An Excursion in Ancient Chinese Mathematics

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Nobody can hope to do justice, in one short article, to the two millennia of indigenous mathematical development in China up to the end of the 16th century.[1] This article attempts only to convey a general flavour of ancient Chinese mathematics and illustrate some of its characteristic features through a few examples. An annotated bibliography is provided at the end for the reader's convenience.[2]

Characteristic Features of Ancient Chinese Mathematics

The characteristic features of ancient Chinese mathematics can best be appreciated by looking at the work of the ancient Chinese mathematicians. Evidenced in their choice of topics is a strong social relevance and pragmatic orientation, and in their methods a primary emphasis on calculation and algorithms. However, contrary to the impression most people may have, ancient Chinese mathematics is not just a "cook-book" of applications of mathematics to mundane transactions. It is structured, though not in the Greek sense exemplified by Euclid's *Elements*. It includes explanations and proofs, though not in the Greek tradition of deductive logic. It contains theories which far exceed the necessity for mundane transactions.

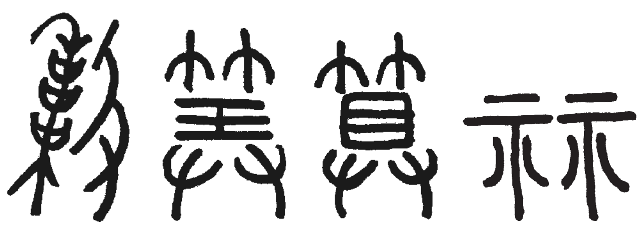


Figure 1

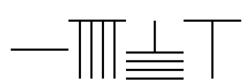
We start with some ideograms (characters) related to mathematics. In ancient classics the term mathematics (數學) was often written as "the art of calculation" (算術) or "the study of calculation" (算學) indicating a deep-rooted basis in calculation. The ideogram for "number" and "to count" (數) appeared on oracle bones about 3000 years ago, in the form of a hand tying knots on a string (see Fig. 1a). The ideogram for "to calculate" (算) appeared in three forms, according to *Shuowen Jiezi* (*Analytic Dictionary of Characters*) by Xu Shen (AD 2nd century). The first is a noun, composed of two parts, "bamboo" on top and "to manipulate" in the bottom, with the bottom part itself in the form of two hands plus some (bamboo) sticks laid down on a board, some placed in a horizontal position and some placed in a vertical position (see Fig. 1b). The second is a verb, also written with the parts of bamboo and hands (see Fig. 1c). The third, somewhat more puzzling, is in the form of a pair of ideogramatic parts pertaining to religious matters (see Fig. 1d). It is a tantalizing thought that the subject of mathematics in ancient China was not exactly the same subject as we understand it today. Indeed, in some ancient mathematical classics we find mention of “internal mathematics” and “external mathematics”, the former being intimately tied up with *Yijing* (*Book of Changes*), the oldest written classic in China.[3]

Besides its appearance in these ideograms, the theme of calculation permeated the whole of ancient Chinese mathematics. This is best illustrated by the calculating device of the counting rods. Ample evidence confirms the common usage of counting rods as early as in the fifth century BCE, and these probably developed from sticks used for fortune-telling in even earlier days. The earliest relics from archaeological findings are dated to the second century BCE These were made of bamboo, wood and even metal, bone or ivory and were carried in a bag hung at the waist. The prescribed length in the literature (verified by the relics) was from 13.86 cm to 8.5 cm, which shortened as time went on. The cross-section changed with time, from circular (of 0.23 cm in diameter) to square so that the rods became harder to roll about. One mathematically extremely interesting feature is the occurrence of a red dot on a counting rod to denote a positive number, and a black dot to denote a negative number. These counting rods were placed on a board (or any flat surface) and moved about in performing various calculations.

The Chinese adopted very early in history a denary positional number system. This was already apparent in the numerals inscribed on oracle bones in the Shang Dynasty (c. 1500 BCE), and was definitely marked in the calculation using counting rods in which the positions of the rods were crucial. Ten symbols sufficed to represent all numbers when they were put in the correct positions. At first only nine symbols were used for the numerals 1 to 9, with the zero represented by an empty space, later by a square in printing, gradually changed to a circle, perhaps when the square was written by a pen-brush. To minimize error in reading a number, numerals were written alternatively in vertical form (for units, hundreds,...) and horizontal form (for tens, thousands,...). In a much later mathematical classic, *Xiahou Yang Suanjing* (*Mathematical Manual of Xiahou Yang*) of the fifth century, this method for writing counting rod numerals was recorded as:

> Units stand vertical, tens are horizontal, hundreds stand, thousands lie down. Thousands and tens look the same, ten thousands and hundreds look alike. Once bigger than six, five is on top; six does not accumulate, five does not stand alone.

For instance, 1996 would have been written as



Calculation using counting rods has several weak points: (1) The calculation may take up a large amount of space. (2) Disruption during the calculation causing disarray in the counting rods can be disastrous. (3) The calculating procedure is not recorded step by step so that intermediate calculations are lost. Counting rods evolved into the abacus in the twelfth-thirteenth centuries, and by the fifteenth century the abacus took the place of counting rods. The weak points (1) and (2) were removed by the use of the abacus, but (3) remained, until the European method of calculation using pen and paper was transmitted in the beginning of the seventeenth century. However, calculation using counting rods had its strong points. Not only did the positions of the counting rods display numerals conveniently, but also the positions in which these rods were placed on the board afforded a means to allow some implicit use of symbolic manipulation, giving rise to successful treatment of ratio and proportion, fractions, decimal fractions, very large or very small numbers, equations, and so on. Indeed, the use of counting rods was instrumental in the whole development of algorithmic mathematics in ancient China.

Even a casual reading of a few mathematical classics will disclose the unmistakable features of social relevance and pragmatic orientation. From the very beginning mathematical development was intimately related to studies of astronomical measurement and calendrical reckoning. The first written text containing serious mathematics, *Zhoubi Suanjing* (*Zhou Gnomon Classic of Calculation*) compiled at about 100 BCE—with its content dated to earlier times, was basically a text in astronomical study. In an ancient society based on agriculture, calendrical reckoning was always a major function of the government. Along with that, mathematics was performed mainly for bureaucratic needs. A sixth century mathematics classic actually carried the title *Wucao Suanjing* (*Mathematical Manual of the Five Government Departments*). The titles of the nine chapters of the most important mathematical classic *Jiuzhang Suanshu* (*Nine Chapters on the Mathematical Art*), which is believed to have been compiled some time between 100 BCE and 100 CE, speak for themselves. These are (1) survey of land, (2) millet and rice (percentage and proportion), (3) distribution by progression, (4) diminishing breadth (square root), (5) consultation on engineering works (volume of solid figures), (6) impartial taxation (allegation), (7) excess and deficiency (Chinese "Rule of Double False Positions"), (8) calculating by tabulation (simultaneous equations), (9) gou-gu (right triangles). The social relevance of the content of mathematical classics was so plentiful that historians have found in the texts a valuable source for tracing the economy, political system, social habits, and legal regulations of the time! The emphasis on social relevance and pragmatic orientation, in line with a basic tenet of traditional Chinese philosophy of life shared by the class of "shi" (intellectuals), viz. self-improvement and social interaction, was also exhibited in the education system in which training in mathematics at official schools was intended for government officials and clerks.[4]

Finally let us come to the issue of mathematical proofs. "If one means by a proof a deductive demonstration of a statement based on clearly formulated definitions and postulates, then it is true that one finds no proof in ancient Chinese mathematics, nor for that matter in other ancient oriental mathematical cultures.... But if one means by a proof any explanatory note which serves to convince and to enlighten, then one finds an abundance of proofs in ancient mathematical texts other than those of the Greeks."[5] The Chinese offered proofs through pictures, analogies, generic examples, and algorithmic calculations. These can be of pedagogical value to complement and supplement the teaching of mathematics with traditional emphasis on deductive logical thinking.

Jiuzhang Suanshu

Jiuzhang Suanshu is the most important of all mathematical classics in China. It is a collection of 246 mathematical problems grouped into nine chapters. There is good reason to believe that the content of *Jiuzhang Suanshu* was much older than its date of compilation, as substantiated by an exciting archaeological finding in 1983 when a book written on bamboo strips bearing the title *Suanshu Shu* (*Book on the Mathematical Art*) was excavated[6]. It is dated at around 200 BC and its content exhibits a marked resemblance to that of *Jiuzhang Suanshu*, including even some identical numerical data which appeared in the problems. The format of *Jiuzhang Suanshu* became a prototype for all Chinese mathematical classics in the subsequent one-and-a-half millennia. A few problems of the same category were given, along with answers, after which a general method (algorithm) followed. In the very early edition that was all and no further explanation was supplied—perhaps it was to be supplied by the teacher. Later editions were appended with commentaries which explained the methods, corrected mistakes handed down from the ancients, or expanded the original text. The most notable commentator of *Jiuzhang Suanshu* was Liu Hui (c. third century), some of whose works will be examined in the next section.

The format of *Jiuzhang Suanshu* may lead one to regard the book as a medley of recipes for solving problems of specific types. Indeed many who studied from the book in accordance with the official system in ancient China might have actually regarded the book as such and thus resorted to rote learning just like in recitation of other classics. This may explain why only a handful of mathematicians of some standing were produced from the tens of thousands of "mathocrats" who went through mathematical training in the official system during two millennia, while most noted mathematicians in history were either self-educated or studied at private academies[7].

However, upon closer scrutiny, the text reveals itself as quite different from a book of recipes. The body of knowledge contained in a classic such as *Jiuzhang Suanshu* is structured around several themes, the two main themes being the concept of "lu" (率, ratio) in arithmetic and the concept of "gou-gu" (勾股) right triangle) in geometry. A brief description on how ratio forms a backbone for most chapters will now be given, while right triangles will be left to the next section. In the commentary of Chapter 1, Liu Hui gave a definition: "a ratio is a relation between numbers."[8] He continued to offer a working definition of ratio by representing it as a reduced fraction. To reduce a fraction the rule of "reciprocal subtraction," known to Westerners as the Euclidean algorithm, was introduced.

If both numerators and denominators are divisible by 2, then halve them both.
If they are not both divisible by 2, then set up the numbers for numerator and

> denominator respectively continually and alternately subtracting the smaller from the larger, and seek their equality.

This is a good illustration of how the calculation itself is already a proof (or convincing argument), as can be seen from Problem 6 of Chapter 1:

> Reduce the fraction $\frac{49}{91}$.

$$\begin{aligned}(49,91) &\to (49,42) \to (7,42) \to (7,35)\\ &\to (7,28) \to (7,21) \to (7,14) \to (7,7).\end{aligned}$$

Hence $49 = 7 \times 7$, $91 = 7 \times 13$, and $\frac{49}{91} = \frac{7}{13}$. At the beginning of Chapter 2 Liu Hui explained the so-called "Rule of Three" (also found in contemporary Indian manuscripts), which enables one to apply the concept of ratio to a number of situations, including distribution in direct proportions or in inverse proportions (Chapters 3, 6), formulation and treatment of problems in excess and deficiency, i.e., the method of "double false positions" (Chapter 7), and systems of simultaneous linear equations (Chapter 8). Although the Chinese terminology "fangcheng" which is the title of Chapter 8, was adopted as a translation for "equation" towards the end of the last century (and has become a standard term today) for a wrong but historically interesting reason,[9] the spirit of Chapter 8 lies rather in the direction of ratio than in the direction of equation. In the light of ratios, the technique amounting to the modern matrix method by Gaussian elimination arises naturally.

In ending this section consider an example after the style of *Jiuzhang Suanshu* which blends together social relevance, ratio and even an application in statistical sampling. It is Problem 6 of Book 12 of *Shushu Jiuzhang* (*Mathematical Treatise in Nine Sections*) by Qin Jiushao, published in 1247:

> When a peasant paid tax to the government granary in the form of 1534 shi of rice, it was found out on examination that a certain amount of rice with husks was present. A sample of 254 grains was taken for further examination. Of these 28 grains were with husks. How many genuine grains of rice were there, given that one shao contains 300 grains?

(In the mensuration system of the Song Dynasty, 1 shi $= 10$ dou $= 100$ sheng $= 1000$ he $= 10000$ shao. According to tradition recorded in *Jiuzhang Suanshu*, a grain of rice with husk was counted as half a grain of rice.) The answer was given to be $4{,}348{,}346{,}456$ grains, out of the original $1534 \times 10000 \times 300 = 4{,}602{,}000{,}000$ grains.

Some Examples and Their Solution Methods[10]

(1) Problem 14 of Chapter 9 of *Jiuzhang Suanshu* is a word problem on right triangles:

> Two persons A (Jia) and B (Yi) stood at the same spot. In the time when A walked 7 steps, B could walk 3 steps. B walked east and A walked south.

> After 10 steps south A turned to walk in a roughly northeast direction to meet B. How many steps had each walked (when they met)?

The rule that follows the problem essentially gives the ratio of the length a, b, c of the three sides of a right triangle with c as that of the hypotenuse, viz.

$$a : b : c = \frac{1}{2}(m^2 - n^2) : mn : \frac{1}{2}(m^2 + n^2),$$

where $m : n = (a + c) : b$. In this problem, $m = 7$, $n = 3$ and $a = 10$. Hence $a : b : c = 20 : 21 : 29$ and $b = 10\frac{1}{2}$, $c = 14\frac{1}{2}$. The mathematical meaning of this result goes much deeper than just an answer to the problem as it stands, for it offers a way to generate the so-called Pythagorean triplets, i.e., (positive) integers a, b, c with $a^2+b^2 = c^2$. While no explicit formula for Pythagorean triplets was stated by the ancient Chinese, they were quite well-versed in these problems in which their Greek contemporaries were also interested, and in ancient Chinese mathematics arithmetic and geometry were intertwined through calculation. The achievement becomes all the more astounding if one notes that the ancient Greeks were aware of the notions of prime number and factorization while their Chinese contemporaries were not. Instead, the Chinese adopted a geometric viewpoint by looking for two quantities with suitable geometric interpretation in terms of which a, b, c can each be rationally expressed. In the case of Problem 14, the two quantities are the sum of the length of one side and the hypotenuse $(a + c)$ and the length of the third side (b). The explanation offered by Liu Hui can be illustrated as in Figure 2. In his commentary Liu Hui actually described in detail how to make use of colored pieces and to reassemble them for a convincing argument. If the original diagrams of the commentary were extant, they would make nice visual aids!

From Figure 2 we can see that

$$\begin{aligned} c : a : b &= S : T : U \\ &= \tfrac{1}{2}\left[(a + c)^2 + b^2\right] \\ &\qquad : (a + c)^2 - \tfrac{1}{2}\left[(a + c)^2 + b^2\right] : (a + c)b. \end{aligned}$$

Hence

$$\begin{aligned} a : b : c &= \tfrac{1}{2}\left[(a + c)^2 - b^2\right] : (a + c)b : \tfrac{1}{2}\left[(a + c)^2 + b^2\right] \\ &= \tfrac{1}{2}(m^2 - n^2) : mn : \tfrac{1}{2}(m^2 + n^2), \end{aligned}$$

where $(a + c) : b = m : n$.

The influence of this prototype classic of *Jiuzhang Suanshu* can be found in later work, for example Problem 2 of Chapter 5 of *Shushu Jiuzhang* by Qin Jiushao, published more than a thousand years later:

> A triangular field has sides of length 13 miles, 14 miles and 15 miles. What is its area?

The solution was given in the book as (in modern day mathematical notations)

$$(\text{Area})^2 = \frac{1}{4}\left[A^2C^2 - \left(\frac{A^2 + C^2 - B^2}{2}\right)^2\right]$$

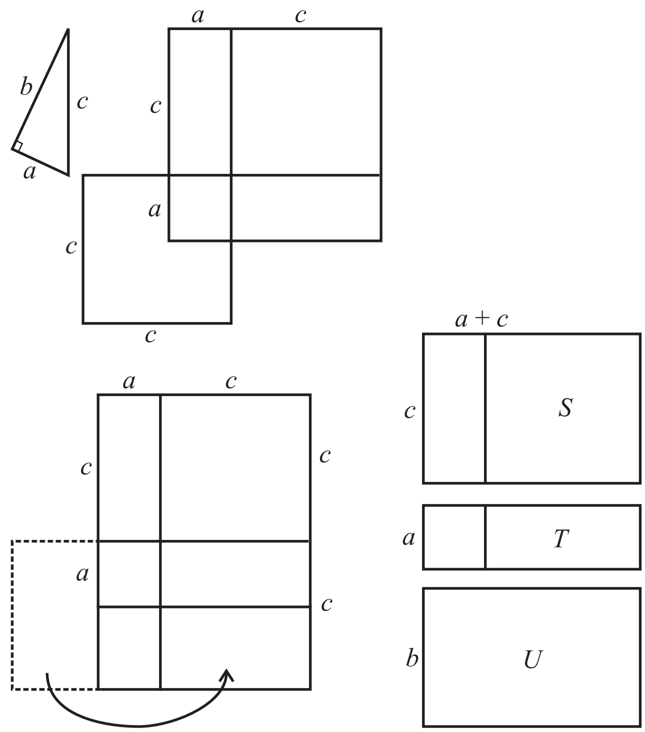


Figure 2

where A, B, C are the length of the three sides in decreasing magnitude. This is a rare gem in Chinese mathematics because this was perhaps the one occurrence of a triangle other than a right triangle in all Chinese mathematical texts before the transmission of Euclid's *Elements* into China.[11] A probable derivation of the formula by Qin Jiushao is as follows.[12] First note that, from our preceding example,

$$\frac{a}{b} = \frac{1}{2}\left[(a+c)^2 - b^2\right]/(a+c)b,$$

so that

$$a = \frac{1}{2}\left[(a+c) - \left(\frac{b^2}{a+c}\right)\right].$$

Construct a right triangle with sides of length a, b, c (c is the hypotenuse) where a, c are lengths as shown in Figure 3. Since $C^2 - a^2 = h^2 = B^2 - c^2$, we have $B^2 - C^2 = c^2 - a^2 = b^2$. Hence

$$\begin{aligned} a = &\ \frac{1}{2}\left[(a+c) - \left(\frac{b^2}{a+c}\right)\right] \\ = &\ \frac{1}{2}\left[A - \frac{B^2 - C^2}{A}\right] \\ = &\ \frac{1}{2}\left[\frac{A^2 + C^2 - B^2}{A}\right]. \end{aligned}$$

Finally,

$$\begin{aligned}(\text{Area})^2 &= \frac{1}{4}h^2A^2 = \frac{1}{4}(C^2-a^2)A^2 = \frac{1}{4}(A^2C^2-a^2A^2)\\ &= \frac{1}{4}\left[A^2C^2-\left(\frac{A^2+C^2-B^2}{2}\right)^2\right].\end{aligned}$$

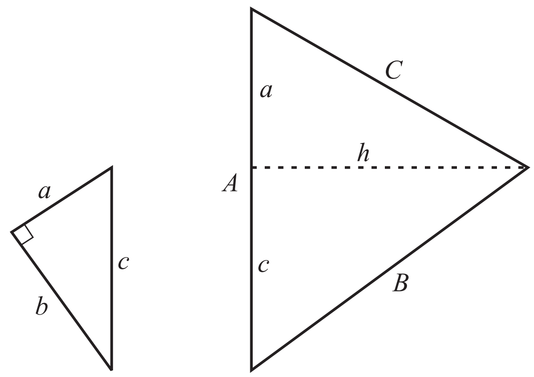


Figure 3

(2) Early Chinese calculation of π is given in Problem 32 of Chapter 1 of *Jiuzhang Suanshu*:

> A circular field has a perimeter of 181 steps and a diameter of 60 and 1/3 steps. What is its area?

The answer was given as "the area equals half the perimeter times half the diameter". This is a correct formula, as one can easily check that $A = (\frac{1}{2}C)(\frac{1}{2}d) = (\frac{1}{2}C)(r) = (\pi r)(r) = \pi r^2$. The data in this problem imply the formula $C = 3d$, which means π was then taken to be 3. In his commentary, Liu Hui explained why the formula is reasonable and pointed out how to obtain a more accurate value for π. He said:

> In our calculation we use a more accurate value for the ratio of the circumference to the diameter instead of the ratio that the circumference is 3 to the diameter's 1. The latter ratio is only that of the perimeter of the inscribed regular hexagon to the diameter. Comparing arc with the chord, just like the bow with the string, we see that the circumference exceeds the perimeter. However, those who transmit this method of calculation to the next generation never bother to examine it thoroughly but merely repeat what they learned from their predecessors, thus passing on the error. Without a clear explanation and definite justification it is very difficult to separate truth from falsity.

In this passage we see a truly first-rate mathematician at work, who probes into knowledge handed down and seeks understanding and clarification, thereby extending the frontier of knowledge. In modern day mathematical language Liu Hui's method is as follows. Put

A_n = area of an inscribed regular n-gon in a circle of radius r,
a_n = length of a side of the inscribed regular n-gon,
C_n = perimeter of the inscribed regular n-gon.

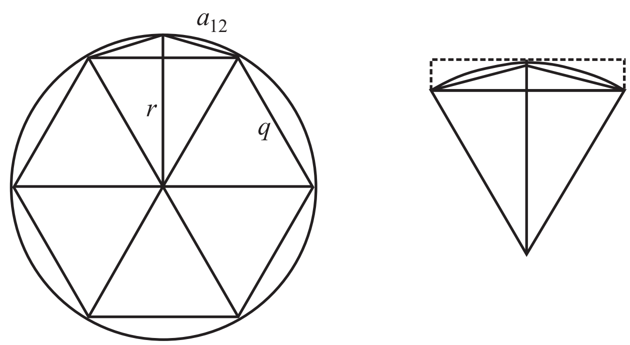


Figure 4

Starting with a regular hexagon $(n = 6)$ and doubling the number of sides, Liu Hui enlarged it to a regular 12-gon, then a regular 24-gon, then a regular 48-gon, and so on, up to a regular 192-gon. He observed that $A_{12} = 3a_6r = \frac{1}{2}C_6r$, $A_{24} = 6a_{12}r = \frac{1}{2}C_{12}r$, $A_{48} = 12a_{24}r = \frac{1}{2}C_{24}r$, etc. He also knew that this was not the end but only the first few steps in an approximation process. He claimed, "the finer one cuts, the smaller the leftover; cut after cut until no more cut is possible, then it coincides with the circle and there is no leftover." We see here the budding concepts of infinitesimal and limit. He even gave an estimate, viz.

$$A_{2m} < A < A_{2m} + (A_{2m} - A_m),$$

as can be seen from Figure 4. With this he concluded that "ultimately" $A = \frac{1}{2}Cr$. He also carried out the computation for finding A_{192}. In doing that he first established the formula

$$a_{2n} = \sqrt{\left[r - \sqrt{r^2 - \left(\frac{a_n}{2}\right)^2}\right]^2 + (a_n/2)^2}.$$

A modern computer can obtain $A_{192} = 3.141032$ (with $r = 1$) with error term 0.001681. Imagine how Liu Hui did it with only the help of counting rods over 17 centuries ago, obtaining $A_{192} = 314\frac{64}{625}$ (with $r = 10$). Effectively he calculated π accurate to two decimal places[13].

(3) The algorithmic feature of ancient Chinese mathematics can best be illustrated by the method of solving simultaneous linear congruence equations. In abstract algebra there is a fundamental result known as the "Chinese Remainder

Theorem". Its name comes from a concrete instance, viz. Problem 26 of Chapter 3 of *Sunzi Suanjing* (*Master Sun's Mathematical Manual*), (c. 4th century):

> There are an unknown number of things. Counting by threes we leave 2; counting by fives we leave 3; counting by sevens we leave 2. Find the number of things.

The problem became quite popular and appeared under different names. In a much later text *Suanfa Tongzong* (*Systematic Treatise on Arithmetic*) of Cheng Dawei, published in 1592, there appeared even a poem about it: "T'is hard to find one man of seventy out of three. There are twenty-one branches on five plum blossom trees. When seven persons meet, it is in the middle of the month. Discarding one hundred and five, the problem is done." The poem conceals the magic numbers 70 (for 3), 21 (for 5), 15 (for 7) of this specific problem, whose general answer is $2 \times 70 + 3 \times 21 + 2 \times 15$ plus or minus any multiple of $105 = 3 \times 5 \times 7$. In general, the problem is to solve a system of linear congruence equations

$$x \equiv a_i \bmod m_i, \quad i \in \{1, 2, \ldots, N\}.$$

Mathematicians were led to investigate linear congruences because of calendrical reckoning and had become quite deft in handling them. Already in this specific problem we can see a very significant step made, viz. reduction of the problem to solving $x \equiv 1 \bmod m_i$, $x \equiv 0 \bmod m_j$ for $j \neq i$ (the solution to the original problem being a suitable "linear combination" of the solutions of these different systems). The investigation was completed by Qin Jiushao who named his method the "Dayan art of searching for unity" in his *Shushu Jiuzhang* (1247). He showed how to find a set of magic numbers for making the "linear combination". Consider the case when the m_i's are mutually relatively prime, using modern notations. (Qin Jiushao also treated the general case.) It suffices to solve separately single linear congruence equations of the form $kb \equiv 1 \bmod m$ by putting $m = m_i$ and $b = (m_1 \cdots m_N)/m_i$. The key point in the method Qin Jiushao employed to find k is to find a sequence of ordered pairs (k_i, r_i) such that $k_i b \equiv (-1)^i r_i \bmod m$ and the r_i's are strictly decreasing. At some point $r_s = 1$ but $r_{s-1} > 1$. If s is even, then $k = k_s$ will be a solution. If s is odd, then $k = (r_{s-1} - 1)k_s + k_{s-1}$ will be a solution. This sequence of ordered pairs can be found by using "reciprocal subtraction" explained in *Jiuzhang Suanshu*, viz., $r_{i-1} = r_i q_{i+1} + r_{i+1}$ with $r_{i+1} < r_i$ (the process will stop before one reaches the case $r_{i+1} = 0$), and put $k_{i+1} = k_i q_{i+1} + k_{i-1}$. (Put $k_{-1} = 0$, $r_{-1} = m$, $k_0 = 1$, $r_0 = b$.) The way the ancient Chinese performed the calculation was even more streamlined and convenient, since they put consecutive pairs of numbers at the four corners of a board using counting rods, starting with

$$\boxed{\begin{array}{cc} 1 & b \\ 0 & m \end{array}}, \quad \text{going to} \quad \boxed{\begin{array}{cc} k & 1 \\ * & * \end{array}}.$$

The procedure was stopped when the upper right corner became a 1, hence the name "searching for unity". A typical intermediate step will look like

$$\boxed{\begin{matrix} k_i & r_i \\ k_{i-1} & r_{i-1} \end{matrix}} \longrightarrow \boxed{\begin{matrix} k_i & r_i \\ k_{i-1} & r_{i+1} \end{matrix}} \longrightarrow \boxed{\begin{matrix} k_i & r_i \\ k_{i+1} & r_{i+1} \end{matrix}}, \quad \text{if } i \text{ is even}$$

or

$$\boxed{\begin{matrix} k_{i-1} & r_{i-1} \\ k_i & r_i \end{matrix}} \longrightarrow \boxed{\begin{matrix} k_{i-1} & r_{i+1} \\ k_i & r_i \end{matrix}} \longrightarrow \boxed{\begin{matrix} k_{i+1} & r_{i+1} \\ k_i & r_i \end{matrix}}, \quad \text{if } i \text{ is odd.}$$

One can see how the positions on a board of counting rods help to fix ideas. In fact, the procedure outlined in *Shushu Jiuzhang* can be phrased word for word as a computer program!

(4) An example on the lighter side is Problem 34 of Chapter 3 of *Sunzi Suanjing*:

> One sees 9 embankments outside; each embankment has 9 trees; each tree has 9 branches; each branch has 9 nests; each nest has 9 birds; each bird has 9 young birds; each young bird has 9 feathers; each feather has 9 colours. How many are there of each?

The problem, an easy exercise in raising a number to certain powers, is not of much interest in itself. What is interesting is the frequent occurrence of such problems of a recreational nature in all mathematical civilizations. The medieval European mathematician, Leonardo Fibonacci posed a problem in his book "Liber Abaci" (1202)[14]:

> Seven old women went to Rome; each woman had seven mules; each mule carried seven sacks; each sack contained seven loaves; and with each loaf were seven knives; each knife was put up in seven sheaths. How many are there, people and things?

It reminds us of a children's rhyme: "As I was going to Saint Ives, I met a man with seven wives. Every wife had seven sacks. Every sack had seven cats. Every cat had seven kits. Kits, cats, sacks and wives, how many were there going to Saint Ives?" And then there was that similar Problem 79 in the oldest extant mathematical text, the Rhind Papyrus of ancient Egypt (c. 17 century BC)[15]:

Houses	7
Cats	49
Mice	343
Heads of wheat	2301
Hekat measures	16807
	19607

David Hilbert (1862–1943) once said, "Mathematics knows no races.. . . For mathematics, the whole cultural world is a single country."[16]

Bibliography

There is a wealth of general accounts and in-depth research works on ancient Chinese mathematics. Many of them are contained in a vast store of books and papers written in Chinese, most of which have not been translated and thus remain inaccessible or even unknown to the non-Chinese-speaking community. For lack of space and in line with the nature of an introductory article, the works of these authors cannot be documented in full here, although liberal use has been made of their scholarship, for which the author feels at the same time thankful and apologetic. For the convenience of readers who do not read Chinese but who wish to go further into the subject, a few helpful references available in English are cited below.

The most well-known and oft-cited standard reference, an immensely scholarly and well-documented work (with access to over 500 items in Chinese and Japanese) on the whole history of Chinese mathematics, is of course

J. Needham (with the collaboration of Wang Ling), *Science and Civilization in China*, vol. 3, Chapter 19, Cambridge University Press, Cambridge, 1959.

A more up to date reference, which also emphasizes the context of ancient Chinese mathematics besides its content, is

J. C. Martzloff, *A History of Chinese Mathematics* (translated from *Histoire des mathématiques chinoises*, Masson, Paris 1987), Springer-Verlag, Heidelberg, 1997.

For a first reading, two good choices are

Li Yan and Du Shiran, *Chinese Mathematics: A Concise History* (translated by J. N. Crossley and A. W. C. Lun from *Zhongguo Gudai Shuxue Jianshi*, Zhonghua, Beijing, 1963), Clarendon Press, Oxford, 1987;

G. G. Joseph, *The Crest of the Peacock: Non-European Roots of Mathematics*, Tauris, London, 1991.

For a quick introduction through several informative and interesting articles, one can read

F. J. Swetz, The evolution of mathematics in ancient China, *Math. Magazine*, 52 (1979), 10–19;

F. J. Swetz, The amazing Chiu Chang Suan Shu, *Math. Teacher*, 65 (1972), 423–430;

F. J. Swetz, The "piling up of squares" in ancient China, *Math. Teacher*, 70 (1977), 72–79.

(All three articles above are collected in *From Five Fingers to Infinity: A Journey Through the History of Mathematics*, edited by F. J. Swetz, Open Court, Chicago, 1994.)

A chronological outline of the development of Chinese mathematics with an accompanying extensive bibliography of references written in Western languages (up to 1984) can be found in

F. J. Swetz and Ang Tian Se, A brief chronological and bibliographic guide to the history of Chinese mathematics, *Historia Mathematica*, 11 (1984), 39–56.

Endnotes

[1] At the end of the sixteenth century the first wave of dissemination of European science in China began. What happened after the sixteenth century, the vicissitude of indigenous mathematical development and its integration with transmitted western mathematics, form a fascinating topic in itself, but will not be discussed in this article.

[2] This article is based on an introductory lecture scheduled for the conference on História e Educação Matemática (Braga, Portugal, 24–30 July 1996). Circumstances prevented the author from attending. The lecture was instead given by Mr. Chun-Ip Fung, to whom the author owes his thanks.

[3] A typical passage can be found in the preface to *Shushu Jiuzhang* by Qin Jiushao (1247). The discussion in this article will be confined to "external mathematics" owing to the author's ignorance of the aspect of "internal mathematics".

[4] For further discussion of mathematics education in ancient China, see: M. K. Siu, Mathematics education in ancient China: What lesson do we learn from it? *Historia Scientiarum*, 4–3 (1995), 223–232. See also Chapter 1 in: F. Swetz, *Mathematics Education in China: Its Growth and Development*, MIT Press, Cambridge 1974.

[5] This is quoted from: M. K. Siu, Proof and pedagogy in ancient China: Examples from Liu Hui's commentary on *Jiuzhang Suanshu*, *Educational Studies in Mathematics*, 24 (1993), 345–357. The paper contains a number of illustrative examples.

[6] It was reported in, for instance: Li Xueqin, A significant finding in the history of ancient Chinese mathematics: A glimpse at the Han bamboo strips excavated at Zhangjiashan in Jiangling (in Chinese), *Wenwu Tiandi*, 1 (1985), 46–47.

[7] Such a statement has to be taken with a grain of salt! A better perspective can only be gained when one views mathematical development against a broader socio-cultural background at the time. In particular, the community of "mathematicians" in ancient China was not a well-defined recognized group of scholars. The author is in the process of studying, in collaboration with A. Volkov, mathematical activities in ancient China in this wider context.

[8] Compare this with Definition 3 of Book 5 of Euclid's *Elements*: "A ratio is a sort of relation in respect of size between two magnitudes of the same kind".

[9] This might have to do with the promulgation of the thesis of "Chinese origin of Western knowledge" in the Qing Dynasty in an effort to reassert the role of indigenous mathematics. A detailed discussion is beyond the scope of this article.

[10] Specific reference to each problem is omitted. Most statements of these problems are translated by the author. But there do exist translated texts for *Sunzi Suanjing* (L. Y. Lam and T. S. Ang, *Fleeting Footsteps: Tracing the Conception of Arithmetic and Algebra in Ancient China*, World Scientific, Singapore, 1992), *Shushu Jiuzhang* (U. Libbrecht, "Chinese Mathematics in the Thirteenth Century: The Shu-shu Chiu-chang of Ch'in Chiu-shao, MIT Press, Cambridge, 1973) and *Jiuzhang Suanshu* (K. Vogel, "Neun Bücher Arithmetischer Technik", Friedr. Vieweg & Sohn, Braunschweig, 1968). A French translation and an English translation of *Jiuzhang Suanshu* are under preparation.

[11] One naturally calls to mind the formula by the Greek mathematician Heron of Alexandria (c. 1st century), viz.

$$(\text{Area})^2 = S(S-A)(S-B)(S-C)$$

where $S = (A+B+C)/2$. Indeed, the two formulas are equivalent.

[12] It is interesting to compare it with the proof of Heron by synthetic geometry, which can be found in, for instance: I. Thomas, *Greek Mathematical Works*, II, Harvard University Press, 1939; reprinted with additions and revisions, 1980, pp. 471–477.

[13] It is interesting to compare this computation of π with that by Archimedes, which can be found in, for instance: R. Calinger (ed), *Classics of Mathematics*, Moore Publishing, 1982; Prentice-Hall, 1995, pp. 137–141.

[14] See: H. Eves, *An Introduction to the History of Mathematics*, 4th edition, Holt, Rinehart & Winston, New York, 1976, pp. 43–44.

[15] See: A.B. Chace, *The Rhind Mathematical Papyrus*, Mathematical Association of America, Oberlin, 1927–29; reprinted by the National Council of Teachers of Mathematics, Reston, 1978, p. 59.

[16] See: Constance Reid, *Hilbert*, Springer-Verlag, Heidelberg, 1970, p.188.