"Less is more" or "Less is less"?:

Undergraduate mathematics education in the era of mass education

Man-Keung SIU

Department of Mathematics

University of Hong Kong

Abstract

This article discusses a teaching philosophy of the undergraduate mathematics curriculum in the era of mass education. It tries to explain through examples why "less is less" can be changed to "less is more", or at least not to let "more is less" happen. The teaching of an introductory course in linear algebra is used as a case study. Besides arguments on a more passive tone on the shortcoming of "more is less" the article also addresses the active side of "less is more", viz. (1) to enable students to see more clearly the central ideas, thereby nurturing a "mathematical taste" in judging the level of significance in the knowledge structure; (2) to allow students the leisure and room to reflect and to make sense of what they have learnt; (3) to afford time to improve skill in communication and presentation through reading, talking and writing mathematics.

1. Prologue: An Examination Question

Last year I set the following examination question in a calculus class:

> The plane $x + y = 1$ intersects the surface $z = xy$ in a curve. Locate the highest and lowest points (measured from the Oxy-plane) on this curve, if any.

It can be solved as an extremum problem with constraint or even as an extremum problem in one-variable calculus after reduction. However, it turned out I got many different "solutions" besides the correct one. Incidentally, these "solutions" provide instructive material for this year's tutorial to incite anxiety, which can be an effective stimulant to learning when felicitously exploited!

"Solution 1": $z = xy$ and $x + y = 1$. Hence $xy - z = x + y - 1$, so $xy - z - x - y + 1 = 0$. Set $F(x, y, z) = xy - z - x - y + 1$. $\frac{\partial F}{\partial x} = y - 1 = 0, \frac{\partial F}{\partial y} = x - 1 = 0, \frac{\partial F}{\partial z} = -1 = 0.$ The final result is impossible, therefore there is no extremum point.

"Solution 2": $z = xy$ and $x + y=1$. Hence $xy - z = x + y -1$, so $z = xy - x - y + 1$. $\frac{\partial z}{\partial x} = y - 1 = 0, \frac{\partial z}{\partial y} = x - 1 = 0.$ Hence (1, 1) is a critical point. Since $\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) = 1^2 - 0 = 1 > 0,$ we conclude (1, 1) is a saddle point.

"Solution 3": $z = xy$ and $x + y = 1$. Hence $(x + y)^2 = 1$, so $x^2 + y^2 + 2xy = 1$, or $x^2 + y^2 + 2z = 1$, or $z = ½(1 - x^2 - y^2)$. $\frac{\partial z}{\partial x} = -x = 0, \frac{\partial z}{\partial y} = -y = 0.$ Hence (0, 0) is a critical point. Since $\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) = 0^2 - (-1)(-1) = -1 < 0$ and $\frac{\partial^2 z}{\partial x^2} = -1 < 0,$ we conclude (0, 0) is a local maximum point.

"Solution 4": $z = xy$ and $x + y = 1$. Hence $z/x = y = 1-x$, so $z = x - x^2$ or $z - x + x^2 = 0$. Set $F(x, z) = z - x + x^2$. $\frac{\partial F}{\partial x} = -1 + 2x = 0, \frac{\partial F}{\partial z} = 1 = 0.$ The final result is impossible, therefore there is no extremum point.

If the student had looked at a geometric picture, the solution would have become quite clear, and so was the mistake occurring in each "solution". However, some students are too much interested in just learning the method (more appropriately called a recipe, and in most cases a one-step or at most two-step recipe) that they unknowingly put on blinkers. I have no doubt that students who gave me those "solutions" studied for the examination and that they can cope with routine calculation with ease, but at the same time they stuffed themselves with too much material undigested.

2. Mathematics Education for the Mass

In the mind of a number of teachers and policy-makers on education, mass education implies an unavoidable decline in the standard, and this can only be remedied by two means: (1) "Dilute" the academic content so as to place a lighter demand on students, with mechanical routines retained while those parts requiring deeper intellectual effort reduced as much as possible. (2) Strengthen the means such as homework and tests to keep students' noses to the grindstone, for it is believed that the more they revise the better they retain the material. It so happens that what are most easy to test on are pieces of information which can be regurgitated without much thought. Hence (1) and (2) are quite "compatible", turning education into a kind of dull training by stuffing into students fragmentary information mainly to be regurgitated in an examination and then forgotten! This is not actually in the spirit of mass education. In the society of today, which is more open, more multifarious, more hi-tech and more information-flowing, all citizens should be adequately prepared for it. Besides possessing a certain amount of knowledge and skills, they should be able to assimilate and analyze information, to synthesize and to communicate effectively. It is not enough if they can only carry out instructions mechanically. Thus, the demand on students in this era of "mass education" is definitely not lower than that in the old days of "elite education". Besides, lowering the standard can only breed laziness and the attitude of shunning difficulty. This is contrary to the goals of mass education. The problem is more serious in the subject of mathematics. By nature mathematics involves abstract thinking. If we strip off abstract thinking in mathematics, what would be left? If we cannot "dilute" the content while students cannot cope with the "un-diluted" content, what can be done? I will propose a teaching philosophy which may help, for which I borrow the slogan "less is more" from the Bauhaus School in architecture (of Ludwig Mies van der Rohe in the 1930s).

3. Less Is Less? More Is Less?

It sounds contradictory to say "less is more"! If it is less, how can it be more? We actually mean to say, "We teach less, but the students learn more." One may ask, "If we teach less, does it not follow that students learn less as well?" The answer will be in the affirmative if students learn just as much as the teacher gives in class, but that is precisely what we do not want to see in an undergraduate education. We wish to nurture the curiosity to know and the capacity for self-learning, in which case the amount the teacher gives in class should not be an upper bound of what students can learn. Unfortunately we usually take too much care of our students, either unconsciously or out of good intention, so that they get the impression that it suffices to learn just that much the teacher gives in class. We do not leave enough room for students to learn by themselves and to think for themselves. We mistakenly believe that as long as we pack the course fully, then students will gain enough even if they master only 40% of it. The opposite to "less is more" is "more is less", which is best paraphrased by a wise saying of Albert Einstein, 'Overburdening necessarily leads to superficiality.'[1] If we simply worry about the many topics that have to be packed into a course, at the end we are merely taking students on a tour of the mathematical museum. Just reading the labels on the exhibits is tiring enough, not to say reading the explanation as well under a tight schedule, and even less can be said about appreciating the exhibits. Those who are at first interested in the exhibits do not feel satisfied with this cursory trip. Those who may get interested in the exhibits get turned off. Those who are not interested in the exhibits to begin with will find them even more disgusting.

The reasons for packing so many topics in a course can only be: (a) We do this out of a kind of “good conscience” for the subject. How can somebody claim to have studied this course without knowing such and such a theorem? (b) We do this out of a kind of “sense of responsibility” for the student. What would happen if the student needs to use such and such a theorem in a subsequent course? This kind of “good conscience” and “sense of responsibility” can easily inflate, particularly when the course is the teacher's favourite subject. Thus the teacher gallops along in a fast pace and students have a hard time in catching up or some of them simply give up or try to cope with it by rote learning, thus making no sense of it. The end result is the same — it does nobody any good! It is a relatively minor matter not to know this theorem or that, but far more disastrous to kill students' interest in the subject and worse yet to breed in them a distorted view of mathematics. Not only does this mistaken impression affect their learning process, it will be transmitted to the next generation when later some of these students become teachers. In fact, both (a) and (b) neglect one point, viz. what we want to teach are the students, not just the mathematical theorems. We should be concerned about whether they can learn something from the course, not just what they should learn in the course. Somebody (Halmos?) has once joked by saying that since students can retain about 40% of what they are taught, we need only increase the taught content by 150%! But every teacher realizes that it is not to be counted as a success if ten theorems are taught and students only know those ten theorems. But this is not to say that we can ignore the knowledge content even though our concern is on the nurturing of the general quality and ability of students, because we do need the knowledge content as a kind of media in education. Besides, one aim of a general education is to enable one to learn how to cope with a complicated mass of information and knowledge, to analyze and synthesize them, to assimilate them and to make good use of them.

4. Linear Algebra As a Case Study

We will elaborate on the theme "less is more" through a first course in linear algebra as an example. (Note that we are using the course to illustrate a teaching philosophy rather than to discuss its syllabus. It should also be emphasized that this is a first introductory course.)

Although students may not have heard about terms such as 2-dimensional or 3-dimensional vector spaces nor seen notations such as $\mathbb{R}^2$ or $\mathbb{R}^3$ in schools, they would certainly have acquired in some form geometric knowledge of entities in these spaces. Some teachers may think that it is simple and natural to generalize these concrete cases in $\mathbb{R}^2$ or $\mathbb{R}^3$ to that in $\mathbb{R}^n$ and even to that in an abstract vector space. Although they themselves might have spent much time and effort in grappling with this problem when they were at that age, they forget about it because they had succeeded in doing that and hence regard it as straightforward. Thus they introduce the definition of a vector space in a flash, explain quickly some basic properties about vectors in a vector space, then follow by a sequence of new notions in succession, such as linear independence, linear dependence, basis, dimension, etc. They want to dispose of such simple preliminaries as quickly as possible, since a host of much more interesting topics are waiting to be discussed, such as linear transformation, matrix, rank, eigenvalue, eigenvector, etc. But this quick march at the initial stage quickly leaves a lot of students behind. Some simply give up and some try to cope with it by rote learning.

Actually, to a novice it is quite some step to go from $\mathbb{R}^2$ or $\mathbb{R}^3$ to $\mathbb{R}^n$, then quite another step to go from $\mathbb{R}^n$ to an abstract vector space. None of these steps is trivial. The second step is even harder and deeper since it involves looking at the situation from an axiomatic viewpoint, which is a novel experience for many beginning undergraduates. Using the description of Guershon Harel and David Tall, [2,3] we can say that the first step belongs to expansive generalization, while the second step belongs to reconstructive generalization. There is still a trace to follow in imagining what $(a_1,\ldots,a_N)$ means when one understands what (a_1, a_2) or (a_1, a_2, a_3) means. But to make the second step one cannot simply try to add onto what one already knows; rather one must reconstruct what one already knows. If one cannot achieve that, one can only accept the piece of new knowledge by rote. Even if one can recite the new definition readily, that piece of new knowledge does not merge with the knowledge acquired in the past and hence has not become part of the knowledge structure of the learner, resulting in what is known as a disjunctive generalization. It is hard to retain information through rote learning, not to mention to know how to make use of it with facility. If we realize this point, we will see that it is not effective to attempt to transmit as much knowledge as possible in an assigned slot of time, especially not so at the initial stage. We should let students see more examples; better yet examples students came across in schools (e.g. system of linear equations and their geometric interpretations) or in other subjects (e.g. "linearization" is a basic theme in calculus, and differential equation is one of the sources of linear algebra in its historical development). By so doing we allow students to build up their own knowledge structure unhurriedly. We can induce students to come up with the notion of a vector space by showing them different "linear" problems. We can lead students see why an axiomatic viewpoint is advantageous. When the time is ripe, the notion of vector space will sound much more natural and even fall out by necessity! On the surface we have taught less and more slowly, but students have learnt more. One main difference between school mathematics and undergraduate mathematics to which many students are at first not accustomed is the emphasis undergraduate mathematics pays to definitions and proofs. Hence it is worthwhile to invest more time on a definition (not just the statement of it). Let us take the definition of linear independence as an example. Most textbooks give a neat definition like the following: N vectors $x_1,\ldots,x_N$ are said to be linearly independent if there do not exist N scalars $a_1,\ldots,a_N$, not all zero, such that $a_1x_1 + \ldots + a_Nx_N = 0$. As it is neat it is also mysterious to a novice. We should take the time to expand on it. What it tries to describe is the fact that none among the vectors $x_1,\ldots,x_N$ can be written as a linear combination of the remaining N- 1 vectors. Geometrically, that means the span of any N- 1 vectors cannot capture the remaining one — that one is "sticking out" from the space spanned by the other vectors. (We may add a caveat here about individual means to build his or her knowledge structure. Helpful as geometric thinking may be, some people may prefer to think algebraically or otherwise. As teachers we can try to facilitate this building process by exposing students to various means, but we should not dictate by which means. Students have to do that by themselves.) But if the neat definition is logically equivalent to what we just said, then why use that neat definition? It has an advantage for the purpose of checking. Given say, x_1, x_2, x_3, to be checked whether they are linearly independent or not, according to what is said above, one would have to check if x_1 can be written as a linear combination of x_2, x_3, then check if x_2 can be written as a linear combination of x_1, x_3, then finally check if x_3 can be written as a linear combination of x_1, x_2. But according to the neat definition one need only check whether $a_1x_1 + a_2x_2 + a_3x_3 = 0$ has a nontrivial solution (a_1, a_2, a_3) or not, and that can be reduced to a mechanical calculation. Getting clear and seeing through the definition of linear independence will facilitate the grasp of new concepts to come later, such as dimension and rank.

"Less is more" does not mean cutting out 1/4 or 1/3 or 1/2 of the course material per se. We must pay attention to the pruning. Although the course material is less, the story along with its climax has to remain. For instance, in a first course in linear algebra, a climax to be retained is the notion of eigenvalues and eigenvectors and some of its applications. If we simply cut out without planning, it may turn out that we set up the stage (all those basic notions in linear algebra) but put on no show (the structure of a linear transformation). What will the audience think? Conversely, if we can let the audience know what the show will be, maybe they will be more patient and motivated to watch the setting up of the stage. But if we want to apply the theory of eigenvalues and eigenvectors to explain in full the theory of canonical forms, time constraint may force us to go through it in a galloping pace. One possible way is to discuss in detail one special case, which though much simplified technically, still contains the basic idea. I would suggest, in this case, to discuss the case when all the N eigenvalues are mutually distinct. As William Blake said in his poem: 'To see a world in a grain of sand and a heaven in a wild flower '[4], this special case contains the basic idea and illustrates well certain basic techniques. Those really motivated students will definitely not be content with this special case. That provides an excellent opportunity for interaction between students and the teacher in further discussion after class. Pruning the coverage will not hinder the growth of the mathematically inclined students. Covering the full theory of canonical forms in the name of completeness will likely cause the majority of students not to see the basic idea at all but only learn a few terms by rote, just to be forgotten after the examination.

5. Less Is More

Besides this seemingly passive argument on the shortcoming of "more is less" there is an active side to "less is more". Even if students can cope with the material, teaching less will still make them learn more.

(1) Although a person's memory can store a vast amount of information, one cannot handle too much information when one concentrates on a particular portion in order to synthesize or analyze them in an organized manner. At the same time as one accumulates knowledge one must reorganize and condense the knowledge so gained.

To be able to do this well one has to rely on a "mathematical taste" which judges the level of significance in the knowledge structure. A student without this "mathematical taste" will see everything as equally important, which means nothing is important! Without a focus, the student would try to commit everything to memory. Without noticing the central ideas, the student does not know how to use what is learnt with facility. If we do not teach that much stuff but concentrate on a coherent theme, students would not be distracted by so many "frills" and can see more clearly the central ideas, thereby gaining a "mathematical taste" as they proceed. "Frills" can be left for further probe to those who are motivated to do so. If a student does not know how to make use of the library but merely regards it as a quiet place for reviewing the class notes, that is a great pity!

(2) If the taught content is made lighter and the number of tests is lessened, then students can have more free time and wider room to turn over what they learn in their minds, to build up their own knowledge structure and to acquire a global view of the subject, in one word to make sense of what they learn. It is not hard to cram a course with stuff, but much more difficult to leave room to induce students to make use of this freedom wisely. As a Chinese painter once commented, 'What has an outer appearance is not actually real; emptiness is the hardest to depict.' Students do needs this "emptiness" for their growth.

(3) If the amount of "hard knowledge" to be stuffed into students is decreased, then we can devote more time to improve their communication skill. At present many students are weak in logical reasoning and in language capacity. This reveals a muddled mind which lacks the necessary "hygiene in thinking". They write down anything that comes to their minds, disconnected, disorganized and perhaps irrelevant pieces. One possible reason for this bad habit is the examination strategy they have adopted since their school days — write down everything you can remember, for you will score certain marks for certain key points (even if these key points are not necessarily presented in a correct logical order!) and the kind-hearted examiner will take the trouble to sift the wheat from the chaff! They take with them this bad habit into the

university. Not only is the exposition illogical, sometimes it does not even make one coherent collection of sentences! Even worse, some take advantage of this confusion to write down the hypothesis at the beginning and the conclusion at the end, write some irrelevant things in between and suddenly come up with "therefore it follows that ..."! André Weil has commented, 'Rigour is to the mathematician what morality is to man.' We should make students aware of the moral value of mathematical rigour. They should distinguish between a casual guess, a conjecture and a theorem. They should know what they say or write down. Francis Bacon said, 'Reading maketh a full man. Conference maketh a ready man. Writing maketh an exact man.'[5] This applies to education in general, so also to mathematics education in particular. By teaching less stuff there will be more time to train students in reading, talking and writing mathematics.

6. Epilogue: Aims of Mathematics Education

Finally I must emphasize the (broad) aims of mathematics education in the three aspects of ability, knowledge and wisdom. An eighteenth century Chinese scholar, Yuan Mei, once said (but referring to a literary context), 'Knowledge is like the bow, ability like the arrow; but it is wisdom which directs the arrow to bull's eye.' These three aspects correspond to the three aims found in most curriculum documents: (i) training of the mind, (ii) transmission of technical knowledge, (iii) awareness of the cultural aspect [6]. Just (ii) alone constitutes mathematics education in the narrow sense. By combining (i), (ii), (iii) we attain mathematics education in a broad sense. In this era of mass education, mathematics education at all levels should try to achieve these aims in the broad sense. Mathematics is not viewed merely as a practical tool, but through its teaching we wish to attain a broader educational function, which includes extending mathematical thinking to general thinking, bringing up an effective working habit and healthy attitude of study, generating a regard for learning in general through an appreciation of mathematics. A teacher should not just feel concerned about how many theorems or formulae a student knows, but should care more about how to motivate students and how to arouse their interest and curiosity, how to make the curriculum more relevant to students' experience (in their daily lives as well as in connection to what they have learnt), to excite the potential of students and help them grow by themselves, to nurture their logical reasoning and critical thinking, to let them appreciate the cultural aspect of mathematics. (For some related discussion see also [7].) This is not achieved by mere transmission of knowledge. Albert Einstein once jokingly said, 'Education is that which remains, if one has forgotten everything learned in school.'[8] Of course we cannot interpret this literally, but this quotation pushes to the extreme the spirit of "less is more". It reminds me of a saying of the ancient Chinese sage Lao Zi, 'The greatest fullness seems empty; yet its use cannot be exhausted.'[9] If we really can achieve "less is more" in our teaching, our students will benefit for life!

References

[1] Einstein, A., 1954, *Ideas and Opinions by Albert Einstein*, edited by C. Seelig (New York: Crown Publishers Inc.), p.67.

[2] Harel, G., Tall, D.O., 1991, The general, the abstract and the generic in advanced mathematical thinking, *For the Learning of Mathematics*, **11(1)**, 38-42.

[3] Tall, D.O., 1991, The psychology of advanced mathematical thinking, in *Advanced Mathematical Thinking*, edited by D.O. Tall (Dordrecht: Kluwer Academic Publishers), pp.3-21.

[4] Blake, W., 1979 (originally published 1905), *The Poems of William Blake*, edited by W.B. Yeats (London: Routledge & Kegan Paul), p.90.

[5] Bacon, F., 1937, *Essays, Civil and Moral and The New Atlantis*, edited by C.W. Eliot (New York: Collier & Sons), p.122.

[6] Siu, M.K., 1985, {(History of [(Mathematics)} Teachers], *Bulletin de l'Association des Professeurs de Mathématiques*, **354**, 309-319.

[7] Fung, C.I., Siu, M.K., 1994, Mathematics for math majors: Loss of its self-esteem, *Humanistic Mathematics Network Journal*, **9**, 28-31.

[8] Einstein, A., 1954, *Ideas and Opinions by Albert Einstein*, edited by C. Seelig (New York: Crown Publishers Inc.), p.63.

[9] Lao Zi, 1982 (originally published 1959), *Tao Te Ching*, translated by T.K. Ch'u (London: Unwin), p.71.

Man-Keung SIU

Department of Mathematics

University of Hong Kong

Hong Kong SAR

CHINA

email: mathsiu@hkucc.hku.hk