Does society need IMO Medalists?

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Abstract

The title of this talk that sounds provocative is not chosen with any intention to embarrass the organizers and participants of the event of IMO (International Mathematical Olympiad). It should be seen as the sharing of some thoughts on this activity, or more generally on mathematical competitions, by a teacher of mathematics who had once helped in the coaching of the first Hong Kong Team to take part in the 29th IMO held in Canberra in 1988 and in the coordination work of the 35th IMO held in Hong Kong in 1994. The speaker tries to look at the issue in its educational context and more broadly in its socio-cultural context.

About the speaker

SIU Man Keung obtained his BSc from the Hong Kong University and went on to earn a PhD in mathematics from Columbia University. Like the Oxford cleric in Chaucer's *The Canterbury Tales*, "and gladly would he learn, and gladly teach" for more than three decades until he retired in 2005, and is still enjoying himself in doing that after retirement. He has published some research papers in mathematics and computer science, some more papers of a general nature in history of mathematics and mathematics education, and several books in popularizing mathematics. In particular he is most interested in integrating history of mathematics with the teaching and learning of mathematics and has been participating actively in an international community of History and Pedagogy of Mathematics since the mid-1980s. He has devoted much of his time in offering a course titled *Mathematics: A Cultural Heritage* in the tradition of liberal studies for undergraduates from various Faculties of the Hong Kong University for a decade during the 2000s as well.

Does society need IMO Medalists?

Does society need IMO Medalists? No, society does not "need" IMO Medalists. Society does not even "need" mathematicians. Does this mean my 30-minute talk will end here? Stopping here and now would amount to an admission of wrong choice of my profession in all these years, so I should go on talking. You will notice that I put the word "need" in quotation marks. Society does not "need" (in quotation marks!) IMO Medalists or mathematicians, but society needs (no quotation mark!) MTR maintenance workers, garbage collectors, street cleaners, plumbers, electricians, etc. . Now, perhaps you know what I mean. Let us get back to IMO.

After 22 years IMO comes back to Hong Kong as the host. Hong Kong hosted the 35th IMO in the summer of 1994. I like to mention two Medalists in that particular IMO. One is Maryam Mirzakhani of Iran, who became the first female mathematician to receive a Fields Medal at the International Congress of Mathematicians in 2014. The other one is Subash Ajit Khot of India, who was awarded the Nevanlinna Prize at the same Congress.

In a talk I gave in 2012 with the title "The good, the bad and the pleasure (not pressure) of mathematics competitions" I outlined certain good and bad points of mathematics competitions. Allow me to repeat them here in summary. [For a more detailed discussion, see: M.K. Siu, Some reflections of a coordinator on the IMO, *Mathematics Competitions*, 8 (1) (1995), 73-77; M.K. Siu, The good, the bad and the pleasure (not pressure!) of mathematics competitions, *Mathematics Competitions*, 26(1) (2013), 41-58.]

Good points: Nurturing of (1) clear and logical presentation, (2) tenacity and assiduity, (3) “academic sincerity”; moreover, arouse a passion for and pique the interest in mathematics.

Bad points: (1) competition problems *versus* research, (2) over-training?
We further ask: Is the passion for the subject of mathematics itself genuine? Can the interest be sustained?

Let me further explain the point about competition problems *versus* research through three examples (see **APPENDIX**).

What do we see from these three examples? It makes me think that there are two approaches in doing mathematics. To give a military analogue one is like positional warfare and the other guerrilla warfare. The first approach, which has been going on in the classrooms of most schools and universities, is to present the subject in a systematically organized and carefully designed format supplemented with exercises and problems. The other approach, which goes on more predominantly in the training for mathematics competitions, is to confront students with various kinds of problems and train them to look for points of attack, thereby accumulating a host of tricks and strategies. Each approach has its separate merit and they supplement and complement each other. Each approach calls for day-to-day preparation and solid basic knowledge. Just as in positional warfare flexibility and spontaneity are called for, while in guerrilla warfare careful prior preparation and groundwork are needed, in the teaching and learning of mathematics we should not just teach tricks and strategies to solve special type of problems or just spend time on explaining the general theory and working on problems that are amenable to routine means. We should let the two approaches supplement and complement each other in our classrooms. In the biography of the famous Chinese general and national hero of the Southern Song Dynasty, Yue Fei (岳飛 1103-1142) we find the description: “陣而後戰，兵法之常。運用之妙，存乎一心。 (Setting up the battle formation is the routine of the art of war. Manoeuvring the battle formation skillfully rest solely with the mind.)”

Sometimes the first approach may look quite plain and dull, compared with the excitement acquired from solving competition problems by the second approach. However, we should not overlook the significance of this seemingly bland approach, which can cover more general situations and turns out to be much more powerful than an *ad hoc* method which, slick as it is, solves only a special case. Of course, it is true that frequently a clever *ad hoc* method can develop into a powerful general method or can become a part of a larger picture. A classic case in point is the development of calculus in history. In ancient time, only masters in mathematics could calculate the area and volume of certain geometric objects, to name just a couple of them, Archimedes (c. 287 B.C.E. – c. 212 B.C. E.) and LIU Hui (劉徽 3rd century). In hindsight their formula for the area of a circle, $A = (1/2) \times C \times r$, embodies the essence of the Fundamental Theorem of Calculus. With the development of calculus since the seventeenth century and the eighteenth century, today even an average school pupil who has learnt calculus will be able to handle what only great mathematicians of the past could have resolved.

Since many mathematics competitions aim at testing the contestants’ ability in problem solving rather than their acquaintance with specific subject content knowledge, the problems are set in some general areas which can be made comprehensible to youngsters of that age group, independent of different school syllabi in different countries and regions. That would cover topics in elementary number theory, algebra, combinatorics, sequences, inequalities, functional equations, plane and solid geometry and the like. Gradually the term "Olympiad mathematics" is coined to refer to this conglomeration of topics. One question that I usually ponder over is this: why can't this type of so-called "Olympiad mathematics" be made good use of in the school classroom as well? If one aim of mathematics education is to let students know what the subject is about and to arouse their interest in it, then interesting non-routine problems should be able to play their part well when used to supplement the day-to-day teaching and learning.

Let us get back to the question in the title of the talk: **Does society need IMO Medalists?** No, society does not "need" IMO Medalists. Society does not even "need" mathematicians. But society needs "friends of mathematics". A "friend of mathematics" may not know a lot of mathematics but would understand well what mathematics is about and appreciate well the role of mathematics in the modern world.

The mathematician Paul Halmos once said, "It saddens me that educated people don't even know that my subject exists." Allen Hammond, editor of *Science*, once described mathematics as "the invisible culture". On the other hand, perhaps it is a blessing to remain not that visible! Two months ago I read in the news (Associated Press, May 7, 2016): "Ivy League Professor Doing Math Equation on Flight Mistaken for Terrorist". An American Airline passenger seated next to Guido Menzio of UPenn suspected the unfamiliar writings of the professor were a code for a bomb. It led to Professor Menzio being taken away from the plane to be interrogated!

In ancient China the third-century mathematician LIU Hui (劉徽) said, "[九數…]至於以法相傳，亦猶規矩度量可得而共，非特難為也。當今好之者寡，故世雖多通才達學，而未必能綜於此耳。(The subject [mathematics] is not particularly difficult by using methods transmitted from generation to generation, like the compasses [*gui*] and gnomon [*j*u] in measurement, which are comprehensible to most people. However, nowadays enthusiasts for mathematics are few, and many scholars, much erudite as they are, are not necessarily cognizant of the subject.)"
Why is it like that?

Here is a passage taken from a book: "Central to my argument is the idea that ***** distinguished by a self-conscious attention to its own ***** language. Its claim to function *as art* derives from its peculiar concern with its own materials and their formal patterning, aside from any considerations about its audience or its social use." Can you guess what the missing words are?

This passage is taken from the book by Julian Johnson, *Who Needs Classical Music? Cultural Choice and Musical Value* (2002). The missing words are "classical music" and "musical". However, the passage would ring equally true if "classical music" is replaced by "mathematics"! In the same book the author says, "… that it [meaning classical music] relates to the immediacy of everyday life but not immediately. That is to say, it takes aspects of our immediate experience and reworks them, reflecting them back in altered form. In this way, it creates for itself a distance from the everyday while preserving a relation to it." **Mathematics is also like that.** This explains why it is not easy to bring mathematics to the general public. To become a "friend of mathematics" one needs to be brought up from school days onward in an environment where mathematics is not only enjoyable but also makes good sense. In the preface to a textbook [*Alice in Numberland: A Students' Guide to the Enjoyment of Higher Mathematics* (1988)] the authors, John Baylis and Rod Haggarty, remark, "The professional mathematician will be familiar with the idea that entertainment and serious intent are not incompatible: the problem for us is to ensure that our readers will enjoy the entertainment but not miss the mathematical point, […]"

My good friend, Tony Gardiner, an experienced four-time UK IMO team leader, once commented that I should not blame the negative aspects of mathematics competitions on the competition itself. He went on to enlighten me on one point, namely, a mathematics competition should be seen as just the tip of a very large, more interesting, iceberg, for it should provide an incentive for each country to establish a pyramid of activities for masses of interested students. It would be to the benefit of all to think about what other activities besides mathematics competitions can be organized to go along with it. These may include the setting up of a mathematics club or publishing a magazine to let interested youngsters share their enthusiasm and their ideas, organizing a problem session, holding contests in doing projects at various levels and to various depth, writing book reports and essays, producing cartoons, videos, softwares, toys, games, puzzles, … .

Finally the question boils down to one in an even more general context: **Does society need ME?** We frequently hear about the cliché "No one is indispensable!" But please bear in mind that everyone has his or her worth and can do his or her part to make this world a better place to live in. An IMO medalist is no exception!

Thank you!

APPENDIX

Example 1

The first example is a rather well-known problem in one IMO. Since I helped with the coaching of the first HK Team that was sent to take part in the 29$^{\text{th}}$ IMO held in Canberra in 1988, naturally I paid some special attention to the questions set in that year. Question 6 of the 29th IMO reads:

"Let *a* and *b* be positive integers such that $ab + 1$ divides $a^2 + b^2$. Show that $\frac{a^2+b^2}{ab+1}$ is the square of an integer."

A slick solution to this problem, offered by a Bulgarian youngster (Emanouil Atanassov) who received a special prize for it, starts by supposing that $k = \frac{a^2+b^2}{ab+1}$ is **not** a perfect square and rewriting the expression in the form

$$a^2 - kab + b^2 = k \qquad (*) .$$

Note that for any integral solution (a, b) for (*) we have $ab > 0$ since *k* is not a perfect square. Let (a, b) be an integral solution of (*) with $a > 0$ and $b > 0$ and $a + b$ ***smallest***. We shall produce from it another integral solution (a', b) of (*) with $a' > 0$, $b > 0$ and $a' + b < a + b$. This is a contradiction! [We omit the argument for arriving at such a solution (a', b).]

Slick as the proof is, it also invites a couple of queries. (1) What makes one suspect that $\frac{a^2+b^2}{ab+1}$ is the square of an integer? (2) The argument should hinge crucially upon the condition that *k* is not a perfect square. In the proof this condition seems to have slipped in casually so that one does not see what really goes wrong if k is *not* a perfect square. More pertinently, this proof *by contradiction* has **not explained** why $\frac{a^2+b^2}{ab+1}$ must be a perfect square, even though it **confirms** that it is so.

In contrast let us look at a much less elegant solution, which is my own attempt. When I first heard of the problem, I was on a trip in Europe and had a 'false insight' by putting $a = N^3$ and $b = N$ so that

$$a^2 + b^2 = N^2(N^4 + 1) = N^2(ab + 1) \ .$$

Under the impression that any integral solution (a, b, k) of $k = \frac{a^2+b^2}{ab+1}$ is of the form (N^3, N, N^2) I formulated a strategy of trying to deduce from $a^2 + b^2 = k(ab + 1)$ another equality

$$[a - (3b^2 - 3b + 1)]^2 + [b - 1]^2 = \{k - [2b - 1]\}\{[a - (3b^2 - 3b + 1)][b - 1] + 1\}.$$

Were I able to achieve that, I could have reduced b in steps of one until I got down to the equation $k = \frac{a^2+1}{a+1}$ for which $a = k = 1$. By reversing steps I would have solved the problem. I tried to carry out this strategy while I was travelling on a train, but to no avail. Upon returning home I could resort to systematic brute-force checking and look for some actual solutions, resulting in a (partial) list shown below.

a	**1**	**8**	**27**	30	**64**	112	**125**	**216**	240	**343**	418	**512**	…
b	**1**	**2**	**3**	8	**4**	30	**5**	**6**	27	**7**	112	**8**	…
k	**1**	**4**	**9**	4	**16**	4	**25**	**36**	9	**49**	4	**64**	…

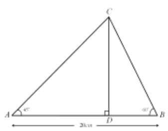
Then I saw that my ill-fated strategy was doomed to failure, because there are solutions other than those of the form (N^3, N, N^2). **However, not all was lost.** When I stared at the pattern, I noticed that for a fixed k, the solutions could be obtained recursively as (a_i, b_i, k_i) with

$$a_{i+1} = a_i k_i - b_i,\ b_{i+1} = a_i,\ k_{i+1} = k_i = k.$$

It remained to carry out the verification. Once that was done, all became clear. There is a set of 'basic solutions' of the form (N^3, N, N^2) , $N \in \{1,2,3, \dots\}$. All other solutions are obtained from a 'basic solution' recursively as described above. In particular, $k = \frac{a^2+b^2}{ab+1}$ is the square of an integer. I feel that I understand the phenomenon much more than if I just learn from reading the slick proof.

Example 2

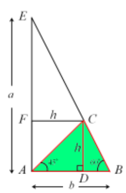
Problem 5 in the 21st Hong Kong Primary School Mathematics Competition held in May of 2010 says: Using only a ruler draw a triangle *ABC* on the A3-size paper so that *AB* is of length 20 cm., *angle BAC* is of measure 45°, and angle ABC is of measure 60°. Find the shortest distance from *C* to *AB* correct to one decimal place.



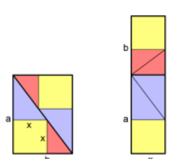
There are various ways to do this problem, which is probably originally set to see if a primary school pupil knows how to make use of paper-folding to arrive at the required triangle, then makes use of paper-folding again to get the perpendicular to the base and measures it by the ruler.

Some "early starters (who jump the gun)" tried to solve it by using trigonometry, which is not normally taught in primary school. They even knew about the Law of Sine! But they were stumped when they came to an angle of measure other than 30, 45 or 60 degrees!

Here is a clever solution which does not rely on secondary school mathematics. We extend the figure to a larger right triangle and borrow from the wisdom of ancient Chinese mathematicians.



Problem 15 in Chapter 9 of the ancient Chinese mathematical classics *Jiuzhang Suanshu* (九章算術) asks for the side of an inscribed square in a given right triangle, which is equal to *ab*/(*a*+*b*). [Let me demonstrate the result by the method of dissect-and-re-assemble credited to the 3rd-century Chinese mathematician LIU Hui (劉徽).]



Hence, the height of the original triangle can be calculated.

This clever solution would not work if the measures of the two base angles are arbitrary, while the not-so-clever "dry" method which relies on the Law of Sine still works well.

Example 3

There is a well-known anecdote about the famous mathematician John von Neumann (1903-1957). A friend of von Neumann once gave him a problem to solve. Two cyclists A and B at a distance 20 miles apart were approaching each other, each going at a speed of 10 miles per hour. A bee flew back and forth between A and B at a speed of 15 miles per hour, starting with A and back to A after meeting B, then back to B after meeting A, and so on. By the time the two cyclists met, how far had the bee travelled? In a flash von Neumann gave the answer — 15 miles. His friend responded by saying that von Neumann must have already known the trick so that he gave the answer so fast. His friend had in mind the slick solution to this quickie, namely, that the cyclists met after one hour so that within that one hour the bee had travelled 15 miles. To his friend's astonishment von Neumann said that he knew no trick but simply summed an infinite series!

For me this anecdote is very instructive. (1) Different people may have different ways to go about solving a mathematical problem. There is no point in forcing everybody to solve it in just the same way you solve it. Likewise, there is no point in forcing everybody to learn mathematics in just the same way you learn it. (2) Both methods of solution have their separate merits. The first method of calculating when the cyclists met is slick and captures the key point of the problem. The other method of summing an infinite series, which is slower (but not for von Neumann!) and is seemingly more cumbersome and not as clever, goes about solving the problem in a systematic manner. It indicates patience, determination, down-to-earth approach and meticulous care. Besides, it can help to consolidate some basic skills and nurture in a student a good working habit.