The world of geometry in the classroom: virtual or real?

SIU Man Keung

Department of Mathematics, The University of Hong Kong

email: mathsiu@hkucc.hku.hk

Abstract

This paper attempts to offer lots of examples to illustrate how we can build up in the classroom a *world of geometry* that helps to bridge the gap between the *virtual (abstract) world* and the *real (concrete) world*. It is hoped that, by so doing, novices (students) may feel more at home instead of being alienated from the subject.

1. Despite the occurrence of the word "virtual" in the title this paper is not on DGE (dynamic geometry environment), as I know far too little on the subject to be able to tell you what you did not already know. However, DGE would unavoidably slip into the discussion. In another plenary talk Alain Kuzniak explained so well how we live in different *worlds of geometry*, and that only experts, in contrast to most novices, can commute with ease from one world to the other. In this paper I attempt to offer lots of examples to illustrate how we can build up in the classroom a *world of geometry* that helps to bridge the gap between the *virtual (abstract) world* and the *real (concrete) world*. It is hoped that, by so doing, novices (students) may feel more at home instead of being alienated from the subject.

Our discussion would involve some perennial controversial pedagogical issues in the learning and teaching of geometry, among which the main ones being:

(1) empirical knowledge ('physical' geometry) and theoretical knowledge ('pure' geometry),

(2) heuristic explanation and formal proof,

(3) intuition and deductive reasoning,

(4) spatial comprehension and computational skill.

[This paper is an expanded text of a talk given at the 5th International Colloquium on Didactics of Mathematics held in Greece (in Rethymnon) in April of 2008. I feel greatly honoured to have the opportunity to deliver a talk on geometry in this great land from which the subject flourished since the 6th century B.C.]

2. Geometry is a very old subject that grew out of practical need. In *A Commentary on the First Book of Euclid's Elements* (c. 5th century) Proclus referred to Herodotus' saying that "geometry was first discovered among the Egyptians and originated in the remeasuring of their lands... necessary for them because the Nile overflows and obliterates the boundaries between their properties " (Morrow, 1970) . The etymology of the very word "geometry", meaning "earth measurement", bears out this claim. In the Western world, the contribution of the ancient Greeks to mathematics in general, and to geometry in particular, is well-known. Saul Stahl summarizes this contribution in the following passage: "Geometry in the sense of mensuration of figures was spontaneously developed by many cultures and dates to several millenia B.C. The science of geometry as we know it, namely, a collection of abstract statements regarding ideal figures, the verification of whose validity requires only pure reason, was created by the Greeks." (Stahl, 1993)

In many textbooks one reads that Euclid's *Elements* (c.300 B.C.) (Heath, 1925/1956) offers a prototype which enshrines the spirit of modern mathematics, more specifically on what is known today as an axiomatic and deductive approach. As the book was written over two thousand years ago, unavoidably it has deficiencies here and there, but one is told that the book is basically sound and can be made adequate by a patch-up job, for instance, the notable accomplishment in *Grundlagen der Geometrie* (1899) by David Hilbert. Such a depiction is not entirely incorrect but gives an over-simplified picture. It is not the intention of this talk to elaborate on this topic. Suffices to say that Euclid's *Elements* is a milestone in the history of mathematics and more generally in the history of thought, and that, as an immensely important mathematical text in the ancient world, it has been exerting enormous influence throughout the Western world ever since it was compiled. Last year was the 400th anniversary of the translation of *Elements* into Chinese, the first European mathematical text that was transmitted into China. Interested readers (for whom Chinese is not Greek to them!) can consult (Siu, 2007).

Many famed scholars have recounted the benefit and impact they received from learning Euclidean geometry in reading Euclid's *Elements* or some variation based on it. I just cite two.

(1) Bertrand Russell wrote in his autobiography (Russell, 1967):
"At the age of eleven, I began Euclid, with my brother as tutor. This was one of the great events of my life, as dazzling as first love. (...) I had been told that Euclid proved things, and was much disappointed that he started with axioms. At first, I refused to accept them unless my brother could offer me some reason for doing so, but he said,

"If you don't accept them, we cannot go on", and as I wished to go on, I reluctantly admitted them *pro temp.*"

(2) Albert Einstein wrote in his autobiography (Schilepp, 1949/1957):
"At the age of twelve I experienced a second wonder of a totally different nature: in a little book dealing with Euclidean plane geometry, which came into my hands at the beginning of a school year. (...) The lucidity and certainty made an indescribable impression upon me. (...) it is marvellous enough that man is capable at all to reach such a degree of certainty and purity in pure thinking as the Greeks showed us for the first time to be possible in geometry."

However, not everybody experienced the same. The English mathematician James Joseph Sylvester once said (Baker, 1904-1910):
"The early study of Euclid made me a hater of Geometry, which I hope may plead my excuse if I have shocked the opinions of any in this room (...) by the tone in which I have previously alluded to it as a schoolbook;" but he continued with the following remark on the immense value of the subject: "and yet, in spite of this repugnance, which had become a second nature in me, whenever I went far enough into any mathematical question, I found I touched, at last, a geometrical bottom."

3. Geometry is not to be equated solely with Euclid's *Elements*. Let us see what others have to say.

John Greenlees Semple and G.T. Kneebone say, (Semple and Kneebone, 1959):
"Geometry is the study of spatial relations, and in its most elementary form it is conceived as a systematic investigation into the properties of figures subsisting in the space familiar to common sense. As mathematical insight grows, however, the 'space' that constitutes the geometer's ultimate object of study is seen to be an ideal object — an intellectual construction that reveals itself to be essentially different from any possible object of naive intuition. Nevertheless, even the most abstract geometrical thinking must retain some link, however attenuated, with spatial intuition, for otherwise it would be misleading to call it geometrical, (...)."

Charles Godfrey stresses the importance of a "geometrical eye", which is 'the power of seeing geometrical properties detach themselves from a figure' (Godfrey, 1910). This viewpoint is implemented in a textbook he wrote with Arthur Warry Siddons (Godfrey and Siddons, 1903). An illustrative example is provided in (Rouche, 2003) in which two triangles are presented first in separate instances, then in a combined context (see Figure 1).

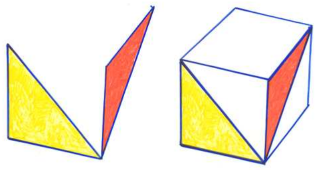

Figure 1

Hans Freudenthal maintains (Freudenthal, 1973):

"Geometry is grasping space (...) grasping that space in which the child lives, breathes and moves. The space that the child must learn to know, explore, conquer, in order to live, breathe and move better in it. (...) Geometry is one of the best opportunities that exists to learn how to mathematize reality. It is an opportunity to make discoveries(...) teaching geometry is an unparalleled struggle between ideal and realization."

In Recommendation 2 in Section 3 of *Teaching and Learning Geometry 11-19, Report of a Royal Society/Joint Mathematical Council Working Group* (Royal Society, 2001) it is written that "We recommend that the title of the attainment target Ma 3 of the National Curriculum be changed from 'Shape, space and measures' to 'Geometry'." An explanatory remark says that "We believe that geometry has declined in status within the English mathematics curriculum and that this needs to be redressed. It should not be the "*subject which dare not speak its name*"".

Thomas J. Willmore says (Willmore, 1970):

"I suppose that the first lesson is that mathematics no longer consists of separate water-tight compartments, and that geometry as such is no longer a subject. What is important is a geometrical way of looking at a mathematical situation — *geometry is essentially a way of life*."

Willmore's remark points to the fact that geometry comes up in different areas, for instance,

- school algebra (graphical solution, coordinate geometry),
- linear algebra ([Euclidean] inner product space, which is based on a Euclid-Hilbert-Dedekind postulate system of geometry),
- calculus (curves and surfaces),
- physics (work of Gauss, Riemann, Poincaré, Einstein),
- abstract algebra, number theory, combinatorics (finite geometries, algebraic structures, diophantine equations).

My personal recollection of a first encounter of this symbiosis involving geometry and algebra is an exercise in a school textbook (Mayne, 1938): "Draw the graphs of $y = x^2$ and $5y = 6x + 4$ on the same diagram for values of x from -2 to 3. From the graphs solve $5x^2 = 6x + 4$. Also find out roughly from the graphs, by drawing the appropriate parallel line, for what value of a the equation $5x^2 = 6x + a$ will have equal roots."

But be careful, synthetic geometry is not always inferior to algebra! Another personal recollection in secondary school mathematics is the question on a convex quadrilateral $ABCD$ (with diagonal BD) with each pair of opposite sides equal. It is rather easy to prove that such a quadrilateral is a parallelogram using synthetic geometry, namely, with the congruence of $\triangle ABD$ and $\triangle CDB$ we conclude that $\angle ABD = \angle CDB$ and $\angle ADB = \angle CBD$, thus AB is parallel to DC and AD is parallel to BC. However, try to prove the result by algebra using either coordinate geometry or vector analysis; you will appreciate what I mean.

Another example is the reflection of a light ray on a mirror. In physics we learn that the angle of incidence is equal to the angle of reflection. Interestingly, this obeys the so-called Fermat's Principle in that light travels by the shortest path. I would call this 'physical' geometry. The same fact can be explained by synthetic geometry in showing that from any point Q other than the point of incidence P, $AQ + BQ > AP + BP$, where the light ray emanates from A and is reflected to B. I would call this 'pure' geometry (see Figure 2).

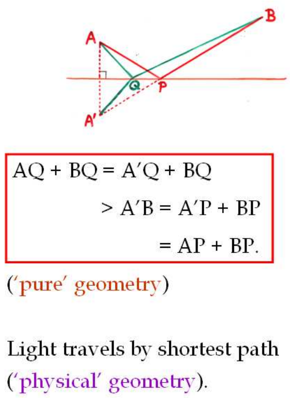

Figure 2

4. The axiomatic and logical aspect of Euclid's *Elements* has long been stressed. Following the reasoning put forth by S.D. Agashe (Agashe, 1989) let us look at an alternative feature of the book, namely, that right from the start metric geometry plays a key role, not just in the exposition itself but even in the motivation of the design of the book.

Proposition 14 of Book II says (Heath, 1925/1956):

> "To construct a square equal to a
> given rectilineal figure."

It seems that the motivating problem of interest is to compare two polygons ("rectilineal figures"), whose one-dimensional analogue of comparing two straight line segments is easy. One simply places one segment onto the other and checks which segment lies completely inside the other or whether the two segments are equal to one another. This is in fact what Proposition 3 of Book I sets out to do (Heath, 1925/1956):

> "Given two unequal straight lines, to cut off from the greater a straight line equal to the less."

To justify this result one relies on Postulate 1, Postulate 2 and Postulate 3. The two-dimensional case is not as straightforward, except for the special case when both polygons are squares. In that case a comparison of area amounts to a comparison of the side, as one places the smaller square at the lower left corner of the larger square. Incidentally, one needs to invoke Postulate 4 in doing that. What Proposition 14 of Book II sets out to do is to reduce the comparison of two polygons to that of two squares.

Let us look a bit further into the proof of Proposition 14 of Book II. It can be separated into two steps: (i) construct a rectangle equal (in area) to a given polygon, (ii) construct a square equal (in area) to a given rectangle. Note that (i) is already explained through Proposition 42, Proposition 44 and Proposition 45 of Book I, by triangulating the given polygon then converting each triangle into a rectangle of equal area. Incidentally, one relies on the famous (notorious?) Postulate 5 on (non)parallelism to prove those results. What about the final kill in (ii)? A preliminary step is to convert a given rectangle into an L-shaped gnomon of equal area, which is illustrated in Proposition 5 of Book II that says (Heath, 1925/1956):

> "If a straight line be cut into equal and unequal segments, the rectangle contained by the unequal segment of the whole together with the square on the straight line between the points of section is equal to the square on the half." (see Figure 3)

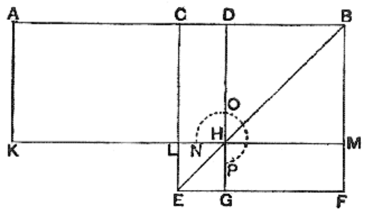

Figure 3

The result asserts that, if $AC = CB$, then the rectangle $AKHD$ is equal (in area) to the gnomon NOP, which is the square $CEFB(c^2)$ minus the square $LEGH(b^2)$. Finally, to accomplish (ii) one wants to construct a square (a^2) equal to the difference of two squares $(c^2 - b^2)$, or equivalently, the square (c^2) is a sum of two squares $(a^2 + b^2)$. This leads naturally to the famous Pythagoras' Theorem, which is Proposition 47 of Book I. The Pythagoras' Theorem epitomizes the relationship between shape and number, between geometry and algebra.

This metric flavour of Euclidean geometry can be and should be conveyed at an early stage through the relationship between shape and number. As an example, the shape of sheets of paper of size A3 and A4 is encountered so often in daily usage that it provides a nice topic for discussion even at the level of primary school. Half a sheet of A3 size paper is of size A4, and the A3 size paper is a blow-up of the A4 size paper, or in geometric terms the two sheets are similar in shape (see Figure 4).

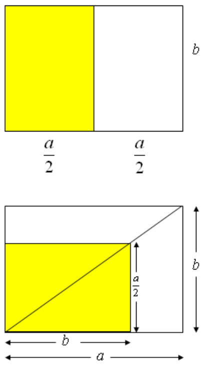

Figure 4

With the sort of argument in 'geometric algebra' prevalent in the ancient world (Greek and Chinese) is mind, we can devise a visual means to explain the ratio between the sides of a sheet of paper of size A3 (and A4) (see Figure 5). This explains why the magnifying factor indicated on a xerox machine (for enlarging A4 to A3) is 141%, while the shrinking factor (for reducing A3 to A4) is 71%.

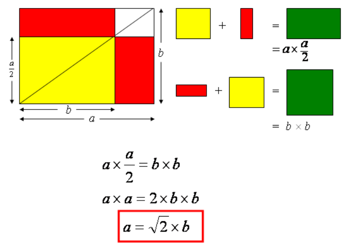

Figure 5

Although it is common knowledge that the area (resp. volume) varies as the square (resp.

cube) of the linear dimension, one often overlooks this simple mathematical fact and allows oneself to be misled by cunning newspaper reporting. One example is a comparison of the diminishing purchasing power of the American dollar during different administrations of the U.S.A. government, from the late 1950s to the late 1970s, that appeared on the *Washington Post* [dated October 25, 1978] (Tufte, 1983) (see Figure 6).

Figure 6

The monetary variation is represented by the linear dimension of a banknote, but our visual attention is drawn to the area of the banknote. Take for example the situations in 1958 and 1973, in which case the actual ratio (of buying power in terms of dollars) is 100:64. In the figure the situations are represented by two banknotes of sides that bear the same ratio. However, if we want to have two banknotes of area bearing the same ratio, then the ratio of the sides should be 100:80 so that the picture would appear less alarming.

On the other hand, the Irish writer and satirist Jonathan Swift paid much attention to this mathematical fact in his famous novel *Gulliver's Travel* of 1726 (Starkman, 1962), when he gave a meticulous account of the adventure of Captain Gulliver in the land of Lilliput, where the inhabitant was of size a twelfth of that of himself, and in the land of Brobdingnag, where the inhabitant was of size twelve times that of himself. Excerpts from the book make for both interesting reading and instructive exercises in mathematics for school pupils. For instance, how many Lilliputans can a daily allowance of meat and drink for Gulliver support? [Answer: 1728.] The author did make one serious mistake in the account of the malicious plot of the jealous dwarf Glumdalclitch in Brobdingnag who shook an apple tree at the foot of which poor Gulliver stood. Gulliver recounted that "one of them [the fallen apples] hit me on the back as I chanced to stoop and knocked me down flat on my face; but I received no other hurt". With a bit of knowledge in physics as well, we can be quite sure that Gulliver would be dead as a doornail!

The same elementary principle can be employed to explain a number of interesting facts.

For instance, one can ask: "If a man weighing 50Kg can normally lift up 30Kg, how much can a man weighing 100Kg normally lift up?" Or, one can ask: "An ant normally measures 0.005m. It can carry a burden that is 5 times its own weight. If a giant ant were a blow-up copy as big as a man, say of height 1.75m, how many times of its own weight would it be able to carry?" To do the calculation we need to know that weight is proportional to the cube of height, while the weight capable of lifting up is proportional to the cross-sectional area of the muscle, hence proportional to the square of height. The comforting news is that the giant ant can only carry a seventieth of its own weight (see Figure 7), so that it can hardly stand on its own feet! That implies it is not possible to have a blow-up copy of a giant ant, or that a giant ant no longer looks like an ant that is simply larger.

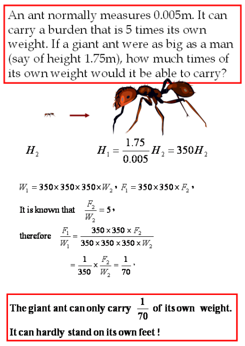

Figure 7

As an example of geometry present in our surroundings it is natural to expect that geometry occurs in architecture. Indeed, the Serpentine Gallery in London, designed by Toyo Ito in 2002, made the news with the title "All angles covered" in the *Financial Times* [dated January 21/22, 2006] (see Figure 8).

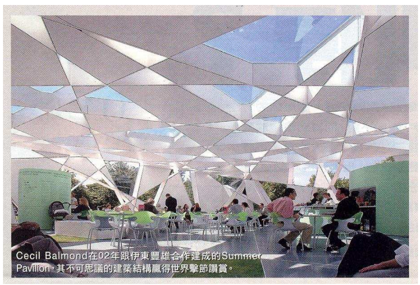

Figure 8

There are numerous examples of buildings designed by masters all over the world to substantiate this close relationship, two notable masters being the Swiss architect Le Corbusier (see Figure 9) and the American-Chinese architect I.M. Pei (see Figure 10).

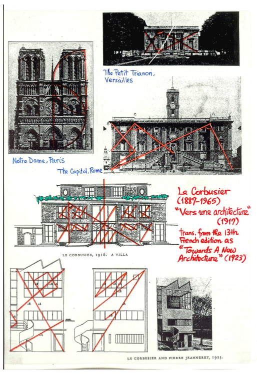

Figure 9

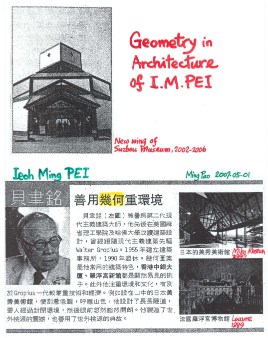

Figure 10

Besides architecture the aesthetic appeal of geometry is easily found in visual art. In Western art history the rational thinking in perspectivity not only produced paintings that better reflect reality, it also led to mathematical development of projective geometry (Andersen, 2007). Mathematical themes permeate the artwork of the Dutch artist Maurits Cornelis Escher (1898-1972) to such an extent that a book (Schattschneider, Emmer, 2003) was published on his centennial. Escher is known for his fascination with symmetry, which is a topic that links up geometry with (abstract) algebra. Interested readers can consult (Armstrong, 1988; Martin, 1982). Even though these topics may go beyond the curriculum in primary/secondary school, it still affords a good opportunity to let students appreciate the beauty of geometry in them. Besides, there are examples that lie within reach of school geometry, for instance, the investigation on the number of ways to tile a plane (tesselation) using regular polygons (in an edge-to-edge manner with every vertex looking the same) (see Figure 11).

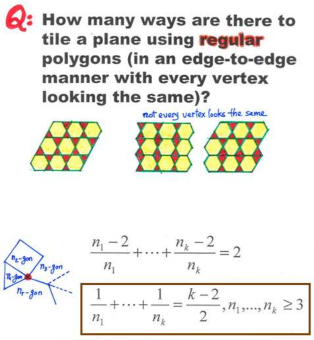

Figure 11

The easier case of using only one kind of regular polygon turns out to be only tiling by an equilateral triangle, or a square, or a regular hexagon. The general case, which requires a bit more patience with the computation by looking at different possibilities, is more interesting and turns out to have eleven patterns, including the aforementioned three (see Figure 12).

The complete solution to the case of using only one kind of convex, but *not necessarily* regular, polygon is still an open problem.

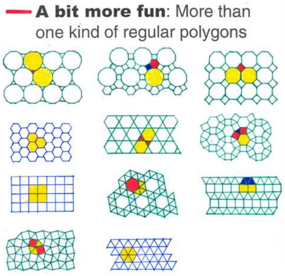

Figure 12

I will now offer a few more example to illustrate how to develop in students a "geometrical eye". The first example is the making of a chimney from a sheet of metal, inspired by what I once saw out of the window of a hotel room in Rome (see Figures 13).

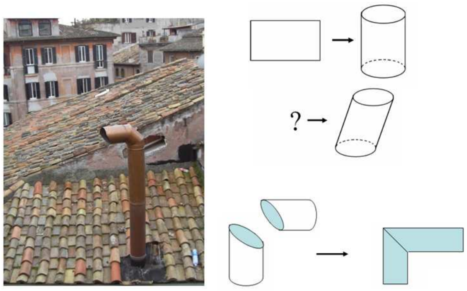

Figure 13

The chimney is composed of two slanting cylinders joined at a certain angle. It requires only school mathematics to calculate the shape of the sheet, but it provides a good exercise in nurturing spatial visualization. It turns out that the sheet has a sinusoidal curve as its boundary.

The second example involves a bit more of mathematics, namely, that of motion. It is about Leonardo da Vinci's cam, which is a device that converts a uniform circular motion to a uniform linear motion (see Figure 14).

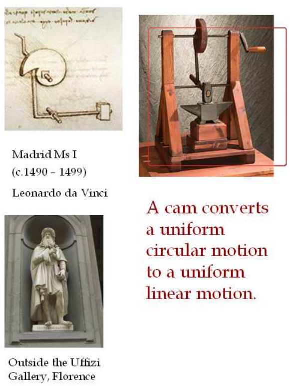

Figure 14

It turns out half of the cam has a semi-circular boundary and the other half has an Archimedes' spiral as the boundary (see Figure 15).

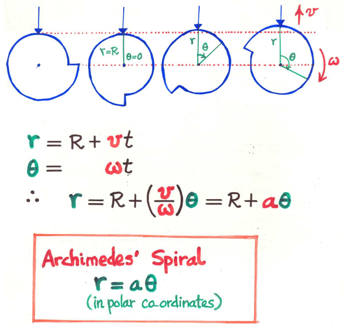

Figure 15

The third example is inspired by an item in a Chinese newspaper *Hong Kong Economic Journal* [dated January 1, 2008] that talks about furniture design involving a lot of geometry (see Figure 16).

Hong Kong Economic Journal (《信報》) 2008.01.05


家具奧運熱

Geometry in design of furniture

Figure 16

That reminds me of an interesting geometric problem in dissection: "Try to dissect an equilateral triangle into pieces, hinged in such a way that the pieces can be reassembled into a square." (see Figure 17) This will make a nice piece of multi-purposed furniture, a triangular table which can be converted to a square table and vice versa within seconds!

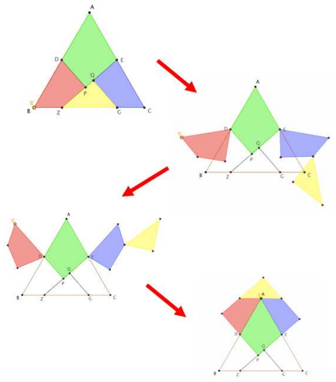

Figure 17

The use of dissection to compute area is a time-honoured geometric means since ancient times. Examples abound in treatises like Euclid's *Elements* or the Chinese classics *Jiu Zhang Suan Shu* (*Nine Chapters of the Mathematical Art*). The mathematical basis of the method was provided in the early 19th century, leading to the famous Hilbert's Third Problem that asks about its three-dimensional analogue (see Figure 18).

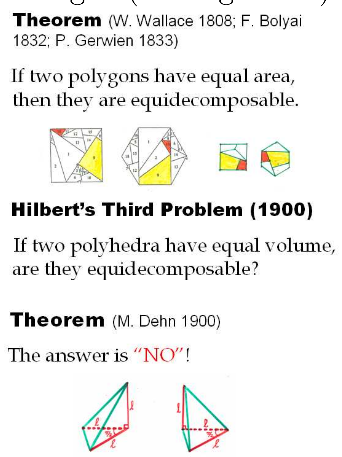

Figure 18

Tracing the proof of the Bolyai-Gerwien-Wallace Theorem we can of course find a dissection of an equilateral triangle into a square of equal area. The geometric detail will be left to the reader as an exercise (see Figure 19), with a hint that BH and PE are parallel.

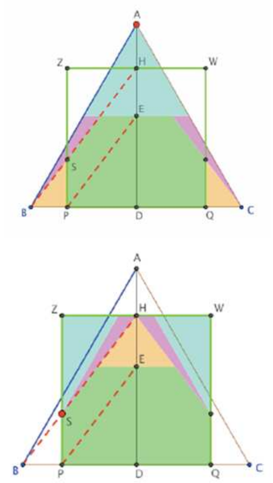

Figure 19

But what is more interesting is to figure out how to carry out the dissection shown in Figure 17. The discussion will lead to some interesting geometry as well as an instructive problem solving activity. Note that $DB = DA$ and $EC = EA$ so that D and E are midpoints of AB, AC respectively (see Figure 20). Angles at P and Q are right angles so that DP and GQ are perpendicular to ZE. If one checks the angles at the various joints, they come out correct. Since $BZ + CG = ZG$, we have $ZG = BC/2$. These much at least guarantee that $PZQGQEQPD$ is a rectangle. But is it a square? It will be a square if $2DP = PZ + QZ = PZ + PE = ZE$. This step of locating Z on BC, perhaps the hardest part, will be left as an exercise for the reader. [Hint: Construct FE perpendicular to DE and set $FE = DE$. If Z is a point on BC that works, then what can we say about FZ?] This example illustrates the power of deductive reasoning and the need for formal proof in addition to heuristic explanation.

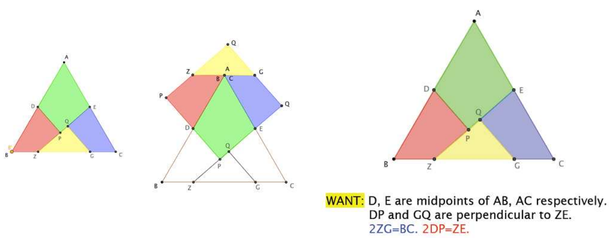

Figure 20

Finally, I like to suggest that three-dimensional geometry is perhaps a more natural setting for students, as we live in a three-dimensional world. In comparison, in some sense plane geometry is more artificial and is an abstraction of reality, hence not necessarily easier. Besides, nowadays there are good softwares on three-dimensional geometry to aid in the learning. It reminds me of an interesting incident involving a question in SAT (Scholastic

Aptitude Test) in the USA that took place in October of 1980. The question shows two polyhedra, a regular tetrahedron $ABCD$ with all edges of length a, and a square-based pyramid $EFGHI$ with all edges of length a. The two polyhedra are glued together at the faces ABC and EGF with vertices A, B, C glued to E, G, F respectively (see Figure 21).

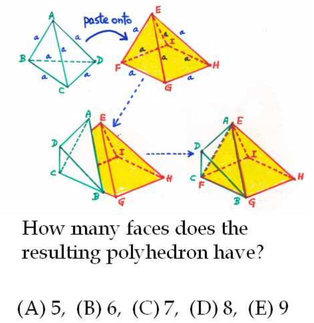

Figure 21

The question asks how many faces the resulting polyhedron has with one choice from: (A) 5, (B) 6, (C) 7, (D) 8, (E) 9. One candidate picked (A) as his answer rather than the 'model answer' (C) – 4 faces plus 5 faces minus 2 (concealed) faces. According to the story reported in *New York Times* [dated March 17, 1981], the student reasoned with SAT authority and got back his marks for that question! It turns out that after gluing, ABD and EGH are coplanar, and so are ACD and EFI. There are different ways to see this, relying on mathematical means at the student's disposal at different levels and involving computation to different degree. But it would be instructive to play around with three-dimensional models and try to understand the result by means of synthetic geometry without having to do any computation (see Figure 22).

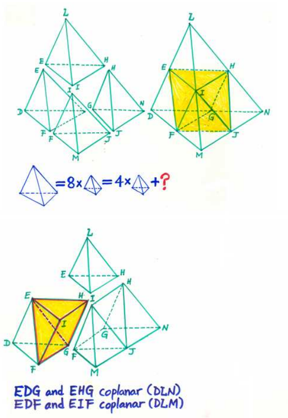

Figure 22

In (Sharygin, 2004) Igor Fedorovich Sharygin expresses this view more eloquently:

"One of the major and difficult problems of school Geometry is the problem of locating the teaching of plane and solid Geometry. On one hand, from the epistemological point of view solid Geometry is prime, the solid body is the fundamental and the most important object of geometrical investigations. (...) And plane figures are but mathematical abstractions, and only a well-trained person with a good level of abstract thinking is prepared to investigate them. On the other hand, from the point of view of mathematical theory it is more natural to move from plane to solid Geometry. Besides this, most of the methods in solid Geometry are based on various tricks, reducing a solid Geometry problem to one or more plane Geometry ones. (......) We've run across a striking contradiction, or I'd rather say, a whole tangle of contradicts, and it is not possible to resolve them all, since just as in quantum Physics, there is an uncertainty principle, saying that while improving one element we will inevitably worsen another ones."

5. The style and content of textbooks in geometry have always been an issue of debate. For a relatively long period of time that spanned many centuries, Euclid's *Elements* or its various versions compiled by later authors, sometimes with added material, were standard textbooks people learnt geometry from.

In the mid 18th century the French mathematician Alexis-Claude Clairaut wrote a textbook *Éléments de géométrie* (1741; 1753) with a style and underlying pedagogical philosophy quite different from that of Euclid's *Elements*. He said at the beginning of the book (Clairaut, 1741/2006):
"Although geometry in itself is an abstract subject, one has to admit that the difficulties that discourage those who begin to study it, mostly occur by the way it is taught in the usual elementary books. One always starts with a large number of definitions, postulates, axioms and preliminary principles, which seems to promise nothing but dryness to the reader. The theorems that follow do not fix the mind on things of interest, or are otherwise difficult to understand, so in the end the beginners tire themselves and are being discouraged, before having any idea at all, about what one is trying to teach them." Clairaut made use of surveying the terrain as an appropriate means to introduce the basic notions and first principles of Euclidean geometry. A more noteworthy feature is how the author carried out his promise that: "(. . .) but I hope that it will have still another important use, that it will accustom the mind to searching and discovering because I have avoided with care to present a proposition in the form of a theorem; that is to say, of propositions of which is shown why it is true, without showing in which way it was discovered." (I like to thank Harm Jan Smid of Delft University of Technology in the Netherlands in sharing with me his translation of the preface of Clairaut's book.)

Let us illustrate with the exposition of the Pythagoras' Theorem in Part II of Clairaut's book. The result appears as Proposition XVIII ("The square on the hypotenuse of a right triangle is equal to the sum of the squares on the two other sides.") with an accompanying figure (see Figure 23).

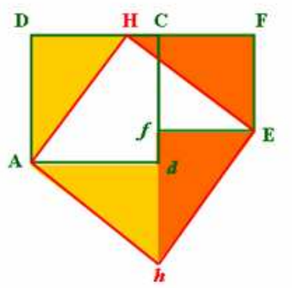

Figure 23

To understand how the result and the figure come about readers are first introduced to Proposition XVI ("To make a square equal in area double that of another square."). Again, an accompanying figure (see Figure 24) yields an easy solution. (I have modified the labelling slightly from that of the original figure to obtain an easier comparison with Figure 23.)

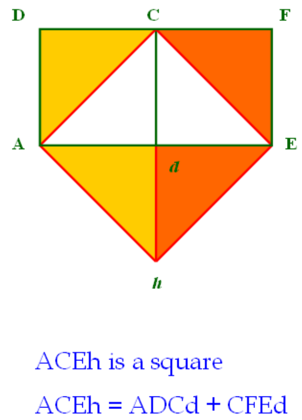

Figure 24

It is now natural to follow up with a more general question, which appears as Proposition XVII ("To make a square equal in area to two other taken together."). The author explained how to borrow from the idea in Proposition XVI with the help of a figure (see Figure 23), "Following the trend of thought in XVI we try to find a point H on DF such that (i) when ADH, EFH are turned around A, E to Adh, Efh [respectively], they join at a point h. (ii) AH, HE, Eh, hA are equal and perpendicular to each other." This can be accomplished by taking H on DF such that $DH = CF = EF$. From this construction Proposition XVIII (Pythagoras' Theorem) falls out as a by-product! It is a nice example of a proof *giving rise* *to a theorem* instead of a proof *validating a theorem*.

Towards the end of the 18th century another French mathematician Adrien-Marie Legendre wrote a textbook on geometry with the same title as that by Clairaut. His *Éléments de géométrie* (1794; many editions in subsequent years) had been a leading textbook for a century, and became a prototype of later geometry textbooks. In principle, Legendre's textbook is nearer in spirit to Euclid's *Elements* than Clairaut's book. However, he greatly rearranged and simplified many of the propositions from Eulcid's *Elements*, making use of knowledge of arithmetic and trigonometry of his time so that readers would find the study going more smoothly. The textbook was made even more famous through its English translation by Charles Davies of the West Point Academy in 1852, so much so that "Davies' Legendre" became a synonym of geometry in 19th century America!

The following chronology of events that took place in 19th century England centered around a debate on the suitability of Euclid's *Elements* as a school textbook. In 1869, in a paper titled "A plea for the mathematician" [*Nature*, 1 (1869-70), James Joseph Sylvester made public his view that Euclid's *Elements* should be abandoned as a textbook. He was supported by a school master James Maurice Wilson, who published a textbook *Elementary Geometry* in the year before. In the other camp, Augustus De Morgan wrote a negative review on Wilson's textbook. Charles Lutwidge Dodgson (better known by his pseudonym Lewis Carroll) published *Euclid and His Modern Rivals* in 1868. Isaac Todhunter wrote *Euclid for the Use of Schools and Colleges: Comprising the First Six Books and Portions of the Eleventh and Twelfth Books* in 1862. They argued for the rentention of Euclid's *Elements* on the ground of its historical and humanistic value. The debate went on. In 1870 a meeting of thirty six headmasters of public schools was called, and in the next year a special committee of the British Association for the Advancement of Science was formed to look into the re-evaluation of Euclid's *Elements* as a textbook. The special committee comprised members who were mostly university mathematician, such as Arthur Cayley, T. Archor Hirst, William Kingdon Clifford, George Salmon, Henry John Stanley Smith, James Joseph Sylvester, James Maurice Wilson.

In 1871 the organization AIGT (Association for the Improvement of Geometrical Teaching) was established, which in 1897 became Mathematical Association with its official journal *Mathematical Gazette* started in 1894. The first two tasks carried out by AIGT in 1875 were to write a syllabus of plane geometry and to issue an AIGT Report, which practice was continued annually into 1893. In 1901 John Perry renewed the reform movement and published an important paper titled "The teaching of mathematics" [*Educational Review*, 23 (1902), 158-181]. By 1903 the sequence of theorems in Euclid was no longer enforced in examinations in Oxford and Cambridge Universities, with accompanying changes in the school curriculum. Interested readers can consult (Richards, 1988) for a much more detailed in-depth account of this episode. An interesting account of the corresponding scene in Italy during the same period is given in (Menghini, 1996).

My own experience in studying geometry in school in the late 1950s is to read *Elements* after a fashion (Siu, 2003):
"I was brought up with a large dose of synthetic geometry replete with lots of proofs and construction problems. Not only was I accustomed to the notion of proof and logic before starting undergraduate study, but in school geometry I tasted the joy of discovery and the joy of succeeding in understanding something which was tangible (you can at least draw some pictures even if you do not know why it has to be like that at first) but not obvious (you do not know why it is like that at first). Geometry is a subject in which one can exercise logical discipline and free imagination *at the same time*. Developing a liking for geometry also enabled me to look at problems in other subjects from a geometric viewpoint. This helped in particular in the study of analysis. After all, calculus is the process of linear approximation, and linear problems fall within in the purview of linear algebra, which is in itself akin to geometry. Of course, it does not work all the time and different persons are accustomed to different ways of thinking; nevertheless it offers an alternative. Many students today are not accustomed to this flexibility in framework in their study of mathematics."

The textbook I used at the time was *The Essentials of School Geometry* (1933) by A.B. Mayne. In the preface the author wrote (Mayne, 1933):
"Those teachers and examiners who are in a position to compare the results obtained by the teaching of Geometry in schools today with those obtained before the dethronement of Euclid agree almost unanimously that there have been both gain and loss. On the one hand, almost all pupils today acquire much more power in applying and reasoning from the fundamental facts of Geometry than did their predecessors, but, on the other hand, their reasoning is often less rigorous, and the average pupil often fails lamentably to reproduce the standard proofs when called upon in examinations. Almost all would agree that the gain outweighs the loss; for the educational value of the subject lies far more in the former than in the latter accomplishment. There are many, however, who think that the loss need not accompany the gain, (...)"

Nobody would deny the benefit of an axiomatic-deductive approach in learning geometry, just as nobody would deny the benefit of a geometrical intuitive approach. The question is when to shift gear from one to the other, and how to make students aware of the shift as well as to let them see the necessity of such a shift.

In a famous conference held in November of 1959, the Royaumont seminar of OEEC, the French mathematician Jean Dieudonné introduced the slogan "Euclid must go!" Another French mathematician, Réne Thom, opposed to this and expressed his view eloquently in a paper, "Les mathématiques modernes: Une erreur pédagogique et philosophique?" (Thom, 1970/1971) while Dieudonné responded with a paper, "Should we teach "modern" mathematics?" (Dieudonńe, 1973) From then on the learning and teaching of geometry have gone through a lot of changes — for better or for worse. Interested readers can consult (Howson, 2003; Pritchard, 2003).

The generation of today are more fortunate in having access to modern learning aids, among them softwares in geometry. As said at the beginning, I am no expert in DGE to offer advice on the topic. But I like to introduce a software project a group of school teachers in Hong Kong develop recently to help investigate how school pupils learn geometry and to help them learn better. (The main designer is Arthur Lee of the Faculty of Education at University of Hong Kong, and the software is placed at http://geometry.eclass.hk/ with further detail obtainable at http://www.hkame.org.hk/)

I will illustrate with three examples.

(1) Drag D so that the quadrilateral $ABCD$ has at least one pair of parallel sides (see Figure 25).

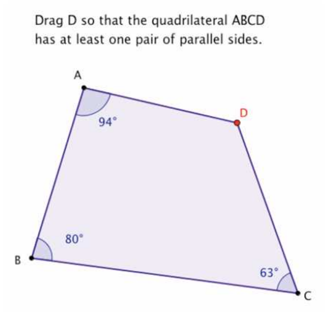

Figure 25

By examining the answers from students one can reflect on the extent students understand the notion of paralleleism.

(2) Drag point C so that the area of the triangle becomes 12 square centimeters (see Figure 26).

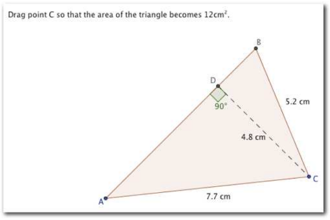

Figure 26

The lengths of the two sides and that of the altitude are displayed, but not the length of the base. The exercise checks the knowledge of students on area. The student needs to find out the length of the base by dragging C to a special position.

(3) Drag points J and K to make the figure the net of a square based pyramid (see Figure 27).

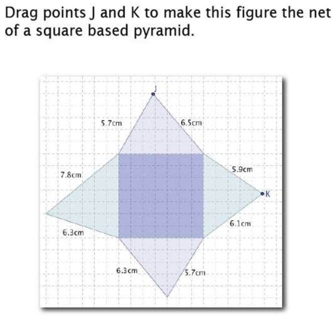

Figure 27

The student learns through doing this exercise certain constraints imposed upon the net. It helps to nurture in the student a sense of spatial visualization.

6. Finally I like to highlight one benefit of learning geometry that is important but is seldom emphasized, namely, the value of geometry in character building.

In 1607 the first translated text of *Elements* (the first six books from a Latin version compiled by Christopher Clavius in the late 16th century) in Chinese was published, a landmark collaboration between the Ming Dynasty Scholar-minister XU Guang-qi and the Italian Jesuit Matteo Ricci. The Chinese title was rendered *Jihe Yuanben.* In an essay on the book [*Discourse on the Jihe Yuanben*] Xu said:

"The benefit derived from studying this book [the *Elements*] is many. It can dispel shallowness of those who learn the theory and make them think deep. It can supply facility for those who learn the method and make them think elegantly. Hence everyone in this world should study the book (...) Five categories of personality will not learn from this book: those who are impetuous, those who are thoughtless, those who are complacent, those who are envious, those who are arrogant. Thus to learn from this book one not only strengthens one's intellectual capacity but also builds a moral base." (The translation is taken from (Siu, 2008).)

This message is echoed in modern time, as the late Russian mathematics educator Igor Fedorovich Sharygin (1937-2004)once said:

"Geometry is a phenomenon of the human culture. (...) Geometry, as well as mathematics in general, helps in moral and ethical education of children. (...) Geometry develops mathematical intuition, introduces a person to independent mathematical creativity. (...) Geometry is a point of minimum for the distance between school mathematics and the mathematics of high level." In addition, Sharygin places emphasis on proof for a moral reason (Sharygin, 2004): "Learning mathematics builds up our virtues, sharpens our sense of justice and our dignity, strengthens our innate honesty and our principles. The life of mathematical society is based on the idea of proof, one of the most highly moral ideas in the world."

References

Agashe, S.D. (1989). The axiomatic method: Its origin and purpose, *Journal of the Indian Council of Philosophical Research*, 6(3), 109-118.

Andersen, K. (2007). *The Geometry of an Art: The History of the Mathematical Theory of Perspective From Alberti to Monge.* New York: Springer.

Armstrong, M.A. (1988). *Groups and Symmetry.* New York: Springer-Verlag.

Baker, H.F. (Ed.) (1904-1910). *The Collected Mathematical Papers of James Joseph Sylvester, 4 volumes.* Cambridge: Cambridge University Press.

Clairaut, A.C. (1741/2006). *Éléments de géométrie.* Paris: Éditions Jacques Gabay (reprinted edition, 2006).

Dieudonné, J.A. (1973). Should we teach "modern" mathematics? *American Scientists*, 61, 16-19.

Freudenthal, H. (1973). *Mathematics as an Educational Task.* Dordrecht: Reidel

Godfrey, C. (1910). The Board of Education circular on the teaching of geometry, *Math. Gazette*, 5, 195-200.

Godfrey, C., Siddons, A.W. (1903). *Elementary Geometry, Practical and Theoretical.* Cambridge: Cambridge University Press.

Heath, T.L. (1925/1956). *The Thirteen Books of Euclid's Elements*, 2nd Edition. Cambridge: Cambridge University Press (reprinted by Dover Publications in 1956).

Howson, G. (2003). Geometry: 1950-70. In D. Coray et al (Eds.), *One Hundred Years of L'Enseignement Mathématique: Moments of Mathematics Education in the Twentieth Century* (pp.115-131). Geneva: L'Enseignement Mathématique.

Martin, G.E. (1982). *Transformation Geometry: An Introduction to Symmetry.* New York: Springer-Verlag.

Mayne, A.B. (1933). *The Essentials of School Geometry.* London: Macmillan.

Mayne, A.B. (1938). *The Essentials of School Algebra.* London: Macmillan.

Menghini, M. (1996). The Euclidean method in geometry teaching. In H.N. Jahnke et al (Eds.), *History of Mathematics and Education: Ideas and Experiences* (pp.195-212). Göttingen, Vandenhoeck & Ruprecht.

Morrow, G.B. (1970). *A Commentary on the First Book of Euclid's Elements by Proclus.* Princeton: Princeton University Press.

Pritchard, C. (Ed.) (2003). *The Changing Shape of Geometry: Celebrating a Century of Geometry and Geometry Teaching.* Cambridge: Cambridge University Press.

Richards, J.L. (1988). *Mathematical Visions: The Pursuit of Geometry in Victorian England.* Boston: Academic Press.

Rouche, N. (2003). Reaction to papers on geometry. In D. Coray et al (Eds.), *One Hundred Years of L'Enseignement Mathématique: Moments of Mathematics Education in the Twentieth Century* (pp.155-159). Geneva: L'Enseignement Mathématique.

Royal Society (2001). *Teaching and Learning Geometry 11-19. Report of a Royal Society / Joint Mathematics Council Working Group.* London: Royal Society.

Russell, B. (1967). *The Autobiography of Betrand Russell*, *Volume 1 (1872-1914).* London: Allen & Unwin.

Schattschneider, D., Emmer, M. (2003). *M.C. Escher's Legacy: A Centennial Celebration.* Berlin: Springer-Verlag.

Schilepp, P.A. (Ed.) (1949/1957). *Albert Einstein: Philosopher-Scientist.* 2nd edition (originally published in 1949). New York: Tudor.

Semple, J.G., Kneebone, G.T. (1959). *Algebraic Curves.* Oxford: Clarendon Press.

Sharygin, I.F. (2004). On the concept of school geometry. In J. Wang, B. Xu (Eds.), *Trends and Challenges in Mathematics Education* (pp.43-51). Shanghai: East China Normal University Press.

Siu, M.K. (2003), Learning and teaching of analysis in the mid-twentieth century: A semi-personal observation. In D. Coray et al. (Eds.), *One Hundred Years of L'Enseignement Mathématique: Moments of Mathematics Education in the Twentieth Century* (pp.177-190). Geneva: L'Enseignement Mathématique.

Siu, M.K. (2007), Mr. Ou [Euclid] in China for four hundred years [in Chinese], *Science & Culture Review*, 4(6), 12-30.

Siu, M.K. (2008). Proof as a practice of mathematical pursuit in a cultural, socio-political and intellectual context, to appear in *Zentralblatt für Didaktik der Mathematik.*

Stahl, S. (1993). *The Poincaré Half-plane: A Gateway to Modern Geometry.* Boston: Jones & Bartlett Publishers.

Starkman, M.K. (Ed.) (1962). *Gulliver's Travels and Other Writings by Jonathan Swift.* New York: Bantam Books.

Thom, R. (1970/1971). Les mathématiques modernes: Une erreur pédagogique et philosophique? *L'Age de la Science*, 3, 225-236; translation in *American Scientists*, 59 (1971), 695-699.

Tufte, E.R. (1983). *The Visual Display of Quantitative Information.* Cheshire: Graphics Press.

Willmore, T.J. (1970). Whither geometry? *Math. Gazette*, 54, 216-224.