Mathematical World (or Worlds?) in the Context of HPM

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Without engaging in a detailed analysis of categorization of separate areas nor going into a deeper philosophical discussion we usually hold the conception that there is one single mathematical world, which is different from, though closely related to, the real world we live in. Experience in this mathematical world can appear so foreign to daily life that many people feel turned off by it. Even though many people are aware of the utility and power of mathematics they do not see any direct relevance of the subject to them. In both the historical and pedagogical aspects it may be worthwhile to look at the issue in a pluralistic way from different angles. Among others there are for instance a world of school mathematics, a world of tertiary mathematics, a world of daily life mathematics, a world of recreational mathematics, a world of mathematical proofs and reasoning, a world of mathematics in science and technology and a world of mathematics in humanities.

In fact, the history of mathematics affords some means to mirror these different worlds in the context of HPM to offer a fuller view of the subject, rather than like what is depicted in the ancient Indian fable of the proverbial elephant felt by the six blind men. "Though each was partly in the right, and all were in the wrong!"

Is mathematics a useful science? a vibrant science? an amusing science? a rigorous science? a heuristic science? an experimental science? a humane science? or even, is mathematics a subject in science or in arts?

This lecture related to Theme 4 (Mathematics and its relation to science, technology, and the arts: Historical issues and interdisciplinary teaching and learning) attempts to examine this issue through examples gleaned from the history of mathematics.

Talking about relation to science and technology one word naturally comes up in the mind, a popular word in the education arena these days, namely, STEM. As is well known, the word STEM is an acronym for Science, Technology, Engineering and Mathematics. Initially the acronym was SMET with the four subjects in that order. It was later changed to STEM on the ground that STEM sounds better than SMET! However, with M placed at the end, very often one finds that mathematics has been side-stepped in most educational activities and projects carried out under the name of STEM, and has been relegated to a less significant role mentioned in passing!

But the history of mathematics tells us a different story. We may as well coin a word ANEG instead of STEM, which stands for four great mathematicians who made great contributions to science, technology and engineering as well: Archimedes, Newton, Euler and Gauss.

In the Eastern world there was a historic figure that can be mentioned likewise, in the person of XU Guang-qi [徐光啟] of the Ming Dynasty, who was best known in the history of mathematics as the collaborator with Matteo Ricci in transmitting the first European mathematical text, the *Elements*, into China in 1607. In an official memorial he submitted to the Emperor in 1629 he gave a list of the Ten Categories of various applications such as weather forecast, irrigation, musical system, and so on, with a remark: "These ten categories are intimately related to the well-being of the people. I learn from *Zhoubi Suanjing* 'what made it possible for *Yu* to set the realm in order was what G*ougu* [the Numbers] engender'. All subjects with shapes and substances can be explained in terms of mathematics." This is STEM in early seventeenth century China!

STEM, as I see it, is not a school subject, still less just a collection of hi-tech products like robots and 3-D printers. STEM is a comprehensive awareness which integrates different subjects so as to learn and utilize knowledge in mathematics and in science, coupled with engineering means in step with modern technology, for the betterment of our lives. STEM embodies a spirit of exploration and a way of thought, combined with mathematical thinking and scientific spirit, to organize theories through experiments and observations in the search for knowledge and further innovation. My friend, Dr H. Y. Law, designed a clever logo of STEM in the form of a keyhole which stresses the key role of mathematics. It is meaningful to look at the world of M in STEM, and more generally at the multifarious aspects of the subject in relationship to other human endeavours.

The American mathematician Raymond Wilder offered in his book *Mathematics as a Cultural System* of 1981 an evolutionary model. Let me briefly sketch a picture based on Wilder's view and idea. There are many different host cultures throughout the ages in different parts of the world. Suppose we look at one such host culture, say the Chinese culture. Mathematics is a "subculture" of a host culture, in which there are the so-called internal (or hereditary) stress and external (or environmental) stress, which are respectively development in the "subculture", in this case mathematics, and the development in other "subcultures", both exerting influence to and from the host culture to mathematics. And then there is diffusion to and from other host cultures.

The German philosopher Oswald Spengler went even further to see in mathematics the exemplification of culture. In his famous book *The Decline of the West* of 1918 he gave a comparative study of many cultures, in which he noted that the histories of various cultures follow a similar pattern likened to the regular and predictable course of birth, growth, maturity and decay of a living organism, or metaphorically analogous to the seasons. He further noted that within each culture, certain basic attitudes, which are exemplified in different expression-forms (one of which is mathematics) give the key or clue to the history of the whole culture.

Spengler viewed mathematics in a rather wide scope. In Chapter 2 of his book (Meaning of Numbers) he wrote, "Nevertheless, mathematics — meaning thereby the capacity to think practically in figures — must not be confused with the far narrower scientific mathematics, that is, the *theory* of numbers as developed in lecture and treatise. […] The mathematical vision and thought that a Culture possesses within itself is as inadequately represented by its written mathematic as its philosophical vision and thought by its philosophical treatises. Number springs from a source that has also quite other outlets. Thus at the beginning of every Culture we find an archaic style, which might fairly have been called geometrical in other cases as well as the Early

Hellenic. Gothic cathedrals and Doric temples are *mathematics in stone.* […] Doubtless Pythagoras was the first in the Classical Culture to conceive number scientifically as the principle of a world order of comprehensible things — as *standard* and as *magnitude* — but even before him it had found expression, as a noble arraying of sensuous-material units, in the strict canon of the statue and the Doric order of columns."

When we learn mathematics in school and university we hold a traditional view of its method and content, comprising of theory, computation, algorithm, modelling, with proof and proving. These aspects should be seen as an interrelated and integrated whole like the *yin* and *yang*, each complementing and supplementing the other. But even at that stage we seem to live in three different worlds --- a world of school mathematics, a world of tertiary mathematics and a world of mathematics of daily life. Instead of going into a fuller discussion for lack of time let me just give one simple example, that on the use of language. When we talk about conversion of currencies we usually say something like 1 € = 8.748 HK$. If I have E € which amount to H HK$, how would I write down an expression to indicate the relationship between E and H? Is it $E = 8.748\ H$, or is it $8.748\ E = H$? It should be the latter, but isn't that look like the opposite of what we say in daily life? Indeed this may confound many school pupils who first encounter algebraic expressions. But to go on with the learning of mathematics we have to realize such difference in the language.

Four years ago in 2017 there appeared a paper on mathematics to be learnt in school in this 21st century. I like to draw your attention to one remark in the paper: "Our focus has mainly been on the practical value of mathematics in the world outside school. The goal of mathematics education, however, is also to prepare students for further education to which they add the importance of understanding and appreciating mathematics as a goal in and of itself."

To understand and appreciate mathematics we cannot avoid entering a world of mathematical proofs and reasoning, which is perhaps a main cause for discouragement to most learners because "mathematical thought concerning proof is different from thought in all other domains of knowledge, including the sciences as well as everyday experience". Again, for the lack of time I won't go into a discussion of it. However, I like to mention yet one more world that is less emphasized in traditional schooling, a world of recreational mathematics. Not only does recreational mathematics lead to significant mathematical results and applications in the past as well as today, it can also be put to good use in the teaching and learning of mathematics in the classroom.

In the very early days our ancestors tried hard to survive. They collected berries and fruits, they hunted, learnt to farm, to raise livestock and poultry and cultivated the land and built their homes. They were kept very busy, but that did not mean they had no spare time when they looked at the sky and studied the stars. Astronomy began. They also played with sticks and pebbles; they counted and engaged in thinking. Mathematics began.

There is a saying which is attributed to various authors, "We don't stop playing because we grow old; we grow old because we stop playing." There is another saying which elaborates upon this: "Groos [Karl Groos] well says that children are young because they play, and not vice versa; and he might have added, men grow old because they stop playing, and not conversely, for play is, at bottom, growth, and at the top of the intellectual scale it is the eternal type of research from sheer love of truth."

The famous Silk Road acted as the main trade route in Central Asia that established links between a cross-cultural mix of religions, civilizations and people of many different regions, and also enabled exchanges of learning and cultures of people of different races. According to the Danish historian of mathematics Jens Høyrup, a "Silk Road Community" of traders interacting in Antiquity along this combined caravan and sea route and its extensions, reaching from China to Cadiz, and encompassing in the Middle Ages the Mediterranean-Islamo-Indian trade network, with mathematical knowledge migrated through "camp fire riddles". Thus we find similar games and puzzles and riddle problems in different civilizations throughout the ages.

A very old puzzle attributed to Pythagoras is the Stomachion. How many different ways to assemble a square from the 14 pieces?

I fondly remember the first toy I had as an infant, the Tangram. It probably originated from the Banquet Table in the 12th century Song Dynasty, was in wide circulation by the 17th century Qing Dynasty, becoming a popular game in Europe by the 19th century.

With two sets of Tangram one can prove (a special case of) the Pythagoras' Theorem (called the *Gougu* Theorem in ancient China). Another interesting mathematical question is to ask: Which convex polygons can be formed from a set of Tangram? (Answer: 13).

One well-known three-dimensional analogue of Tangram is the SOMA Cube, a puzzle the Danish polymath Piet Hein created in 1936 while attending a lecture on quantum mechanics by Werner Heisenberg! There are 240 different ways to assemble the cube, disregarding rotation or reflection (obtained by hand "one wet afternoon" in 1961 by John Conway and Michael Guy). Here is a much simpler mathematical question: Assemble the configuration as shown. (Hint: How many pieces must be used? Which pieces?)

In fact, the idea of dissection and re-assembling was a well-known technique in ancient Chinese mathematics, which was cleverly and skillfully employed to explain many geometric results. As one example let us look at this recruitment advertisement in a Dutch magazine dated April 27, 1999. One of the puzzles is a famous problem with a long history of transmission between different parts of the world. The original problem appears as Problem 13 of Chapter 9 of *Jiuzhang Suanshu* [九章算術 Nine Chapters on the Mathematical Art]: "Now given a bamboo 1 *zhang* [丈] high, which is broken so that its tip touches the ground 3 *che* [尺] away from the base. Tell: what is the height of the break?" The problem also appeared (with different data) in the Indian mathematical treatise Bhāskara's *Lilavati* of the 12th century, and also in the text of the Italian mathematician Filippo Calandri of the 15th century.

The Chinese mathematician YANG Hui [楊輝] gave his explanation of the solution in his book of 1261. In today's mathematical language familiar to a school pupil this problem can be readily formulated thus: Given a and $c + b$ in the right-angled triangle ABC, calculate b. A school pupil of today probably solves it like this: Let
$L = c + b$, then $c = L - b$. By the Pythagoras' Theorem we have from $a^2 + b^2 = c^2$ the expression $2Lb = L^2 - a^2$ so that $b = (L^2 - a^2)/2L$.

How was it done at the time, which was more than 2000 years ago? Note that this symbolic mathematical language that allows us to work with ease was developed as late as the 16th century, only about 500 years ago! Let us watch a clever "proof without words" reconstructed from the commentary of ancient Chinese mathematicians. The area of the rectangle is equal to that of the big square of side L minus that of the rectangular strip, which is a^2. But the area of the rectangle is given by $2bL$, hence again we arrive at the expression $2Lb = L^2 - a^2$, so that $b = (L^2 - a^2)/2L$.

Now, let me start with another riddle problem and pass on to STEM, namely, the so-called Radon-Kaczmarz Puzzle: You are given the three row sums, three column sums and the two diagonal sums (all equal to 15), fill in the nine cells in the 3 x 3 grid with suitable positive integers chosen from 1 to 9. One thing that may immediately come up in your mind is the 3 x 3 magic square, which is an old myth in the very ancient Chinese text *Yijing* [易經 Book of Changes], the so-called *Hetu* [河圖 Map of *He*] and *Luoshu* [洛書 Writing of *Luo*]. The Chinese mathematician Yang Hui [楊輝] formulated the solution (actually unique up to rotation and reflection) in his book of 1275: "2 and 4 are on the shoulder; 6 and 8 are at the feet. 7 is on the left; 3 is on the right. Put on 9 as the hat; put on 1 as the shoe. 5 is at the centre."

But in this puzzle we do not require all entries to be mutually distinct. There can be many solutions, like this trivial one, or this, or this, or this. There are altogether 41 solutions to the puzzle, falling into essentially 9 types with the remaining ones obtained via rotation or reflection. There are 9 unknowns and 12 equations. Why can't the given conditions pin down the solution? Even if in addition we are given the five *NW-SE* diagonal sums the answer is still not unique. If we are further given all the ten diagonal sums, then the answer will be unique. In general we are looking at a system of 8 linear equations with 9 unknowns, the rank of the coefficient matrix being equal to 7. Of the 9 unknowns there are 7 pivotal unknowns and 2 free unknowns. A question is: What sort of conditions will guarantee a unique solution if one exists?

Why the puzzle is named after the Polish mathematician Stefan Kaczmarz and the Austrian mathematician Johann Radon? It is because the puzzle is closely related to their works, Kaczmarz's algorithm (in 1937) for solving a system of linear equations and the Radon (Inverse) Transform (in 1917). This much is mathematics, which was later applied to medical imaging. In 1979 the Nobel Prize for Physiology or Medicine was awarded to Godfrey Hounsfield and Allan Cormack for their work in developing diagnostic technique of X-ray computed tomography. A patient is passed through the hole in a device which contains an X-ray source that rotates around the patient. As X-ray passes through the patient it is attenuated so that its intensity is reduced to a degree depending upon what material the ray passes through --- bone, muscle, an internal organ, or in the unfortunate event a tumor. The attenuated ray is detected on the other side as collected data. Basically we try to figure out the entries of a large grid knowing the row sums, column sums, diagonal sums, etc. Is this Mathematics? Science? Engineering? or Technology?

To achieve in science a demand on curiosity and imagination is called for. But by that alone it is not sufficient; our ancestors were good at that from very early times in creating a rich store of mythology. For science to emerge an additional demand on disciplined and critical thinking, with precision in mathematics as well as in words, is called for.

In an ancient Chinese mathematical text *Sunzi Suanjing* [孫子算經 Master Sun's Mathematical Manual] of the 4^{th} /5^{th} century we find a preface which sounds almost like mythology: "Mathematics governs the length and breadth of the heavens and the earth; affects the lives of all creatures; forms the alpha and omega of the five constant virtues; [and goes on and on …] "

A similar sentiment was expressed in the Western world by Galileo in *The Assayer* of 1623: "Philosophy is written in this grand book, the universe, which stands continually open to our gaze. But the book cannot be understood unless one first learns to comprehend the language and reads the letters in which it is composed. It is written in the language of mathematics, and its characters are triangles, circles, and other geometric figures without which it is humanly impossible to understand a single word of it; without these, one wanders about in a dark labyrinth."

By adopting this view Galileo made a significant step forward in modern science by switching the focus from trying to answer "why" to "how (much)", that is, from a qualitative aspect to a quantitative aspect. To keep on asking "why" we may end up in speculative discussion but find it impossible to arrive at an ultimate source of reason. By asking "how (much)" we have a feeling that we are in grasp of a better understanding because we can predict and carry out an experiment to check if the theory agrees with the observed facts.

Philosophers would enquire further. The physicist Max Tegmark goes so far as to ask, "But why has our physical world revealed such extreme mathematical regularity […] we'll explore […] a crazy-sounding belief of mine that our physical world not only is *described* by mathematics, but that it *is* mathematics, making us self-aware parts of a giant mathematical object." I am not as philosophical and will not go as far as that, but I do believe that we are living in a world (or worlds?) of mathematics.

A triumphant page in the history of science of the late 19th century depicted how science progressed. This page is the famous equations developed by James Clerk Maxwell with a beautiful and integrated blending of mathematics (differential equations) and science (waves), thereby unifying three physical phenomena --- electricity, magnetism and light. This page of powerful mathematical study supplemented with strong physical reasoning by Maxwell was preceded by a brilliant page of experimental exploration supplemented with intuitive and figurative reasoning by Michael Faraday and followed by a deep theoretical investigation based on simple physical hypothesis by Albert Einstein.

This triumphant page in physics of the late 19th century illustrates a standard pattern of the three stages of progress in science: from basic science to research and development, then to technology, in this case respectively Maxwell's equations in 1865, Hertz's discovery of the electromagnetic wave in 1886, then Marconi's invention of telecommunication in 1896.

I like to illustrate the indispensable role mathematics plays in STEM by borrowing from an investigation by Galileo, namely, that of a projectile motion: What is the trajectory of a projectile? According to traditional (Aristotelian) theory, the projectile will shoot up in a straight line until the force vanishes so that the object falls straight downward. But in reality the projectile does not look like that at all! In the famous book *Dialogues Concerning Two New Sciences* of Galileo we find a dialogue between the fictitious characters Salviati and Simplicio: "Imagine any particle projected along a horizontal plane without friction; […] the moving particle, which we imagine to be a heavy one, will on passing over the edge of the plane acquire, in addition to its previous uniform and perpetual motion, a downward propensity due to its own weight; so that the resulting motion which I call projection, is compounded of one which is uniform and horizontal and of another which is vertical and naturally accelerated." Salviati then announced a theorem: *A projectile which is created by a uniform horizontal motion compounded with a naturally accelerated vertical motion describes a path which is a semi-parabola*." He mentioned that the ancient Greek mathematician Apollonius introduced the parabola and other conic sections, and proceeded to derive some mathematical properties of a parabola required for explaining a projectile motion.

You can see that by physics alone, perhaps with a little bit of mathematics in dressing up the formulation, one can arrive at a precise description of the projectile, but isn't it amazing to see that the description fits so well with some geometric object that was studied by the Greeks about two millennia before, namely, the intersecting curve of a cone by a plane parallel to its generator? The next step in our discussion of this topic will show that mathematics is indispensable. Now that we know the range of the projectile, we would be interested to know when this range is maximal. We need mathematics to find this out. The angle of elevation of the projectile affects the range. The range is maximal when the angle of elevation of the projectile is at 45 degrees. But an athlete in shot-put won't throw the shot lying on the ground but standing up, which causes some variation in the computation. An angle of elevation at 45 degrees may not make the range maximal. What about if one throws a ball standing on a slope? All these require an understanding of the mathematics involved, not just the physics.

Throughout history we witness a beautiful blending of arithmetic and geometry, or numbers and shapes. The 17^{th} century Irish scientist Robert Boyle once said, "Arithmetic and geometry, those wings on which the astronomer soars as high as heaven."

Are you familiar with the world of arithmetic? Having been immersed in it from primary school all the way to secondary school most people will not feel unfamiliar with it, even though they may find it dry and are not drawn to it! Are you familiar with the world of geometry? You are living within a (3-dimensional) world of geometry. It is all around you, but do you feel comfortable with it?

What do you see in this picture? Two triangles? Now, what do you see? Why can we depict a three-dimensional situation by a drawing on a two-dimensional piece of paper? How do you interpret what you see? Light from an object hits the retina, which is like a two-dimensional piece of light-sensitive paper which turns the light into electrical signals to be processed by your brain into a three-dimensional image you see. So we feel comfortable with our three-dimensional world. What about a four-dimensional world? See how you feel about the next question?

What is the radius of a circle touching four mutually touching circles on a plane? The answer can be readily obtained through the Pythagoras' Theorem. What is the radius of a sphere touching eight mutually touching spheres in space? This can still be readily obtained through the Pythagoras' Theorem. It looks like the same goes for the higher-dimensional cases. Let us look at the calculated figures.

Radius of the *n*-ball touching all those *n*-balls situated at the corners of the *n*-cube with a side of length 4 = $\sqrt{n} - 1$

n	$\sqrt{n} - 1$
2	0.4142•••
3	0.7320•••
4	1
5	1.2360•••
6	1.4494•••
7	1.6457•••
8	1.8284•••
9	2
10	2.1622••• > 2

The *n*-ball in the middle of the *n*-cube pokes out of the *n*-cube, when $n > 9$! How can that be? Well, it is! In fact, many people, including myself, are not really that familiar with a simpler object, the four-dimensional cube, sometimes called a tesseract (or hypercube).

One way to visualize a 3-D object is to look at its slices in 2-D to piece up a picture in mind. (Remember the computed tomography technique?) Similarly this can be done for an object in the *N*-D case, $N > 3$ (stepwise down to the 2-D case). Naturally, it is a much harder process.

Here is how a tesseract can be visualized by its 3-D slices.

A second way to visualize a 3-D object is to look at its net in 2-D, again, to piece up a picture in the mind. To pave the way, let us look at the net of a cube. One interesting question is: Is this a net of a cube? There are altogether only 11 different nets of a cube. Similarly this can be done for an object in the *N*-D case, $N > 3$ (with the net in an $(N-1)$-dimensional world). Naturally, it is even harder to imagine!

The famous 1954 painting by Salvador Dalí called *Crucifixion* (*Corpus Hypercubus*) provides such an illustration. The crucifix is a 3-D net which folds up to form a 4-D hypercube.

A pair of worlds closely related to the world of arithmetic and geometry is the world of continuous and discrete mathematics. The persistent tension between the two worlds filled up a long and intriguing chapter in the history of mathematics and science.

The standard story goes like this: The early Pythagoreans based their mathematics on commensurable magnitudes, but their discovery of incommensurable magnitudes dealt them a blow which shook the foundation of their mathematics. An early Greek belief (*ca* 6th/5th century B.C.E.) was: All Is Number. What was at stake after the discovery of incommensurable magnitudes (*ca* 430 B.C.E.) was the theory of proportion and with it the theory of similar triangles! Eudoxus' theory of proportion (*ca* 370 B.C.E.) came to the rescue, and was written into Book V of Euclid's *Elements*.

A reconstructed story from Wilbur Knorr and David Fowler goes like this: Incommensurability did not cause a foundational crisis for the early Greeks. Early Greek mathematicians worked with several different concepts of "ratio", the most significant concept being that of a*nthyphairesis*. In between the work of the early Pythagoreans and that of Eudoxus there was a page of an unsuccessful attempt of the anthyphairesis theory *ca* 400 B.C.E. Book II and Book X of *Elements* contain a trace of an anthyphairetic investigation of quadratic ratios. Here is a summary of this episode.

Eudoxus' theory of proportion in the first half of the 4th century B. C. E. led to Richard Dedekind's theory of irrational numbers in the latter part of the 19th century, while the anthyphairesis theory of the early 4th century B. C. E. led to the theory of continued fractions in the 16th/17th century. No wonder the great French mathematician Henri Poincaré once said, "If arithmetic had remained free from all intermixture with geometry, it would never have known anything but the whole number. It was in order to adapt itself to the requirements of geometry that it discovered something else."

Our ancestors started with discrete mathematics knowing only the whole numbers, later they came up with fractions, then irrational numbers and moved into the realm of continuous mathematics. A turning point came when René Descartes introduced the idea of a variable, which ushered in dynamic motion and necessitated the invention of the differential and integral calculus. Then came the epoch-making contribution of Isaac Newton in making use of mathematics to explain the physical laws of Nature, put in this way by the early 18th century

English poet Alexander Pope: "Nature and Nature's laws lay hid in night: God said, Let Newton be! And all was light."

With the immense success of differential equations it seems that the realm of real numbers gained an upper hand in science instead of the realm of whole numbers as in the ancient world with the Pythagorean dictum of "All is number". However, history proceeds in a more intricate manner. Two years ago (2019) it was the 150th anniversary of the publication of the Periodic Table by Dimitri Mendeleev (in 1869), in which chemical elements were classified in increasing order of atomic weights --- whole numbers! Predictions and discoveries of new elements followed, as well as attempts to explain some anomalies in the initially designed Periodic Table. This important event in science opened up a new page in modern chemistry and modern physics — a fascinating page of "Pythagoreanism" with the theory of atomic structure and subsequently quantum mechanics emerging in the early 20th century.

It is now time to end the lecture by going back to the P of HPM, that is, about teaching. Teaching is to tell a story, a good story which arouses curiosity and excites imagination, a story about the long quest by the human mind for an understanding of the world around us in all respects.

At the beginning of the lecture we mentioned the household word STEM, which has undergone several modifications. It became STEAM with an A added for Arts, then STREAM by throwing in an R for Reading, then *i*STREAM with an *i* added for Information Science, or STREAiM with an Ai for both Information Science and Artificial Intelligence. By and by the word would cover all subjects taught in school! But why not modify it to THAMES with an H added for Humanities, which is sorely needed? With a humanistic and caring mind and by learning from the long history of the human race, its success and failure, its rise and fall, we would breed a feeling of humbleness and tolerance. Science is not almighty in taking command of everything. Instead, we should learn how to live in harmony with Mother Nature and with others.

Let me conclude with a remark I presented in a panel at the ICM-2014 seven years ago: "Mathematics is part of culture, not just a tool, no matter how useful this tool might prove to be. As such, the history of its development and its many relationships to other human endeavours from ancient to modern times should be part of the subject. […] Despite its importance, history of mathematics is not to be regarded as a panacea to all pedagogical issues in mathematics education, just as mathematics, though important, is not the only subject worth studying. […] It is the harmony of mathematics with other intellectual and cultural pursuits that makes the subject even more worth studying. In this wider context, history of mathematics has yet a more important role to play in providing a fuller education of a person."

Thank you!