Some reflections on the teaching and learning of calculus: From both historical and pedagogical viewpoints

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This is an introduction to a seminar talk I gave to colleagues of the Hanlun Information Limited, which is a company that works on producing e-learning material. The theme of my talk is that calculus (in school) should be taught neither as a cookbook nor as a baby course in mathematical analysis with its full and meticulous rigour. Rather we can learn from what historical development tells us. For that matter I think the same goes with a calculus course for beginning undergraduates as well. That was how I taught calculus to first year students from the mid-1970s to the mid-1980s. The rest of my talk is on the historical aspect with illustrative examples. Exchange of views on teaching and sharing of teaching experience are certainly welcome.

Ordering of topics in the traditional pedagogical approach:
Function
Limit
Differentiation
Integration
Series

Order of themes (roughly) in historical development:
Integration
Function
Differentiation
Series
Limit

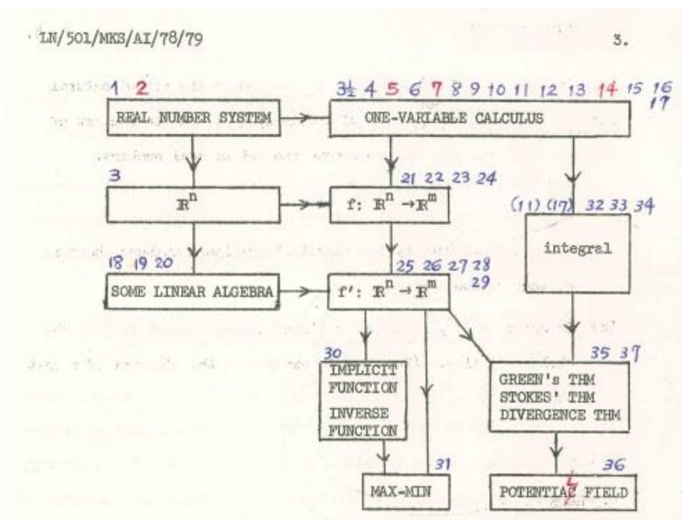
Should we follow the historical development? Or should we modify the traditional pedagogical approach by benefiting from knowledge of the historical development?
Obviously we cannot follow the historical development *strictly*. The development of calculus is a classic case of clever *ad hoc* methods being turned into powerful general methods that became part of a much larger picture. Only masters in mathematics in ancient times could calculate the area and volume of certain geometric objects. With the development of calculus since the 17th and 18th centuries a school pupil today who has learnt calculus will be able to handle what only great mathematicians of the past could have resolved.

Those five themes (and more) cannot be treated one by one separately in turn, but design is needed to interweave them into a beautiful branch called calculus, with careful thought and planning.

In the traditional curriculum calculus was regarded (maybe it still is today) as the pinnacle of school mathematics. BUT we should note that (i) some of the key ideas of calculus are not confined to the advanced level of school mathematics but can already be introduced at the elementary level; (ii) in today's curriculum discrete mathematics should be as important (if not more) as continuous mathematics, so that attention should be paid to that aspect as well, in which case calculus would no longer be emphasized as occupying a supreme position as before.

In order to know how calculus became what we nowadays learn in school and university we would do well to look at its history. In August of 2015 I gave a talk titled “微積分的「史前期」(The “prehistoric age“ of Calculus).

In the preface to my set of Lecture Notes on Sc 501 (First Year Calculus) in 1978-1979 I wrote: “We won’t follow a strictly rigorous path (maybe to the distaste of some of you!) You will pick that up later, and hopefully by that time you will understand why we need rigour. We won’t follow a “cookbook” approach either (maybe to the distaste of the rest of you!) because that won’t help you to use the knowledge as a tool intelligently. Rather, we shall explain new concepts, try to understand (not necessarily equivalent to “to prove”) theorems, discuss some touchy points and work on problems. […]
“The foundation of calculus rests on the concept of real numbers. I shall not go into its history (which is, by the way, quite interesting), and I shall assume you know what real numbers mean. To be honest, you probably don’t, but we won’t press upon this matter. We shall be content with our “common sense” experience with it. We know how to operate with real numbers and how to compare them. In particular it is helpful to visualize them as points on the so-called real number line […] .”



A chart of relationship between sections on various topics
Lecture Notes on Sc501 (First Year Calculus) by M. K. Siu (1978-1979)

We usually hear the remark: “Differentiation and integration are inverse (operations) to each other.” What is meant by that? It is easy to see what is meant by differentiating a function. What is obtained is a function. What about integrating a function? What is obtained is a family of functions. Furthermore, there are two different notions related to integration, one is called the indefinite integral, and the other is called the definite integral. By integrating a function, which integral is to be employed? But the definite integral of a function (on a certain closed interval) is a number, not a function! What is really meant by the two operations being inverse to each other?
What is a differential such as dy, dx, …? What is an increment such as Δy, Δx, …? How is it related to the derivative $\frac{dy}{dx}$? Why is the notion of differential needed if we already know what the derivative is? (What happened in history?)
Differentials → Differential Forms (but *not* for a first course)

Some useful references are:

- M. Baron, *The Origins of the Infinitesimal Calculus*, Dover, 1969.
- C. Boyer, *The History of the Calculus and Its Conceptual Development*, Dover, 1949.
- C.H. Edwards, Jr., *The Historical Development of the Calculus*, Springer-Verlag, 1979.
- O. Toeplitz, *The Calculus, A Genetic Approach*, University of Chicago Press, 1963; reprinted in 2007; original German edition, 1949.
- Michael Spivak, T*he Hitchhiker's Guide to Calculus*, Polished Pebble Press, 1995.
- Qun Lin (林群), *Free Calculus:A Liberation From Concepts and Proofs*, World Scientific,2008.
- David Marius Bressoud, *Calculus Reordered: A History of the Big Ideas*, Princeton University Press, 2019.
- Michael Spivak, *Calculus*, W.A.Benjamin,1967; 4th Edition, Publish or Perish Press, 2008.