MAN-KEUNG SIU

PROOF AND PEDAGOGY IN ANCIENT CHINA: EXAMPLES FROM LIU HUI'S COMMENTARY ON JIU ZHANG SUAN SHU

ABSTRACT. Through the discussion of several examples from Liu Hui's commentary on the ancient Chinese mathematical classics JIU ZHANG SUAN SHU this article attempts to illustrate the pedagogical implications embodied therein, mainly the aspects of proof in ancient Chinese mathematics and more generally the role of proofs in mathematics.

1. INTRODUCTION

Matteo Ricci (1552–1610), who was the most prominent Jesuit missionary in China at the time of the Ming Dynasty, and who collaborated with Xu Guang-qi (1562–1633) in rendering the first six books of Euclid's ELEMENTS into Chinese – the first Western mathematical text ever to be so translated – said in his journal [14, p. 476]: "nothing pleased the Chinese as much as the volume on the Elements of Euclid. This perhaps was due to the fact that no people esteem mathematics as highly as the Chinese, despite their method of teaching, in which they propose all kinds of propositions but without demonstration. The result of such a system is that anyone is free to exercise his wildest imagination relative to mathematics, without offering a definite proof of anything." This comment by Ricci on the complete absence of proof in Chinese mathematics indicates his lack of understanding of that mathematics' ancient tradition. In fact, even the Chinese scholars of that age did not realize the value of their mathematical legacy, which, by the time of the Ming Dynasty, was no longer preserved and nurtured [25, p. 1666]. Even in this century many Western historians of mathematics still share the view expressed in that comment by Ricci.

If one means by a proof a deductive demonstration of a statement based on clearly formulated definitions and postulates, then it is true that one finds no proof in ancient Chinese mathematics, nor for that matter in other ancient oriental mathematical cultures. In the words of Szabo [18, p. 1]: "Before the development of Greek culture the concept of deductive science was unknown to the Eastern people of antiquity. In the mathematical documents which have come down to us from these peoples, there are no theorems or demonstrations, and the fundamental concepts of deduction, definition, and axiom have not yet been formed. These fundamental

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concepts made their first appearance only with the Greek mathematics." But if one means by a proof any explanatory note which serves to convince and to enlighten, then one finds an abundance of proofs in ancient mathematical texts other than those of the Greeks. In any case, it is good to keep in mind what Wilder once said [23, p. 69]: "We must not forget that what constitutes 'proof' varies from culture to culture, as well as from age to age."

In this article I shall confine my choice of examples to ancient Chinese mathematics, indeed only to one single source, viz. the commentary by Liu Hui on JIU ZHANG SUAN SHU. Therefore, it will not be an in-depth scholarly study such as Berggren's paper on Islamic mathematics [2]. It is also not meant to be a documented refutation of the aforesaid view of many Western historians of mathematics. After all, ancient Chinese mathematics and Greek mathematics developed in their respective but different environments, with different styles and characteristics, so that it is as meaningless to argue which is superior to which as it is to argue whether the Chinese language is superior to the English language or vice versa. A critical comparative study of the two mathematical cultures is beyond me and, in any case, is not my concern here. (For a comparative study of one particular theorem, see [3].) My modest aim is to sample some proofs in the commentary by Liu Hui on JIU ZHANG SUAN SHU with a view to using them for the enrichment of the teaching of mathematics. A general theme is the interpretation of the role of proof in mathematics. An even more general theme is the nature of mathematical activities. (See [4, 5, 16].)

JIU ZHANG SUAN SHU (Nine Chapters on the Mathematical Art) is a very old Chinese mathematical text written probably between 100 BC and 100 AD. It is a collection of 246 mathematical problems on various topics, grouped into 9 chapters [1, 9, 13, 15]. The belief that these problems had been handed down through the ages was substantiated by an exciting archaelogical find in Hubei province in 1984, when a book written on bamboo strips bearing the title SUAN SHU SHU (Book on the Mathematical Art) was excavated [8, p. 12]. This book, dated at around 200 BC, exhibits a marked resemblance to JIU ZHANG SUAN SHU, including even the numerical data which appear in the problems! The format of JIU ZHANG SUAN SHU is typical of most ancient oriental mathematical texts: a few problems of one particular type along with the answers, followed by a description of the method used to solve them but without explanation or justification of the method. Later editions of JIU ZHANG SUAN SHU included commentaries by other mathematicians, most notably those by Liu Hui, who flourished in the 3rd century [24]. This article is not concerned with the finer details of the authenticity of these commentaries (interested readers can consult [20]). The commentaries on the examples given in the sequel are generally attributed to Liu Hui, and the original text can be found in [1]. A standard Western reference for the original is the German translation by Vogel [19]. The English quotations in this article were translated by the author. They may not meet the demands of a professional historian but, it is hoped, are adequate for our purpose. We are more interested in the pedagogical implication of these examples than in their historical examination.

2. CIRCULAR FIELD

Problem 32 of Chapter 1 says: "A circular field has a perimeter of 181 steps and a diameter of 60 and 1/3 steps. What is its area?" The answer is: "the area equals half the perimeter times half the diameter". This gives the correct formula for the area A of a circle with radius r, viz. $A = \pi r^2$. The data in Problem 32 imply the formula $C = 3d$ for the perimeter C of a circle with diameter d, i.e. they imply that the value of π was taken to be 3. We shall have more to say about this at the end of this section. The answer is accompanied by a commentary due to Liu Hui.

Let us see how Liu Hui explained the formula in his commentary, which can be summarized in 3 steps. (1) He stated that the area of an inscribed regular 12-gon equals 3 times the radius times one side of an inscribed regular 6-gon, then the area of an inscribed regular 24-gon equals 6 times the radius times one side of an inscribed regular 12-gon. (2) He claimed that "the finer one cuts, the smaller the leftover; cut after cut until no more cut is possible, then it coincides with the circle and there is no leftover". By this he meant that the excess of the circle over an inscribed regular polygon will become smaller and smaller as the number of sides is increased, and that the full circle is reached in the limit. The claim is accompanied by a passage which explains how the circle is sandwiched between an inscribed regular polygon and the same polygon with certain added pieces. (3) He gave an essentially general formula relating the area of an inscribed regular 3.2^k-gon and the perimeter of an inscribed regular 3.2^{k-1}-gon. Using the formula in (3) and the claim in (2) he concluded that the area of the circle is half the perimeter times half the diameter.

With the benefit of better notations and the additional knowledge we possess today we can rewrite (1), (2), (3) in more accurate form. (This will make a good class exercise.) Let A_n, C_n and a_n denote, respectively, the area, the perimeter and a side of an inscribed regular n-gon. (1) says

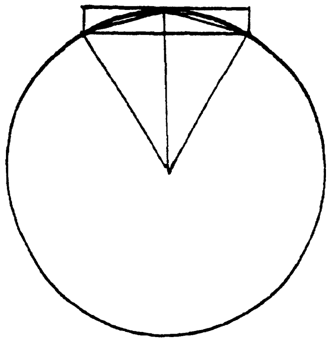

Fig. 1.

that $A_{12} = 3a_6 r = (C_6/2)r$ and $A_{24} = 6a_{12} r = (C_{12}/2)r$. More generally, $A_{2m} = (m/2)a_m r = (C_m/2)r$. The values of m are given by 3.2^k, with $k = 1, 2, 3, \ldots$. (2) says that A_{2m} tends to A as the limit as m increases indefinitely. Liu Hui even gave an estimate for A by noting that $A_{2m} < A < A_{2m} + (A_{2m} - A_m)$. The diagrams in the original commentary are lost, and we can only guess from the text what they looked like. The diagram accompanying our inequality probably looked like Fig. 1. (3) concludes from $A_{3.2^k} = (C_{3.2^{k-1}}/2)r$ that in the limit, $A = (C/2)r = (C/2)(d/2)$.

An even more interesting passage in the commentary follows. Liu Hui said: "In our calculation we use a more accurate value for the ratio of the circumference to the diameter instead of the ratio that the circumference is 3 to the diameter's 1. The latter ratio is only that of the perimeter of the inscribed regular 6-gon to the diameter. Comparing arc with the chord, just like the bow with the string, we see that the circumference exceeds the perimeter". This is apparent from Fig. 2. He continued: "However, those

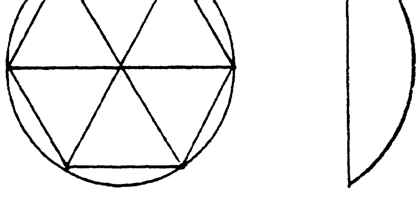

Fig. 2.

who transmit this method of calculation to the next generation never bother to examine it thoroughly but merely repeat what they learned from their predecessors, thus passing on the error. Without a clear explanation and definite justification it is very difficult to separate truth from fallacy." What follows is the famous calculation of π by Liu Hui. Basically, he computed $a_6, a_{12}, \ldots, a_{96}$ and hence, by what was discussed above, $A_{12}, A_{24}, \ldots, A_{192}$ to obtain $314 + 64/625 < A < 314 + 169/625$ (with radius equal to 10). This yields the value 3.14 for π [11]. We will not go into this calculation in this article. We only wish to emphasize how thorough and determined Liu Hui was to probe results handed down from his predecessors. In looking for proofs and explanations he extended the frontier of knowledge.

3. CIRCLE INSCRIBED IN A RIGHT TRIANGLE

Problem 16 in Chapter 9 says: "A right triangle has sides of 8 steps and of 15 steps. What is the diameter of its inscribed circle?" The answer is given, together with the method. The answer is: "compute the Xian from the Gou and the Gu, then add the three together and divide this sum into twice the product of the Gou and the Gu". Gou and Gu are, respectively, the shorter and the longer legs of a right triangle, while Xian is the hypotenuse. (The study of Gou-Gu was a prevalent theme in ancient Chinese mathematics and Gou-Gu is actually the title of Chapter 9 in JIU ZHANG SUAN SHU. Interested readers can consult [6, 10, 17].) The method gives the correct formula for the diameter d of the inscribed circle of a right triangle with sides a, b, c with c, the hypotenuse, viz., $d = 2ab/(a + b + c)$.

It is of some interest to examine how Liu Hui explained this formula by providing three different proofs. The first proof is by a beautiful (and "colourful") method of dissection, a method frequently employed by Liu Hui [22, 23]. If the original diagrams of his commentary were extant, they would form a set of useful visual aids. (We know from the text that Liu Hui used different colours to enhance understanding through visual images.) A "proof without words", displayed in Fig. 3, follows from his instructions on how to dissect the triangle into coloured pieces which are then reassembled to form a rectangle. (In the figure, a dotted region is supposedly coloured yellow, a shaded region is supposedly coloured crimson, and a plain region is supposedly coloured indigo.)

The second proof uses knowledge about proportional quantities, whose study is called Cui-Fen (this is also the title of Chapter 3 in JIU ZHANG SUAN SHU). From Liu Hui's commentary we know that he constructed

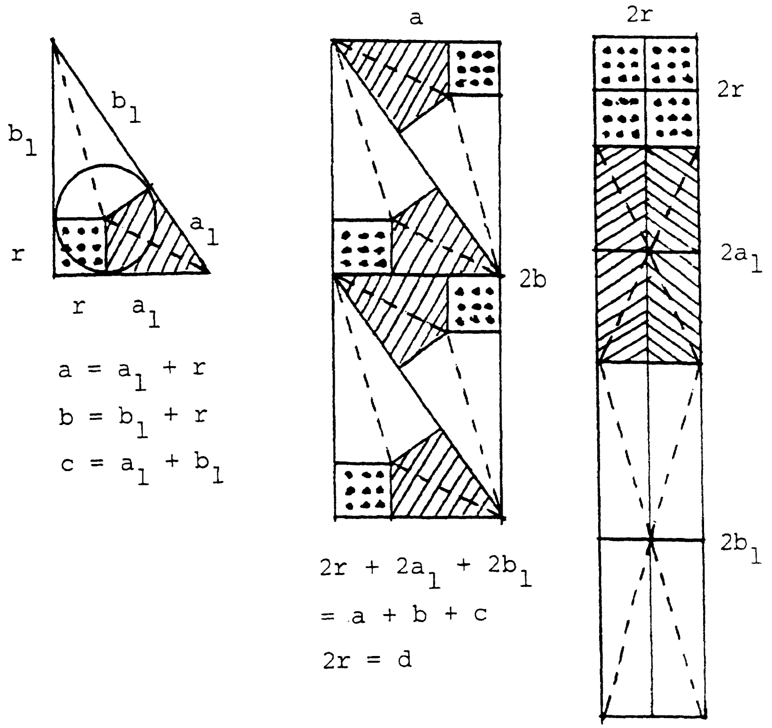

Fig. 3.

an auxiliary line segment to the figure, viz., a hypotenuse through the centre of the inscribed circle forming with the two legs two (similar) right triangles, each having one side equal to the radius r of the inscribed circle, as shown in Fig. 4. Since $a:b:c = DX:r:OX$, we have $b:(a+b+c) = r:(DX+r+OX) = r:(CX+OX) = r:(CX+XB) = r:a$. (Exercise: Show that $OX = XB$, which is not explained in this commentary.) Hence we have $r = ab/(a+b+c)$ and $d = 2r = 2a/(a+b+c)$.

The third proof is most interesting from the viewpoint of our discussion on the role of a proof. We shall say more about this at the end of this section. Strictly speaking, the proof is merely hinted at in the commentary, by the statement of four formulae: (i) $d = a-(c-b)$, (ii) $d = b-(c-a)$, (iii) $d = (a+b)-c$, (iv) $d = \sqrt{2(c-a)(c-b)}$. Let us do some guesswork. (i) to (iii) were probably obtained by noting that $c-b = a_1 - r = a-d$, $c-a = b_1 - r = b-d$ and $a+b = a_1 + b_1 + 2r = c+d$. These relations are obvious from Fig. 3. (iv) amounts to the formula $2(c-a)(c-b) =$

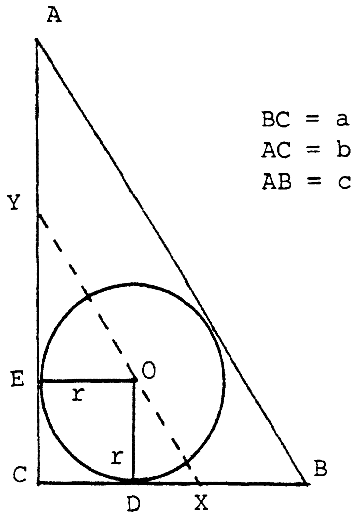

Fig. 4.

$(a+b-c)^2$. Actually, this formula is given in the method to Problem 12 in the same chapter. Problem 12 asks: "A door and a stick have unknown dimensions. The stick exceeds the width of the door by 4 feet, and exceeds the height of the door by 2 feet, but measures exactly the diagonal of the door. What are the width and height of the door?" In our notation, we are given a right triangle with sides a, b, c, with c the hypotenuse, and we want to express a, b in terms of $c-a$, $c-b$. The formula given in the method says that $a = \sqrt{2(c-a)(c-b)} + (c-b)$ and $b = \sqrt{2(c-a)(c-b)} + (c-a)$, i.e., $2(c-a)(c-b) = (a+b-c)^2$. Liu Hui again used a "colourful" method of dissection to prove the formula in his commentary. The original diagram reconstructed from the text would look like Fig. 5. (Dotted or shaded or plain regions are supposedly coloured yellow, crimson, and indigo, respectively. A black region is supposedly not coloured.) From the pictures we see that

$$(\text{indigo } L\text{-shape}) = (\text{square of side } a) - (\text{yellow square})$$

and

$$(\text{crimson } L\text{-shape}) = (\text{square of side } b) - (\text{yellow square}).$$

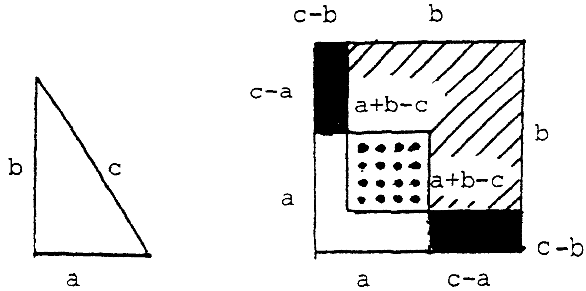

Fig. 5.

Hence

$$\text{(square of side } c) - 2 \times \text{(rectangle)} - \text{(yellow square)}$$

$$= \text{(square of side } a) + \text{(square of side } b) - 2 \times \text{(yellow square)},$$

i.e.

$$c^2 - 2 \times \text{(rectangle)} = a^2 + b^2 - \text{(yellow square)}.$$

But we also know that $a^2 + b^2 = c^2$. Hence (rectangle) = (yellow square). The rectangle has sides $c - a$ and $c - b$, while the yellow square has side $a + b - c$. Hence $2(c - a)(c - b) = (a + b - c)^2$.

Finally, to see the connection between (iv) and $d = 2ab/(a + b + c)$ we have to see why $(a + b + c)(a + b - c) = 2ab$. This is easy for us since the expression on the left-hand side is nothing but $(a + b)^2 - c^2 = a^2 + b^2 + 2ab - c^2$, which is the expression on the right-hand side (recall that $a^2 + b^2 = c^2$). Yang Hui (13th century) actually supplied a proof of this result that resembles the line of thought underlying Liu Hui's method of dissection [6, pp. 104–105]. This lends credibility to both formulae, because (i) is related to (iv) which, in turn, is related to another formula (established in Problem 12). Liu Hui was probably looking for "consistency [of the theorem] with the body of accepted mathematical results" [5, p. 70] in devising alternative proofs. This discussion also helps us to appreciate the role a proof plays in the enhancement of understanding. If the only role of a proof were verification, nothing would be gained by giving different proofs of the same theorem. But different proofs serve not merely to convince but also to enlighten.

4. VOLUME OF A SPHERE

Problem 24 in Chapter 4 says: "The volume of a sphere is 1644866437500 cubic feet. What is its diameter?" The answer is given, together with the method. The answer is: "diameter equals the cube root of (16/9 times the volume)". Assuming the ancients took 3 for π, we see that this formula gives the volume V of a sphere with diameter d as $V = (\pi/4)^2 d^3$. This is not correct, as the correct formula is $V = (\pi/6)d^3$. Liu Hui pointed out this error in his commentary.

He first explained how the error could arise. The ancients knew that the ratio of the area of a circle to that of its circumscribed square is $\pi:4$ (π taken to be 3). From this they inferred (correctly) that the volume of a cylinder to that of its circumscribed cube is $\pi:4$. If they thought (incorrectly) that the ratio of the volume of a sphere to that of its circumscribed cylinder is also $\pi:4$, then they would obtain the formula $V = (\pi/4)^2 d^3$. Liu Hui commented that the latter ratio must be different, because $\pi:4$ gives the ratio of the volume of the sphere to that of another object M which lies strictly inside the circumscribed cylinder. Hence the volume of the sphere is strictly less than $(\pi/4)^2 d^3$. With a strong sense of spatial visualization Liu Hui correctly described what M should look like, viz., the common portion of two cylinders of equal diameter placed with their axes perpendicular to each other, as shown in Fig. 6. He named M a Mou-he-fang-gai (literally, "box with a closely fitted square lid"). One can imagine M as a pile of squares with their sides continuously shrinking to zero from the middle on both sides. The pile of corresponding inscribed circles will then form the sphere. The claim that the ratio of the volume of the sphere to that of M is $\pi:4$ indicates that Liu Hui was well aware of a rudimentary version of the following principle: two objects have equal volume if their respective cross-sections at equal height have equal area. (This principle is usually

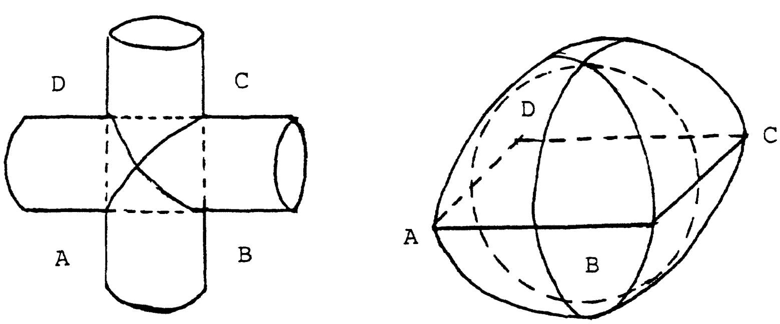

Fig. 6.

referred to in the West as Cavalieri's Principle and is attributed to B. Cavalieri (1598–1647) who stated it in 1635, but it was already explicitly stated by Zu Chong-Zhi and his son Zu Geng in the late 5th century [1, p. 121].)

Although Liu Hui successfully reduced the calculation of the volume of a sphere to that of a Mou-he-fang-gai, he could not determine the latter. It was left to Zu and his son to determine it completely by an ingenious application of the aforesaid principle. We shall not go into this fascinating piece of mathematics. (Interested readers can consult [11, 21].) What is more interesting for us is a passage by Liu Hui, made into an amusing English doggerel by Wagner [21, p. 72]:

"Look inside the cube
And outside the box-lid;
Though the diminution increases,
It doesn't quite fit.

The marriage preparations are complete;
But square and circle wrangle,
Thick and thin make treacherous plots,
They are incompatible.

I wish to give my humble reflections,
But fear that I will miss the correct principle;
I dare to let the doubtful points stand,
Waiting
For one who can expound them".

This attitude reminds us of a saying by A. Weil, a noted mathematician of our time: "Rigour is to the mathematician what morality is to man."

5. EXCESS AND DEFICIENCY

After three examples in geometry, perhaps we should see one example in another area, lest readers obtain the wrong impression that JIU ZHANG SUAN SHU is mostly concerned with the computation of areas and volumes. Problem 1 in Chapter 7 says: "A certain number of persons purchase an article. If each contributes 8 dollars, the excess is 3 dollars; if each contributes 7 dollars, the deficiency is 4 dollars. How many persons are there and what is the cost of the article?" A modern high school student will be able to solve this problem without much difficulty. Let there be x persons and let the cost be y dollars; then we have $8x - y = 3$ and $7x - y = -4$. Solving this system of simultaneous linear equations, we obtain $x = 7$ and $y = 53$. More generally, if the individual contributed amounts are m, m' with respective excess n and deficiency n', then

$mx - y = n$ and $m'x - y = -n'$ so that $x = (n + n')/(m - m')$ and $y = (mn' + m'n)/(m - m')$. The last two formulae are precisely what is written down in the method (in prose form) following several problems of the same type. Although simultaneous linear equations constitute the content of Chapter 8, it seems that the formulae in Chapter 7 were not obtained by their use. A remark in Liu Hui's commentary is illuminating: "Excess and deficiency make up the discrepancy in the total contribution from all persons, while the difference between the two individual contributed amounts is the discrepancy in the contribution from one person. Dividing the former by the latter, we obtain the number of persons." Risking the accusation of committing an anachronism by "misused hindsight" (from an historian's point of view) I would say that Liu Hui's explanation has a tinge of the modern concept of functional dependence, which helps one to understand the formulae better. The following table will illustrate this point. The seventh line locates the answer.

x	$mx(=8x)$	$m'x(=7x)$	difference
1	8	7	1
2	16	14	2
3	24	21	3
4	32	28	4
5	40	35	5
6	48	42	6
7	56	49	$7 = 3 - (-4)$
8	64	56	8

6. CONCLUSION

I hope that the four examples above convey the flavour of Liu Hui's mathematical reasoning. (A more detailed and scholarly account can be found in [7, 24].) Indirectly, they will convey the flavour of proofs in ancient Chinese mathematics. In the preface to his commentary, Liu Hui said [1, p. 4]: "I studied JIU ZHANG at an early age and perused it when I got older. I see the separation of the Yin and the Yang and arrive at the root of the mathematical art. In this process of probing I comprehend its meaning. Despite ignorance and incompetence on my part I dare expose what I understand in these commentaries. Things are related to each other through logical reasons so that like the branches of a tree, diversified as they are, they nevertheless come out of a single trunk. If we elucidate by prose and illustrate by pictures, then we may be able to attain conciseness as well as comprehensiveness, clarity as well as rigour. Looking at a part we will understand the rest." One can read into his message the belief in a balanced employment of rigorous argument and heuristic reasoning with the aim of achieving better understanding. It reminds us of a saying by G. Polya [12, p. 72]: "to prove formally what is seen intuitively and to see intuitively what is proved formally". This is a message worthy of being emphasized in the classrooms of today.

NOTE

I wish to thank Professor Abe Shenitzer for numerous stylistic improvements of this article.

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Department of Mathematics,
University of Hong Kong,
Hong Kong