Proof as a practice of mathematical pursuit in a cultural, socio-political and intellectual context

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Abstract

Through examples we explore the practice of mathematical pursuit, in particular on the notion of proof, in a cultural, socio-political and intellectual context. One objective of the discussion is to show how mathematics constitutes a part of human endeavour rather than stands on its own as a technical subject, as it is commonly taught in the classroom. As a 'bonus', we also look at the pedagogical aspect on ways to enhance understanding of specific topics in the classroom.

1. Prologue: Question 6 of the 29th IMO

Proof is, to some extent, as much an individualized activity as a social activity. It is an individualized activity in that a breakthrough or an igniting spark arises from the mental exertion of oneself, though sometimes aided or stimulated through the discussion with fellow mathematicians. It is a social activity in that a proof has to pass the scrutiny of other mathematicians in order to gain approval and acceptance by the mathematical community. Hence, we will begin with an example this author experienced in person. This example highlights the main function of a proof, which is to elucidate rather than just to verify. After that, we will explore, through four examples (in Sections 3,4,5,6), the practice of mathematical pursuit, in particular on the notion of proof, in a cultural, socio-political and intellectual context.

Question 6 of the 29th International Mathematical Olympiad, held in Canberra in 1988, reads:

> "Let a and b be positive integers such that $ab+1$ divides a^2+b^2. Show that $\dfrac{a^2+b^2}{ab+1}$ is the square of an integer."

A slick solution to this problem, offered by a Bulgarian youngster who received a special award for it, starts by supposing that $k = \dfrac{a^2+b^2}{ab+1}$ is **not** a square and rewriting the expression as an equation.

$$a^2 - kab + b^2 = k, \text{ where } k \text{ is a given positive integer} \qquad (*).$$

Note that any integral solution (a, b) of $(*)$ must satisfy $ab \geq 0$, or else $ab \leq -1$, and $a^2 + b^2 = k(ab + 1) \leq 0$, implying that $a = b = 0$ so that $k = 0$! Furthermore, since k is not a square, we have $ab > 0$, that is, none of a or b is 0. Let (a, b) be an integral solution of $(*)$ with $a > 0$ (and hence $b > 0$) and $a + b$ *smallest*. We may assume $a \geq b$ by symmetry. Regard $(*)$ as a quadratic equation with roots a and a'. Then $a + a' = kb$ and $aa' = b^2 - k$. Hence a' is also an integer and (a', b) is an integral solution of $(*)$. Since $b > 0$, we have $a' > 0$. But

$$a' = \frac{b^2 - k}{a} \leq \frac{b^2 - 1}{a} \leq \frac{a^2 - 1}{a} < a,$$

so that $a' + b < a + b$, contradicting the choice of (a, b)!. This proves that $\dfrac{a^2 + b^2}{ab + 1}$ must be the square of an integer.

Slick as the proof is, it also invites a couple of queries. (1) What makes one suspect that $\dfrac{a^2 + b^2}{ab + 1}$ is a square? (2) The argument by *reductio ad absurdum* should hinge crucially upon the condition that k is not a square. In the proof this condition seems to have slipped in casually so that one does not see what really goes wrong if k is *not* a square. More pertinently, this proof *by contradiction* has **not explained** why $\dfrac{a^2 + b^2}{ab + 1}$ must be a square, even though it **confirms** that it is so. (For a discussion on the cognitive and didatic aspects of students' difficulty with proof by contradiction, see (Antonini, Mariotti, 2008).)

In contrast let us look at a less elegant solution, which is my own attempt. When I first heard of the problem, I had a 'false insight' by putting $a = N^3$ and $b = N$ so that $a^2 + b^2 = N^2(N^4 + 1) = N^2(ab + 1)$. Under the impression that any integral solution (a, b, k) of $k = \dfrac{a^2 + b^2}{ab + 1}$ is of the form (N^3, N, N^2) I formulated a strategy of trying to deduce from $a^2 + b^2 = k(ab + 1)$ the equality

$$[a - (3b^2 - 3b + 1)]^2 + [b - 1]^2 = \{k - [2b - 1]\}\{[a - (3b^2 - 3b + 1)][b - 1] + 1\}.$$

Were I able to achieve that, then I could have reduced b in steps of one until I got down to the equation $k = \dfrac{a^2 + 1}{a + 1}$ for which $a = k = 1$. By reversing steps I would have solved the problem. I tried to carry out this strategy while I was travelling on a train, but to no avail. Upon returning home I could resort to systematic brute-force checking and look for some actual solutions, resulting in a (partial) list shown below.

a	**1**	**8**	**27**	30	**64**	112	**125**	**216**	240	**343**	418	**512**	$\cdots$
b	**1**	**2**	**3**	8	**4**	30	**5**	**6**	27	**7**	112	**8**	$\cdots$
k	**1**	**4**	**9**	4	**16**	4	**25**	**36**	9	**49**	4	**64**	$\cdots$

Then I saw that my ill-fated strategy was doomed to failure, because there are solutions other than those of the form (N^3, N, N^2). However, not all was lost. When I stared at the pattern, I noticed that for a fixed k, the solutions could be obtained recursively as (a_i, b_i, k_i) with

$$a_{i+1} = a_i k_i - b_i,\ b_{i+1} = a_i,\ k_{i+1} = k_i = k.$$

It remained to carry out the verification. Once that was done, all became clear. There is a set of 'basic solutions' of the form (N^3, N, N^2) where $N \in \{1, 2, 3, \cdots\}$. All other solutions are obtained from a 'basic solution' recursively as described above. In particular, $k = \dfrac{a^2 + b^2}{ab + 1}$ is a square. I feel that I understand the phenomenon much more than if I just learn from reading the slick proof.

2. Relevance to learning and teaching of mathematics in the classroom

The explanatory power of a proof, as exemplified in the example in Section 1, has long been recognized and discussed at length by many authors. Instead of giving a list of references, which is bound to be incomplete in view of the vast size of the relevant body of literature (and many more that will be written), I would only mention one survey paper (Hanna, 2000), two websites (see the two items at the end of the references) and three books (Davis, Hersh, 1980; Hanna, 1983; Siu, 1990/2007/2008), with their numerous bibliographical references thereof.

Why then do I write this paper, realizing that it would be like adding one drop of water to a huge ocean of existing works? What specific point of view do I try to offer in this paper? As the title suggests, I like to explore the practice of mathematical pursuit known as a proof in a cultural, socio-political and intellectual context. A broader message I like to convey is that mathematics constitutes a part of human endeavour rather than stands on its own as a technical subject, as it is commonly taught in the classroom.

In (Siu, 2006) one of the reasons given to account for a general hesitation of teachers to integrate history of mathematics with the learning and teaching of mathematics in the classroom is the concern that students lack enough knowledge on culture in general to appreciate history of mathematics in particular. This is probably quite true, but we can look at it from the reversed side. We can regard the integration of history of mathematics with day-to-day learning and teaching as an opportunity to let students know more about culture in general. In particular, proof is so much an important ingredient in a proper education in mathematics that we can ill afford to miss such an opportunity in this regard. Although the evolution of the standard of rigour or the epistemological aspect of mathematical proofs

(Lakatos, 1976; Rav, 1999)) are not our main focal points in this paper as far as learning and teaching of mathematics in the classroom are concerned, they will invariably come into the picture.

I will discuss four examples: (1) the influence of the exploratory and venturesome spirit during the 'era of exploration' in the 15th and 16th centuries C.E. on the development of mathematical practice in Europe, (2) the influence of the intellectual milieu in the period of the Three Kingdoms and the Wei-Jin Dynasties from the 3rd to the 6th centuries C.E. in China on the mathematical pursuit as exemplified in the work of Liu Hui (flourished in the mid 3rd century C.E.), (3) the influence of Daoism on mathematical pursuit in ancient China with examples on astronomical measurement and surveying from a distance. (4) the influence of Euclid's *Elements* in Western culture compared to that in China after its transmission through the first Chinese translation by Matteo Ricci (1552-1610) and Xu Guang-Qi (1562-1633) in 1607.

Example (1) touches on a broad change of mentality in mathematical pursuit, not just affecting its presentation but more importantly bringing in an exploratory spirit. Example (2) is about a similar happening, for a different reason, in the oriental part of the world, with more emphasis on the aspect of argumentation. Example (3) concerns the possible role religious, philosophical (or even mystical) teachings may play in mathematical pursuit. Example (4) points to such kind of influence but in the reversed direction, namely, how the thinking in mathematical pursuit may breed thinking in other areas of human endeavour. As a 'bonus', these examples sometimes suggest ways to enhance understanding of specific topics in the classroom.

3. 'Era of exploration'

¿From the mid 15th century C.E. into the 16th century C.E. the European world saw the emergence of a group of 'ocean explorers' (some would see them as 'crusaders' or 'colonialists' or even 'pirates', depending on one's stand and view on history) who travelled to far-off shores hitherto unheard of. Those who left their marks in history by such ventures include names like Christopher Columbus (1451-1506), Vasco da Gama (1460-1524), Ferdinand Magellan (1480-1521), Francis Drake (1540-1596), Walter Raleigh (1554-1618). Regardless of their motive, one has to admire their exploratory and venturesome spirit.

This exploratory and venturesome spirit became a model and an inspiration for promoters of modern science (Alexander, 2002). In his book *Novum Organum* (1620/2000) Francis

Bacon wrote:

> "We should also take into account that many things in nature have come to light and been discovered as a result of long voyages and travels (which have been more frequent in our time), and they are capable of shedding new light in philosophy. Indeed it would be a disgrace to mankind if wide areas of the physical globe, of land, sea and stars, have been opened up and explored in our time while the boundaries of the intellectual globe were confined to the discoveries and narrow limits of the ancients." (Bacon, 1620/2000, Book I, Section LXXXIV)

He also wrote in the same book:

> "And therefore we should reveal and publish our conjectures, which make it reasonable to have hope: just as Columbus did, before his wonderful voyage across the Atlantic Sea, when he gave reasons why he was confident that new lands and continents, beyond those previously known, could be found; reasons which were at first rejected but were afterwards proven by experience, and have been the causes and beginnings of great things." (Bacon, 1620/2000, Section XCII)

It may seem that mathematics, as a pure and abstract subject, would not fit in well with this trend. Metaphorically speaking, mathematics rests on the sure stable ground of Euclidean geometry while explorers should go out to the rough ocean to explore and to discover the unknown world. However, even in mathematics change occurred in the 17th century C.E., supplanted by philosophical consideration expressed in the words of Galileo Galilei:

> "Logic, it appears to me, teaches us to test the conclusiveness of an argument already discovered and completed, but I do not believe that it teaches us to discover correct arguments and demonstrations." [The translation is adopted from (Kline, 1977, p.118).]

and of René Descartes:

> "I saw that, regarding logic, its syllogisms and most of its other precepts serve more to explain to others what one already knows, or even, like the art of Lully, to speak without judgement of those things one does not know, than to learn anything new." [The translation is adopted from (Descartes, 1637/1968, p.40).]

One outcome is the bold venture to explain and to discover results in the unknown realm of the infinite by the method of infinitesimals, for instance by Johannes Kepler (1571-1630) in his *Nova Stereometria Doliorum Vinariorum* (1615), and the method of indivisibles, for instance by Bonaventura Cavalieri (1598-1647) in his *Geometria Indivisibilus Continuorum Nova Quadam Ratione Promota* (1635). In the former a geometric object is considered to be made up of 'very small' objects of the same dimension, while in the latter a geometric object is considered to be made up of objects of dimension one lower (Calinger, 1982/1995; Mancosu, 1996). In fact, similar ideas already appeared in earlier works of mathematicians in both the West (for instance, Archimedes in the 3rd century B.C.E.) and the East (for instance, Liu Hui in the 3rd century C.E., Zu Chong-Zhi and his son Zu Geng in the late 5th century C.E.). These examples provide stimulating and instructive didactical material for the classroom (Calinger, 1982/1995; Shen, 1997; Siu, 1993; Wagner, 1978; Wagner, 1979).

The story on the computation of the volume of a sphere is particularly noteworthy, not only because of the ingenious use of the method of indivisibles to compute the volume of the closely related object 'Mou-he-fang-gai' (literally, it means "box with a closely fitted square lid"), which is the common portion of two cylinders of equal diameter placed with axes orthogonal to each other, but also because of the kind of intellectual integrity and humbleness Liu Hui displayed. After sketching his brilliant idea on how to proceed, he said:

> "I wish to give my humble reflections,
> But fear that I will miss the correct principle;
> I dare to let the doubtful points stand,
> Waiting
> For one who can expound them."
> [The translation is adopted from (Wagner, 1978)]

It reminds one of a saying from the late Russian mathematics educator, Igor Fedorovich Sharygin (1937-2004):

> "The life of mathematical society is based
> on the idea of proof, one of the most highly
> moral ideas in the world."

The learning of proof does have its value in 'moral education'!

It is interesting to note that, after two centuries of exciting discoveries in calculus, the subsequent development in the 19th century C.E. gradually reverted to a more conservative style dominated by the 'notorious' (among generations of undergraduates!) "epsilon-delta" analysis.

The pendulum may be swinging again. Since the latter part of the 20th century C.E. the increasing power of computers and the increasing versatility of numerous software enable mathematicians to enter into another 'era of exploration'. Some even start to question whether the role of a proof should be reconsidered, leading to debates which are both philosophical and controversial (Davis, 2006).

4. Intellectual milieu in China in the period from the 3rd to 6th centuries

The eminent British mathematician Godfrey Harold Hardy (1877-1947) once made the comment (Hardy, 1940), "The Greeks first spoke a language which modern mathematicians can understand; as Littlewood said to me once, they are not clever schoolboys or 'scholarship candidates', but 'Fellows of another college'." He was speaking from a viewpoint that holds the time-honoured axiomatic-deductive tradition inherited from the ancient Greeks, exemplified in Euclid's *Elements*, to be the only proper mode of a proof. Some authors offer examples from other mathematical cultures to counterbalance this view (Chemla, 1996; Chemla, 1997; Joseph, 1991/1994/2000; Siu, 1990/2007; Wilder, 1968/1978).

In (Siu, 1993) I described in details several specific examples from the commentaries of Liu Hui on *Jiuzhang Suanshu* (*The Nine Chapters on the Mathematical Art*). Many of these can be rendered, with the help of visual aids, into useful didactical material for the classroom. One notable example is Problem 16 in Chapter 9 of the book, which says: "A right triangle has sides of 8 steps and of 15 steps. What is the diameter of its inscribed circle?" The method in the book gives the correct formula for the diameter, namely, $d = 2ab/(a + b + c)$ where c is the hypotenuse and a, b are the other two sides. In his commentary, Liu Hui offered three different proofs. The first proof is by a colourful (literally, as the text indicates pieces of different colours) method of dissection. (See (Siu, 1993, Figure 3) for a "proof without words".) The second proof uses knowledge about proportional quantities. The third proof is most interesting from the viewpoint of the role of a proof, because Liu Hui was probably looking for "consistency [of the theorem] with the body of accepted mathematical results" (Hanna, 1983, p.70). (See (Siu, 1993, Section 3) for a detailed discussion on the mathematics.)

At the beginning of his commentary Liu Hui wrote:

> "I studied *Jiuzhang* [*The Nine Chapters on the Mathematical Art*] at an early age and perused it when I got older. I see the separation of the *Yin* and the *Yang* and arrive at the root of the mathematical art. In this process of probing I comprehend its meaning. [......] Things are related to each other through logical reasons so that like the branches of a tree, diversified as they are, they nevertheless come out of a single trunk. If we elucidate by prose and illustrate by pictures, then we may be able to attain conciseness as well as comprehensiveness, clarity as well as rigour. Looking at a part we will understand the rest." [The translation is adopted from (Siu, 1993).]

This passage not only indicates a different style in mathematical practice but also exhibits a different mentality from that of the traditional school of Confucianism, which by the latter part of the 1st century B.C.E. was made the orthodoxy belief of the Han Dynasty at the expense of other schools of thought.

To put this trend in its historical perspective we should note that the 3rd century C.E., in which Liu Hui flourished, fell into an exceptionally interesting period of Chinese history. The four centuries, beginning with the collapse of the Han Dynasty in 220 and ending with the establishment of the Sui Dynasty in 581, were a "prolonged period of disunity and confusion, [...] marked by frequent warfare and political cleavage between a series of dynasties that ruled in Central and South China, and another series that had control in the North" (Feng, 1948). Although this prolonged period of disarray and strife was politically and socially a 'dark age', it was also "an age in which, in several respects, we reach one of the peaks of Chinese culture" (Feng, 1948). Ironically, the collapse in political and social order brought with it a weakening of the orthodoxy belief, giving way to free and uninhibited thinking. The period was known for a predilection for rhetoric and dialectic, characterized by a refined intellectual activity referred to as *qing-tan*, which literally means "pure conversation". According to the historian Yu Ying-Shih, the intellectual milieu of this period was a result of a kind of "self-awareness", both as an individual and as a community, that was formed since the latter part of the Late Han Dynasty (25-220) among the class of *Shi*[1] (Yu, 1987, Chapter 6, Chapter 7). It is natural to propose that, in the area of science and mathematics, the predilection

[1] *Shi* is a rather peculiar but extremely important social class throughout the whole cultural history of China. It is sometimes rendered in translation as 'literati', 'scholar', 'scholar-official', 'intellectual', but none of these terms individually can capture a holistic meaning of the word.

for rhetoric and dialectic engaged under an atmosphere of free and uninhibited thinking was conducive to the promotion of a notion of proof. (See also (Horng, 1982) for a lengthy discussion on this thesis.)

5. Daoism and mathematical development in China

Daoism is a school of Chinese philosophy that came into being by the 4th century B.C.E.. Towards the end of the Late Han Dynasty (25-220) there was a related development as a religion, usually referred to as *daojiao*. But in this section we refer mainly to the philosophical aspect of Daoism. A central theme is the *Dao* (the Way) or the flow of forces of Nature by which things come together and transform, which reflects a deep-seated Chinese belief that change is a basic characteristic of things. The relationship between Daoism and Chinese science and mathematics in the ancient and medieval times is a topic of scholarly investigation by several authors. (See (Volkov, 1996a) for a survey as well as a new interpretation from the author.)

In particular, the treatise *Huainanzi* (*The Book of the Prince of Huai Nan*) was a Daoist book commissioned by Prince Liu An (179 B.C.E. - 122 B.C.E.), a grandson of the founding emperor of the Han Dynasty (206 B.C.E. - 220 C.E.), in the 2nd century B.C.E.. It is a compendium on different areas, one of which is astronomy. In Chapter 3 titled *Tianwenxun* (*Treatise on the Patterns of Heaven*) we find the following problem on measuring the height of heaven:

> "To find the height of heaven (i.e. of the sun) we must set up two 10-*che* gnomons and measure their shadows on the same day at two places situated exactly 1000 *li* apart on a north-south line. If the northern one casts a shadow of 2 *che* in length, the southern one will cast a shadow 1 and 9/10 *che* long. And for every thousand *li* southwards the shadow diminishes by one *cun*. [In the Chinese system, 10 *cun* amount to 1 *che*]. At 20,000 *li* to the south there will be no shadow at all and that place must be directly beneath the sun. (Thus beginning with) a shadow of 2 *che* and a gnomon of 10 *che* (we find that Southwards) for 1 *che* of shadow lost we gain 5 *che* in height (of gnomon). Multiplying therefore the number of *li* to the south by 5, we get 100,000 *li*, which is the height of heaven (i.e. of the sun)." [The translation is adopted from (Needham, 1959).]

Phrased in modern notations, the calculation can be explained in the figure shown below.

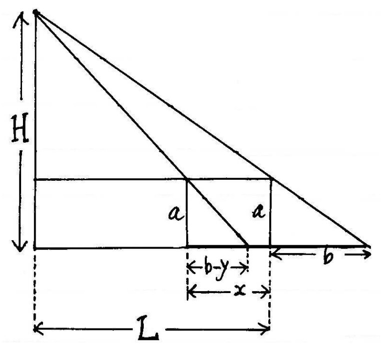

Figure 1

y (decrease in length of shadow) is a function of x (distance moved by the gnomon), say $y = f(x)$. What is x that makes $f(x) = 2$? That x should be L. If we know what $f(x)$ is, then we can calculate L, hence H.

Let us try to find out what $f(x)$ is. We know that

$$\frac{a}{b-y} = \frac{H}{L-(x-b+y)} \qquad (1),$$

$$\frac{a}{b} = \frac{H}{b+L} \qquad (2).$$

¿From (1) and (2) we obtain $y = \left(\frac{a}{H-a}\right) x = \alpha x$, where α is a constant. When $x = 1000, y = 0.1$. Hence $\alpha = 0.0001$, i.e. $y = 0.0001x$. [Note that x is measured in *li* and y is measured in *che*.] When $x = 20,000(li), y = 2$ (*che*), so there is no shadow. Hence, $L = 20,000$ (*li*).

$$H = (b+L)\frac{a}{b} = \left(\frac{2}{180} + 20000\right)\left(\frac{10}{2}\right) = 100,000 + \frac{1}{18} \text{ (in } li.)$$ [2]

The calculation is based on an over-simplified model of 'heaven and earth', so it does not measure the 'height of heaven'. However, the same calculation can be used to measure the height and distance of an inaccessible object. This method of using two gnomons for measurement was explained in detail in Liu Hui's *Haidao Suanjing* (*Sea Island Mathematical Manual*) of the 3rd century C.E.. The same method was also explained by the Indian mathematician Aryabhata in the early 6th century C.E.. In the West the instrument in surveying known as cross-staff, believed to be invented in the beginning of the 14th century C.E., relies on this same method. The way the answer was presented by Liu Hui, which was explained by Yang Hui (flourished in the mid 13th century C.E.) in his *Xugu Zhaiqi Suanfa* (*Continuation of Ancient Mathematical Methods for Elucidating the Strange Properties of Numbers*) of 1275, is based on an elegant use of area computation as shown in the figure below (Figure 1).

[2] A conversion from *che* to *li* accounts for the factor $\frac{1}{180}$.

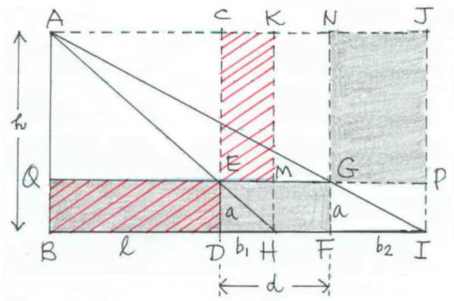

Figure 2

$$
\begin{aligned}
EGFD &= QGFB - QEDB \\
&= NJPG - CKME, \\
\text{hence} \quad ad &= b_2(h-a) - b_1(h-a) \\
&= (b_2 - b_1)(h-a), \\
h &= \frac{ad}{b_2 - b_1} + a. \\
QEDB &= CKME, \\
\text{hence} \quad a\ell &= b_1(h-a) = b_1\left(\frac{ad}{b_2-b_1}\right), \\
\ell &= \frac{b_1 d}{b_2 - b_1}.
\end{aligned}
$$

(It is an easy exercise to compare h, ℓ with H, L to see that the answers agree.) When a schoolboy of today faces this problem, very likely he will make use of similar triangles to set up simultaneous equations in ℓ and h. The answers would come out to be the same, but Yang Hui's solution seems to be much more elegant. The solution in *Huainanzi*, arrived at through yet another approach, is elegant in its own way, for it is a dynamic version using a functional dependence, which is perhaps more akin to the thinking of change or transformation in Daoism.

It seems that both Liu Hui and Yang Hui were unaware of the method explained in *Huainanzi*, but the same problem and method appeared earlier in *Zhoubi Suanjing* (*The Arithmetical Classic of the Gnomon and the Circular Paths*) of 100 B.C.E. and later in another treatise written by a Daoist in 1230, namely *Gexiang Xinshu* (*New Writing On the Image of Alteration*) of Zhao You-Qin (Volkov 1996b). This may be explained by the historical happening in the early Han Dynasty. Prince Liu An, who commissioned the writing of *Huainanzi* by convening a group of Daoist scholars around himself, was later forced to commit suicide for treason. As a result, the book was banned; probably the proof using functional dependence was also lost to the public except possibly within the Daoist circle.

6. Influence of *Elements* in Western culture and in Chinese culture

It is well-known that Euclid's *Elements* exerts significant influence on western culture, both as an exemplar of axiomatic approach and as an exemplar in logical proof (Grabiner, 1988). This mathematical classic of all times was transmitted into China through a collaboration in translation by the Italian Jesuit Matteo Ricci and the Chinese scholar, later appointed to high-ranking officials in charge of various important duties in the imperial court, Xu Guang-Qi of the Ming Dynasty (1368-1644), published in 1607 as *Jihe Yuanben*

(Siu, 1995/1996).

In an essay *Jihe Yuanben Zayi* (*Discourse on the Jihe Yuanben*) Xu Guang-Qi commented:

> "The benefit derived from studying this book is many. It can dispel shallowness of those who learn the theory and make them think deep. It can supply facility for those who learn the method and make them think elegantly. Hence everyone in this world should study the book.... Five categories of personality will not learn from this book: those who are impetuous, those who are thoughtless, those who are complacent, those who are envious, those who are arrogant. Thus to learn from this book one not only strengthens one's intellectual capacity but also builds a moral base." [The translation is by this author.]

Xu Guang-Qi felt rather disappointed when he saw that few people paid attention to the translated text, but surmised that everybody would study it a hundred years later. However, in 1681 Li Zi-Jin of the Qing Dynasty (1644-1911), said in the preface to *Shuxue Yao* (*Key to Mathematics*) written by Du Zhi-Geng:

> "Even those gentlemen in the capital who regard themselves to be erudite scholars keep away from the book [*Jihe Yuanben*], or close it and do not discuss its content at all, or discuss it with incomprehension and perplexity." [The translation is by this author.]

Even though *Elements* had little influence on mathematics in China, surprisingly it bore fruit in another arena, exerting influence on Chinese liberals like Kang You-Wei (1858-1927) and Tan Si-Tong (1865-1898), who were main figures in the futile attempt of the "Hundred-Day Reformation Movement" of 1898 that ended in tragedy for many concerned. Little would Xu Guang-Qi imagine that his somewhat over-optimistic prediction of the influence of Euclid's *Elements* came true in the political arena! A more detailed account of the influence of *Elements* in China can be found in (Siu, 2007), written on the occasion of the 400th anniversary of the translation of *Elements* in China.

7. Conclusion

How would the message conveyed in this paper contribute to the learning and teaching of proof? It would not yield specific tactics nor a comprehensive theory. But it serves to remind us that, to make the subject more 'humanistic' so that students feel that it makes good sense to spend time on it, mathematics is best studied along with its influence to and from other human endeavour. Proof, as a characteristic component of the subject, shares the same fate. As a 'bonus', in viewing proof in this light we may be able to pick up good suggestions to enhance understanding of specific topics to make the learning of mathematics a more interesting activity.

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