Maths for ALL
(a script for two 8-minute videos)

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Let us first play a number trick. Scramble the digits of your phone number. Subtract the smaller number from the larger. Add 1, then add up the digits of the resulting number and keep doing this until a one-digit number is obtained.

What number do you obtain? Is it ONE?

With a little bit of mathematics we can explain this number trick. First of all, the difference between a number and the number with its digits scrambled is always a multiple of 9.

Here comes the crucial step! Add up the digits of a number and keep doing this until a one-digit number is obtained. If you begin with a number that is divisible by 9, then the number finally obtained is 9. If you begin with a number that is not divisible by 9, then the number finally obtained is not 9, but will be the remainder when the original number is divided by 9.

This crucial step is called the method of *casting out nines* [棄九法]. This principle was already known and recorded in the 3rd century by the Roman scholar Iamblichus residing in Syria. In the 10th century the Indian mathematician Aryabhata II applied the method to check calculation in doing arithmetic.

In 1613 LI Zhi-zao [李之藻] of the Ming Dynasty, in collaboration with the Italian Jesuit Matteo RICCI [利瑪竇], compiled the treatise *Tongwen Suanzhi* [同文算指], literally meaning “Rules of Arithmetic Common to Cultures”, which first transmitted into China in a systematic and comprehensive way the art of written calculation [筆算] that had been in common practice in Europe since the 16th century.

When I was in primary school we learnt this method of checking calculation in doing arithmetic, quite useful and interesting!

An explanation of the principle of *casting out nines* rests on the theory of congruence, which has a long and illustrious history. It started with a famous problem in an ancient Chinese mathematical text, Problem 26 in Chapter 3 of *Sunzi Suanjing* [Mathematical Manual of Master Sun 孫子算經] of the 4th/5th century: “There are an unknown number of things. Counting by threes we leave two; counting by fives we leave three; counting by sevens we leave two. Find the number of things.”

A method for solving this kind of problems involving a system of linear congruence equations was devised and extended to a general algorithm known as the *Dayan Qiuyi Shu* [*Great Extension Art of Searching for Unity* 大衍求一術] by the Chinese mathematician Qin Jiu-shao [秦九韶] in his mathematical treatise *Shushu Jiuzhang* [Mathematical Treatise in Nine Sections 數書九章] of 1247.

In 1801 the German mathematician Carl Friedrich Gauss, at the age of twenty-four, independently established a much more encompassing theory of congruence in his important and fundamental mathematical treatise *Disquisitiones Arithmeticae* [Arithmetical Investigation].

What you have just seen is not merely a number trick. It has important applications in what is known as a check digit. For instance, the last digit in the International Standard Book Number (ISBN) and the digit within bracket of your Hong Kong Identity Card are check digits to ensure a more reliable transmission of data electronically.

Well, mathematics is such a subject.

(1) It makes sense where results can be explained and can be understood.
(2) It has a long history filled with human wisdom from people in all parts of the world.

(3) It is useful to many areas of human endeavour.

Mathematics is therefore always an integral part of any curriculum in both the East and the West from ancient to modern times. In ancient Greece mathematics formed an important part of the curriculum for higher education in Plato's Academy. This group of four subjects was later named by the Roman philosopher Boethius of the 6th century as the Q*uadrivium*, comprising arithmetic and logistic, plane geometry and solid geometry, astronomy, harmonics (music theory). In the text *The Republic* of Plato, the foremost Greek philosopher in the 4th century B.C.E., he said, "When they reach thirty they will be promoted to still higher privileges and tested by the power of Dialectic, to see which can dispense with sight and the other senses and follow truth into the region of pure reality."

Another Greek scholar of the 5th century Proclus said of the term "mathematics" in his commentary on the mathematical classic *Elements* of Euclid which was compiled in around 300 B.C.E., "Its name [μαθηματική] thus makes clear what sort of function this science performs. It arouses our innate knowledge, awakens our intellect, purges our understanding, brings to light the concepts that belong essentially to us, takes away the forgetfulness and ignorance that we have from birth, sets us free from the bonds of unreason; […]"

In the Chinese classic text *Zhou Li* [Rites of Zhou 周禮] that was probably compiled by the 3rd/4th century B.C.E. we find the *Liuyi* [Six (Gentlemanly) Arts 六藝]: *li* [Rites 禮], *yue* [Music 樂], *she* [Archery 射], *yu* [Charioteering / Horsemanship 御], *shu* [History (Writing) 書], *shu* [Mathematics (Arithmetic) 數], where mathematic is one of the six subjects.

In the medieval time in Europe, the *Quadrivium* (arithmetic, geometry, music, astronomy) and the *Trivium* (rhetoric, dialectic, grammar) together formed the traditional academic programme known as the seven liberal arts that was regarded as a well-rounded education for a free person, in Latin *liberalis,* to acquire.

The English philosopher Francis Bacon of the 16^{th} century wrote in his essay *Of Studies*, "Histories make men wise; poets, witty; the mathematics, subtle; natural philosophy, deep; moral, grave; logic and rhetoric, able to contend."

The German philosopher Oswald Spengler went even further to see in mathematics the exemplification of culture. In his famous book *Der Untergang des Abendlandes* [The Decline of the West] of 1918 he gave a comparative study of many cultures, in which he noted that the histories of various cultures follow a similar pattern likened to the regular and predictable course of birth, growth, maturity and decay of a living organism, or metaphorically analogous to the seasons. He further noted that within each culture, certain basic attitudes, which are exemplified in different expression-forms (one of which is mathematics) give the key or clue to the history of the whole culture.

The American mathematician Raymond Wilder offered in his book *Mathematics as a Cultural System* of 1981 an evolutionary model. Let me briefly sketch a picture based on the view and idea of Wilder. There are many different host cultures throughout the ages in different parts of the world. Suppose we look at one such host culture, say the Chinese culture. Mathematics is a "subculture" of a host culture, in which there are the so-called internal (or hereditary) stress and external (or environmental) stress, which are respectively development in the "subculture", in this case mathematics, and the development in other "subcultures", both exerting influence to and from the host culture to mathematics. And then there is diffusion to and from other host cultures.

Let us continue to look at the "subculture" of mathematics in the host culture of ancient/medieval China and sample a couple of examples in more mathematical detail.

What can be said about characteristic features of ancient/medieval Chinese mathematics? Evidenced in the choice of topics is a strong social relevance and pragmatic orientation [經世致用], and in the methods a primary emphasis on calculation and algorithms [算法]. However, ancient Chinese mathematics is not just a "cookbook" of applications of mathematics to mundane transactions. It is structured, though not in the Greek sense exemplified by Euclid's *Elements*. It includes explanations and proofs, though not in the Greek tradition of deductive logic. It contains theories which far exceed the necessity for mundane transactions.

Maybe we can summarize by saying that ancient Chinese mathematics displays a beautiful blending of arithmetic and geometry, or numbers and shapes [寓數於形, 表形以數. 數形結合, 雙翼齊飛.] Indeed, the English scientist Robert Boyle of the 17th century once said, "Arithmetic and geometry, those wings on which the astronomer soars as high as heaven."

The ancient Chinese mathematical treatise *Jiuzhang Suanshu* [Nine Chapters on the Mathematical Art 九章算術] was compiled between 100 B.C.E. and 100 C.E. Problem 15 of Chapter 9 of the treatise says, "Now given a right triangle whose *gou* is 5 *bu* and whose *gu* is 12 *bu*. What is the side of an inscribed square? The answer is 3 and 9/17 *bu*." The description of a method follows: Let the sum of the *gou* and the *gu* be the divisor; let the product of the *gou* and the *gu* be the dividend. Divide to obtain the side of the square."

There is an elegant and clever explanation of this formula given by the Chinese mathematician LIU Hui [劉徽] of the mid-3rd century in his commentary on *Jiuzhang Suanshu*, making use of the method of dissect-and-re-assemble. Let us watch this "proof without words" unfolding before our eyes!

The two rectangles have the same area. The area of one rectangle is given by ab, the other by $(a + b)\,x$, and so $ab = (a + b)\,x$. Hence, $x = ab/(a + b)$.

For a second example let us look at this recruitment advertisement in a Dutch magazine dated April 27, 1999. One of the puzzles is in fact a famous problem with a long history of transmission between different parts of the world. The original problem appears as Problem 13 of Chapter 9 of *Jiuzhang Suanshu*: "Now given a bamboo 1 *zhang* high, which is broken so that its tip touches the ground 3 *chi* away from the base. Tell: what is the height of the break?"

The problem also appeared (with different data) in the Indian mathematical treatise Bhāskara's *Lilavati* of the 12th century, and also in the text of the

Italian mathematician Filippo Calandri of the 15th century. The Chinese mathematician YANG Hui [楊輝] gave his explanation of the solution in his book of 1261 *Xiangjie Jiuzhang Suanfa* [A Detailed Analysis of the Mathematical Methods in the 'Nine Chapters' 詳解九章算法].

In today's mathematical language familiar to a school pupil this problem can be readily formulated thus: Given a and $c + b$ in the right-angled triangle ABC, calculate b. A school pupil of today probably solves it like this:

Let $L = c + b$, then $c = L - b$.
But $a^2 + b^2 = c^2 = (L - b)^2 = L^2 - 2Lb + b^2$,
so $a^2 = L^2 - 2Lb$, or $2Lb = L^2 - a^2$, or $b = (L^2 - a^2)/2L$.

How was it done at the time, which was more than 2000 years ago? Note that the advantage of working with ease by adopting this symbolic mathematical language was developed only as late as the 16th century, about 500 years ago!

Let us again watch a clever "proof without words" reconstructed from the commentary of ancient Chinese mathematicians.

The area of the rectangle is equal to that of the big square of side L minus that of the rectangular strip, which is a^2. But the area of the rectangle is given by $2bL$, hence again we arrive at $2Lb = L^2 - a^2$, or $b = (L^2 - a^2)/2L$.

Finally, I would like to say this. Many people do not need nor make use of a lot of mathematical knowledge in their workplaces. But mathematics is all around us; it is our cultural heritage, and can build up our rational thinking. Thus, Mathematics is for all！