The confluence of the Yellow River and the Mediterranean: Synthesis of European mathematics and Chinese mathematics during the seventeenth, eighteenth and nineteenth centuries. In L. Radford, F. Furinghetti, & T. Hausberger (Eds.), *Proceedings of the 2016 ICME Satellite Meeting* (pp. 581-592). Montpellier: IREM de Montpellier.

Swetz, F. (2012). *Mathematical expeditions: Exploring word problems across the ages*. Baltimore: Johns Hopkins University Press.

Wang, Y. W. [王雲五] (Ed.) (1978) *Shiku quanshu zhenben baji* [四庫全書珍本八集 Complete Library in Four Sections:eighth collection in 400 volumes]. Taipei: Commercial Press.

Wu,W.J. [吳文俊] *et al* (Eds.). (1998-2004). *Zhongguo shuxueshi daxi* [中國數學史大系 A Compendium of History of Chinese Mathematics], Books 1-8, Supplementary Books 1-2. Beijing: Beijing Normal University Press.

Zhongguo diyi lishi dangan guan [中國第一歷史檔案館 First office on historical documents of China].(Ed.) (2003). *Qing zhongqianqi xiyang tianzhujiao zai hua huodong dangan* [清中前期西洋天主教在華活動檔案 Archives concerning Western Catholic missions from early to mid Qing dynasty in China]. Beijing: Zhonghua Publishing.