Ergodicity, bounded holomorphic functions and geometric structures in rigidity results on bounded symmetric domains

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ICCM, Hangzhou, Dec 17-22, 2007

Hermitian Metric Rigidity (Mok 87,To 89)

Ω irr. bounded symmetric domain,
rank$(\Omega) \geq 2$

$\Gamma \subset \mathrm{Aut}(\Omega)$ torsion-free lattice

$X := \Omega/\Gamma$, $g =$ canonical Kähler-Einstein metric on X

$h =$ Hermitian metric on X

$\Theta(h) =$ Curvature of (T_X, h)

$\Theta(h) \leq 0$ in the sense of Griffiths, i.e.,

$\Theta_{\alpha\overline{\alpha}\beta\overline{\beta}}(h) \leq 0 \quad \forall \alpha, \beta \in T_x(X)$.

> Then,
>
> $h \equiv cg$ for some constant $c > 0$.

Theorem. (Rigidity on Holomorphic Maps)

Ω irr. bounded symmetric domain,
$\mathrm{rank}(\Omega) \geq 2$

$\Gamma \subset \mathrm{Aut}(\Omega)$ torsion-free lattice,

$X := \Omega/\Gamma$, $g =$ canonical Kähler-Einstein metric

$(N, h) =$ Hermitian manifold of nonpositive curvature in the sense of Griffiths.

$f : X \to N$ nonconstant holomorphic map

$$\boxed{\begin{array}{l}\text{Then,} \\ \qquad f : X \to N \text{ is an immersion }, \\ \text{totally-geodesic if } (N,h) \text{ is Kähler }.\end{array}}$$

Remarks:

(1) When $N = \Omega'/\Gamma'$, is Hermitian locally symmetric, $f : X \to N$ lifts to

$$F : \Omega \to \Omega' \ , \ \text{totally geodesic} \ .$$

In particular, F is an *embedding*.

(2) The same as in (1) can be asserted if we assume that (N, h) is a complete Kähler manifold of nonpositive *Riemannian* Sectional curvature, by Comparison Theorems.

Ω irr. BSD, $\mathrm{rank}(\Omega) \geq 2$

$G = \mathrm{Aut}_o(\Omega)$, $o \in \Omega$, $K = Isot_o(\Omega; o) \subset G$

$\exists$ polydisk $P \cong \Delta^r \subset \Omega$, totally geodesic

$\bigcup\limits_{k \in K} kP = \Omega$. (Polydisk Theorem).

$D = \Delta \times \{(0, \ldots, 0)\}$ *minimal disk* (e.g.)

$\eta \in T_x(\Omega)$ is called a *characteristic vector*

$\Leftrightarrow \eta$ is tangent to a minimal disk.

$$\mathcal{S}_\Omega = \{[\eta] \in \mathbb{P}T_\Omega : \eta \text{ is a char. vect.}\}$$

$\mathcal{S} = \mathcal{S}_\Omega / \Gamma$, the minimal *characteristic* bundle on X.

- Hermitian Metric Rigidity for the compact case is proven by an integral curvature identity on $\mathcal{S}$.
- For the noncompact case. To studied the asymptotic behavior of Hermitian metrics.

Ergodic actions

$(\mathfrak{X}, \mu)$ σ-finite measure space

$\mathfrak{G}$ group acting on $(\mathfrak{X}, \mu)$ as measure-preserving transformations

> We say that <u>$\mathfrak{G}$ acts ergodically on (X, μ)</u>
>
> if and only if
>
> every $\mathfrak{G}$-invariant subset S is either of 0 or full measure, i.e.
>
> $$\mu(S) = 0 \text{ or } \mu(\mathfrak{X} - S) = 0 \ .$$

More generally, we do not require $\mathfrak{G}$ to be measure-preserving. Two measures μ and μ' on a Borel space $(\mathfrak{X}, \mathcal{B})$ is said to be equivalent if and only if they have the same null sets, i.e. $\mu(S) = 0 \Leftrightarrow \mu'(S) = 0$.

Denote by $(X, \{\mu\})$ the measure class, i.e. identifying equivalent measures on $(X, \mathcal{B})$, where $\mathcal{B}$ is understood.

We consider actions of $\mathfrak{X}$ on $(X, \{\mu\})$ such that $\gamma^* \mu \sim \mu$ for every $\gamma \in \mathfrak{G}$. Then, $\mathfrak{G}$ acts ergodically on $(X, \{\mu\})$ if and only if the space of null-sets is preserved under any $\gamma \in \mathfrak{G}$.

Example

G semisimple Lie group $H \subset G$ closed subgroup. Then G/H carries a canonical measure class, and G acts ergodically on G/H.

Density Lemma. (Raghunathan)

$\Omega = \Omega_1 \times \cdots \times \Omega_m$ *reducible* bounded symmetric domain

$I = (i(1), \ldots, i(p))$, $1 \leq i(i), \ldots, i(p))$,
$1 \leq i(1) < \cdots < i(p) \leq m$

$\mathrm{pr}_I : \mathrm{Aut}_o(\Omega) \to \mathrm{Aut}_o(\Omega_{i(1)}) \times \cdots \times \mathrm{Aut}_o(\Omega_{i(p)})$
canonical projection

Then,

$$\overline{\mathrm{pr}_I(\Gamma)} = \mathrm{Aut}_o(\Omega_{i(1)}) \times \cdots \times \mathrm{Aut}_o(\Omega_{i(p)})$$

whenever $1 \leq p < m$.

Moore's Ergodicity Theorem

G semisimple Lie group over $\mathbb{R}$

$\Gamma \subset G$ irreducible lattice,
i.e., $\mathrm{Vol}(\Gamma \setminus G) < \infty$

$H \subset G$ closed subgroup

H acts on $\Gamma \setminus G$ by right multiplication. Then,

$$H \text{ acts ergodically } \Leftrightarrow H \text{ noncompact } .$$

Corollary. Γ acts ergodically on $G/H \Leftrightarrow H$ noncompact.

Lemma. $\exists E \subset G/H$ null subset such that for any $gH \in G/H - E$, $\Gamma(gH)$ is dense in G/H.

Integral Curvature Identity

Ω irr. BSD $\mathrm{rank}(\Omega) \geq 2$

$\Gamma \subset \mathrm{Aut}(\Omega)$ torsion-free lattice, $X := \Omega/\Gamma$,
$g =$ KE metric on X, $\omega =$ Kähler form

$(L, \hat{g}) \to \mathbb{P}T_X$ taugological line bundle

$\theta = -c_1(L, \hat{g}) \geq 0$, $\mathrm{Ker}\theta([\alpha]) \subset T_{[\alpha]}(\mathcal{S})$
$\forall [\alpha] \in \mathcal{S}$, $\mathrm{rank}(\mathrm{Ker}\theta([\alpha])) = q$,

$\pi : \mathbb{P}T_X \to X$; $\nu = \pi^*\omega - c_1(L, \hat{g}) > 0$
loc. homogeneous Kähler form. Then,

$$\begin{aligned} 0 &= \int_{\mathcal{S}} [-c_1(L, \hat{g})]^{2n-2q} \wedge \nu^{q-1} \\ &= \int_{\mathcal{S}} [-c_1(L, h)] \wedge [-c_1(L, \hat{g})]^{2n-2q-1} \wedge \nu^{q-1} \end{aligned}$$

for any Hermitian metric h on L. The integrand ≥ 0, hence $\equiv 0$, if $c_1(L, h) \leq 0$.

The minimal characteristic bundle as a foliated manifold

- $\Omega =$ an irreducible bounded symmetric domain of rank ≥ 2.
- $\Gamma \subset \mathrm{Aut}(\Omega)$ torsion-free discrete subgroup $X := \Omega/\Gamma$ of finite volume.
- $\mathcal{S} :=$ minimal characteristic bundle on X.
- There is a canonical foliation $\mathcal{N}$ on $\mathcal{S}$, as follows.
- For any $[\eta] \in \mathbb{P}T_o(\Omega)$, $N_\eta := \{\zeta \in T_o(\Omega) = R_{\eta\overline{\eta}\zeta\overline{\zeta}} = 0\}$, the null-space of η. Write $q = \dim(N_\alpha)$ for $[\alpha] \in \mathcal{S}$. Let $\triangle \subset \Omega$ be the unique minimal disk passing through o such that $T_o(\triangle) = \mathbb{C}\alpha$. Then, there exists a unique totally geodesic complex submanifold Ω_o passing through o such that $T_o(\Omega_o) = N_\alpha$. Moreover, $\mathbb{C}\alpha \oplus N_\alpha$ is tan-

gent to a unique totally geodesic $(q+1)$-dimensional complex submanifold $\cong \triangle \times \Omega_o$.

- Identify $\{o\} \times \Omega_o$ with Ω_o. For every $z \in \Omega_o$ write $[\alpha(z)] := \mathbb{P}T_z(\triangle \times \{z\}) \in \mathcal{S}_z(\Omega)$. As z runs over Ω_o, this defines a lifting of Ω_o to a complex submanifold $F \subset \mathcal{S}(\Omega)$ which is by definition the leaf of the lifting of $\mathcal{N}$ to $\mathcal{S}(\Omega)$ passing through $[\alpha]$. Note that G acts transitively on $\mathcal{S}(\Omega)$. Let $H \subset G$ be the closed subgroup which fixes Ω_o as a set. The leaf space of the lifted foliation on $\Omega \cong G/H$. Set-theoretically the leaf space of $\mathcal{N}$ is given by $\Gamma \setminus G/H$.
- For $[\alpha] \in \mathcal{S}$, $T_{[\alpha]}(\mathcal{N}) = \mathrm{Ker}\theta([\alpha])$.

<u>Complex Finsler metrics</u>

$\|\eta\|$ defined, $\|\lambda\eta\| = |\lambda|\|\eta\|$ whenever $\lambda \in \mathbb{C}$, no inner products

Complex Finsler metric on T_X
$=$ Hermitian metric h on the tautological line bundle $L \to \mathbb{P}T_X$

$\|\cdot\|$ continuous $\Leftrightarrow h$ continuous, etc.

$(X, \|\cdot\|)$ is of nonpositive curvaure $\overset{\text{def}}{\Leftrightarrow} (L, h)$ is of nonpositive curvature.

For $\|\cdot\|$ smooth, $\Theta(L,h) \leq 0$ defined.
h given by $e^{-\varphi}$ locally, $\Theta(L,h) = \sqrt{-1}\partial\overline{\partial}\varphi$, which makes sense also for h (hence φ) continuous. For $\|\cdot\|$ continuous

- We say that $(X, \|\cdot\|)$ is of nonpositive curvature iff φ is plurisubharmonic.

Carathéodory Pseudometric

M complex manifold,

$\mathcal{H}(M) := \{\text{holo. functions } f : M \to \Delta\}$

$\eta \in T_x(M)$, $\|\eta\|_\kappa := \sup_{f \in \mathcal{H}(M)} \|f_*\eta\|_{ds^2_\Delta}$,

where ds^2_Δ = Poincaré metric on Δ

κ = Carathéodory pseudometric on M

κ nondegenerate for $M = D \Subset \mathbb{C}^n$ a bounded domain

κ is invariant under $\mathrm{Aut}(M)$. It descends to any quotient of M by a torsion-free discrete group of automorphisms. The quotient pseudometric is called the induced Carathéodory pseudometric.

κ agrees with the Bergman metric on B^n (up to a constant).

On Δ^n, $\eta \in T_x(\Delta^n)$, $\eta = (\eta_1, \ldots, \eta_n)$

$$\|\eta\|_\kappa = \sup_k \|\eta_k\|_{ds^2_\Delta} \ .$$

For Ω any bounded symmetric domain,
$P \subset \Omega$ maximal polydisk,
$x \in P$, $\eta \in T_x(P)$, we have

$$\|\eta\|_{\kappa(P)} = \|\eta\|_{\kappa(\Omega)} \ .$$

In other words,

> $P \subset \Omega$ is an isometric embedding with respect to Carathéodory metrics.

<u>Finsler Metric Rigidity</u> (Mok 2002)

Ω bounded symmetric domain,
$\mathrm{rank}(\Omega) \geq 2$.
$\Gamma \subset \mathrm{Aut}(\Omega)$ torsion-free lattice
$X := \Omega/\Gamma$, g = canonical Kähler-Einstein metric on X
h = continuous complex Finsler metric of non-positive curvature
$\Omega = \Omega_1 \times \cdots \times \Omega_m$, Ω_k irr. factor,
$T_\Omega = T_1 \oplus \cdots \oplus T_M$

> Then,
>
> $\exists c_1, \ldots, c_m > 0$ such that
>
> $\|\eta^{(k)}\|_h = c_k \|\eta^{(k)}\|_g$
>
> for any $\eta^{(k)} \in T_x(X)$ whose lifting to Ω belongs to T_k.

Theorem 1. (Mok, Invent Math 2004)
Ω irreducible bounded symmetric domain, $\mathrm{rank}(\Omega) \geq 2$

$\Gamma \subset \mathrm{Aut}(\Omega)$ torsion-free lattice

$X := \Omega/\Gamma$, $N =$ complex manifold. Assume

- $f : X \to N$ holomorphic map,
  $F : \Omega \to \widetilde{N}$ lifting to universal covers.
- $\exists$ *bounded* holomorphic function on $\widetilde{N}$ such that $F^*h \not\equiv$ Constant.

Then,

$$F : \Omega \to \widetilde{N} \text{ is an } \textit{embedding}.$$

In particular,

$$f : X \to N \text{ is an immersion}$$

Theorem 1'.

Analogue for locally reducible case,
e.g., irr. quotients of the polydisk Δ^n

$\Omega =$ bounded symmetric domain

$\Omega = \Omega_1 \times \cdots \times \Omega_m, \Omega_k$ irreducible factor

$\Omega_1' := \Omega_1 \times \{(x_2, \ldots, x_m)\}$ called an

irreducible factor subdomain, etc.

Then, the analogue of Theorem 1 holds under the assumption
(†)For any $k, 1 \leq k \leq m, \exists$ bounded holomorphic function h_k on $\widetilde{N}$ such that $F^* h_k \not\equiv$ Constant on some Ω_k' .

Embedding Theorem $=$ Theorems $1 + 1'$.

Theorem 2.

Ω bounded symmetric domain,
$\operatorname{rank}(\Omega) \geq 2$

$\Gamma \subset \operatorname{Aut}(\Omega)$ torsion-free irreducible lattice

$X := \Omega/\Gamma$

D *arbitrary* bounded domain,
$\Gamma' \subset \operatorname{Aut}(D)$ torsion-free, discrete

$N := D/\Gamma'$

$f : X \to N$ nonconstant holomorphic map, $F : \Omega \to \widetilde{N}$ lifting to universal covers

$$\boxed{\begin{array}{l}\text{Then,}\\ \qquad F : \Omega \to D \text{ is an } \textit{embedding.}\end{array}}$$

Theorem 3.

Ω bounded symmetric domain,
$\mathrm{rank}(\Omega) \geq 2$

$\Gamma \subset \mathrm{Aut}(\Omega)$ torsion-free irreducible lattice

$X := \Omega/\Gamma$, Z = normal complex space

$f : X \to Z$ proper holomorphic map

Then, either

(a) f is an unramified covering, OR

(b) $|\pi_1(Z)| < \infty$.

$\mathcal{F}$-extremal bounded holomorphic functions

$f : X = \Delta^n/\Gamma \to N$

$F : \Delta^n \to \widetilde{N}$

$\mathcal{H} = \{h : \widetilde{N} \to \Delta \text{ holomorphic}\}$

$\mathcal{F} := F^*\mathcal{H} = \{h \circ F : \Delta^n \to \Delta\}$

Proposition.
$D^{n-1} = \{0\} \times \Delta^{n-1} \subset \Delta^n$; $\eta \in T_o(\Delta^n)$, $\eta \perp T_o(D^{n-1})$. $g = h \circ F$, $g \in \mathcal{F}$. Then,

$$\boxed{\begin{array}{l} \|df(\eta)\|_{\kappa(\widetilde{N})} = \|ds(\eta)\|_{\text{Poin.}} \\ \Rightarrow \quad s\big|_{D^{n-1}} \equiv \text{ Constant }. \end{array}}$$

Here, s is called an $\mathcal{F}$-extremal bounded holomorphic function adapted to η.

Proof. Take $s(0;0) = 0$. Write

$$\|\eta\|_{\mathcal{F}} = \sup\{\|ds(\eta)\|_{\text{Poin.}} : s \in \mathcal{F}\} \ ,$$

so that

$$\|\eta\|_{\mathcal{F}} = \|df(\eta)\|_{\kappa(\widetilde{N})} \ .$$

Then,

$$\bullet \ \|\eta(0;z)\|_{\mathcal{F}} \geq \|ds(\eta)(0;z)\|_{\text{Poin.}}$$
$$= \frac{|ds(\eta)(0;z)|}{1 - |s(0;z)|^2} \geq |ds(\eta)(0;z)| \ ;$$

$$\bullet \ \|\eta(0;0)\|_{\mathcal{F}} = |ds(\eta)(0;0)| \ .$$

Finsler metric rigidity

$$\Rightarrow \|\eta(0;z)\|_{\mathcal{F}} = \lambda \ ,$$
$$\textit{independent of } \ z \in \Delta^{n-1} \ .$$

Thus,

$$\begin{cases} \log|ds(\eta)(0;z)| \leq \log\lambda \\ \log|ds(\eta)(0;0)| = \log\lambda \end{cases}$$

$\log|ds(\eta)(0;z)|$ pluriharmonic in z

$$\Rightarrow \quad \log|ds(\eta)(0;z)| \equiv \log\lambda$$

$$\|\eta(0;z)\|_{\mathcal{F}} \equiv \log\lambda \equiv \log|ds(\eta)(0;z)| \ .$$

$$\Rightarrow \quad s(0;z) = 0 \text{ for any } z \in \Delta^{n-1} \ .$$

Proof that $f : X \to N$ is an immersion in Thm. 1' for $X = \Delta^n/\Gamma$ irreducible:

Suppose $\eta \in \mathrm{Ker}\ dF(o)$. $\eta = \eta_1 + \eta'$, orthogonal decomposition, $\eta_1 = \mathrm{Const.} \times \frac{\partial}{\partial z_1}$.

Let $s \in \mathcal{F}$ be $\mathcal{F}$-extremal, adapted to η_1, $s \equiv h \circ F$. (by Prop.)

s constant on $D^{n-1} \Rightarrow ds(\eta') = 0$

$\eta \in \mathrm{Ker}\ dF(o) \Rightarrow ds(\eta) = dh(F_*\eta)dF(\eta) = 0$. Hence,

$$ds(\eta_1) = ds(\eta) - ds(\eta') = 0\ .$$

If $\eta_1 \neq 0$, then, $ds(\frac{\partial}{\partial z_1})(0;0) = 0$.

By Proposition,

$$ds\Big(\frac{\partial}{\partial z_1}\Big)(0;z) = 0 \text{ for any } z \in \Delta^{n-1} \ ,$$

which contradicts Finsler metric rigidity.

We have proven:

$$\eta \in \text{Ker } dF(o) \Rightarrow \eta_1 = 0 \ .$$

Same argument gives $\eta = (\eta_1, \ldots, \eta_n) = 0$, so that Ker $(dF) \equiv 0$, *i.e.*

$$f \text{ is a holomorphic immersion} \ .$$

Proof that $F : \Delta^n \to \widetilde{N}$ *is an embedding in the cocompact case:*

By normal family argument, $\exists$ $\mathcal{F}$-extremal bounded holomorphic function s such that $s(x_1; z') \equiv s(x_1)$, $s = F^* h$.

Suppose $F(x) = F(y)$, $x_1 \neq y_1$.

$s(x) = h(F(x)) = h(F(y)) = s(y)$.

Hence, $s(x_1) = s(y_1)$. May assume $x_1 = 0$, $y_1 = |y_1| = r > 0$.

Density Lemma $\Rightarrow s(0) = s(re^{i\theta})$ for all $\theta \in \mathbb{R}$. Contradiction!

Difficulty in general case:

(1) Normal family argument fails for $\Gamma \subset \mathrm{Aut}(\Delta)^n$ non-uniform.

(2) For Ω irreducible, $\mathrm{rank}(\Omega) \geq 2$, $\Gamma \subset \mathrm{Aut}(\Omega)$ cocompact normal family argument may lead to maximal polydisk $P \subset \Omega$ such that $\{\gamma P : \gamma \in \Gamma\}$ is discrete.

In (2), there is an exceptional set when we apply Moore's Ergodicity Theorem on some moduli space of maximal polydisks. We cannot apply density argument on P.

Partial complex Finsler metric on Ω by averaging over geodesic circles

$$x \in \Omega\ , \quad \eta \in T_x(\Omega)$$

$$\|\eta\|_{\mathcal{F}} = \sup\{\|ds(\eta)\|_{\text{Poin.}} : s \in \mathcal{F} = F^*\mathcal{H}\}\ .$$

Suppose α characteristic. D_α minimal disk such that $\alpha \in T_x(D_\alpha)$. $\delta > 0$ fixed. Define

$$\|\alpha\|_{e(\mathcal{F})} = \sup\{\text{Average}(\|ds(\widetilde{\alpha}(y))\|_{\text{Poin.}}) : y \in \partial B_\alpha(x;\delta)\}$$

$\|\widetilde{\alpha}(y)\| = \|\alpha(x)\|$. $B_\alpha(x;\delta)$ geodesic disk on D_α.

Main Proposition. *s an $e(\mathcal{F})$-extremal bounded holomorphic function adapted to α. P maximal polydisk through x, $\alpha \in T_x(P)$. Then,*

$$s(z;z') \equiv s(z)\ .$$

Remarks:

(1) $\|\alpha\|_{e(\mathcal{F})}$ is defined on Ω and hence on $X = \Omega/\Gamma$ only for characteristic vectors α. It corresponds to a continuous Hermitian metric on the tautological line bundle $L \to \mathcal{S}$ over the characteristic bundle $\mathcal{S}$.

(2) $e(\mathcal{F})$, as a continuous Hermitian metric on $L \to \mathcal{S}$, is <u>not</u> *a-priori* of nonpositive curvature. It is only of nonpositive curvature when restricted to liftings of certain totally geodesic product complex submanifolds e.g. maximal polydisks P.

Proposition.

(Z, ω) compact Kähler manifold, $\dim_{\mathbb{C}} Z = m$.

$\theta \geq 0$ on Z, smooth closed (1, 1)-form.

Ker θ of constant rank on Z.

$\mathcal{K} =$ foliation on Z defined by Re(Ker θ).

Leave $\mathcal{L}$ of $\mathcal{K}$ automatically holomorphic.

$u : Z \to \mathbb{R}$ continuous such that

$$u|_{\mathcal{L}} \text{ is plurisubharmonic for any leaf } \mathcal{L}\ .$$

Then,

$u|_{\mathcal{L}}$ is *pluriharmonic* for every leaf $\mathcal{L}$.

If u is Lipschitz, then

$$u|_{\mathcal{L}} \equiv \text{ Const. for every } \mathcal{L}\ .$$

If $\exists$ a dense leaf of $\mathcal{K}$, then

$$u \equiv \text{ Const. on } Z\ .$$

Lemma.

$U \subset \mathbb{C}^n$ open; $a < b$.

$u : [a, b] \times U \longrightarrow \mathbb{R}$ continuous; $u_t(z) := u(t; z)$.

$u_t : U \longrightarrow \mathbb{R}$ plurisubharmonic

$\varphi, \psi : U \longrightarrow \mathbb{R}$ given by

$$\varphi(z) := \log \int_a^b e^{u_t(z)} dt$$
$$\psi(z) := \int_a^b u_t(z) dt \ .$$

Then

$$\boxed{\begin{array}{l} e^{\varphi}\sqrt{-1}\partial\overline{\partial}\varphi \geq e^{\psi}\sqrt{-1}\partial\overline{\partial}\psi \geq 0 \ . \\ \text{In particular,} \\ \quad \varphi, \psi \text{ are plurisubharmonic .} \end{array}}$$

Proof.

$$
\begin{aligned}
&(e^{u_1}+e^{u_2})\sqrt{-1}\partial\overline{\partial}\log(e^{u_1}+e^{u_2})\\
&= e^{u_1}\sqrt{-1}\partial\overline{\partial}u_1 + e^{u_2}\sqrt{-1}\partial\overline{\partial}u_2 +\\
&\qquad \frac{e^{u_1+u_2}}{e^{u_1}+e^{u_2}}\sqrt{-1}(\partial u_1 - \partial u_2)\wedge\overline{(\partial u_1 - \partial u_2)}\ .
\end{aligned}
$$

Apply now to finite Riemann sums and take limits. $\square$

The foliated minimal characteristic bundle with a transverse measure

- The closed (1,1)-form $\lambda := -c_1(L, \widehat{g})|_{\mathcal{S}}$ is a real 2-form on the real $2(n+p)$-dimensional underlying smooth manifold of the minimal characteristic bundle $\mathcal{S}$.
- As a skew-symmetric bilinear form on $\mathcal{S}$, λ is of constant rank $4p+2$.
- The foliation $\mathcal{N}$ is precisely defined by the distribution $\mathrm{Ker}(\lambda)$, which is integrable because λ is d-closed. The leaves of $\mathcal{N}$ are holomorphic, $\dim_{\mathbb{C}} L = (n+p)-(2p+1) = n-p-1 = q$.
- For the corresponding foliation $\widetilde{\mathcal{N}}$ on $\mathcal{S}(\Omega)$, the leaves are closed, and the leaf space can be given the structure of a smooth real $(4p+2)$-dimensional manifold of G/H.

- The real skew-symmetric bilinear form λ corresponds to some $\widetilde{\lambda}$ on $\mathcal{S}(\Omega)$.
- G/H is then endowed with a quotient skew-symmetric bilinear form $\overline{\lambda}$, which is G-invariant and non-degenerate everywhere on G/H.
- $\Lambda^{4p+2}\overline{\lambda} = d\mu$ is a G-invariant volume form on the homogeneous space G/H.
- Since Γ acts ergodically on G/H the leaf space $\Gamma \setminus G/H$ of $\mathcal{N}$ on $\mathcal{S}$ does not carry the structure of a smooth manifold. In this sense $\overline{\lambda}$ does not descend to the leaf space of $\mathcal{N}$.
- However, the structure of $\mathcal{S}$ as a foliated manifold in the small lifts to $\mathcal{S}(\Omega)$, and as far as integration on $\mathcal{S}$ is concerned we can sometimes make use of the volume form $d\mu$ on local pieces of $\mathcal{S}$.

Fix a triple $(P, P'; \alpha)$, $P = \Delta \times P'$, and consider the subgroup $H \subset G$ consisting of $\mu \in G$ such that $\mu(P) = P, \mu(P') = P'$ and such that $d\mu(\alpha)$ projects to the same vector as α under the canonical projection $\pi : P \to \Delta$.

Lemma.
Suppose $\gamma_i \in \Gamma$ are such that $\gamma_i H$ converges to $\tau_\theta H$ in G/H. Then, $s \circ \gamma_i^{-1}$ converges to $s \circ \tau_{-\theta}$ on P, i.e., $s\big(\gamma_i^{-1}(z; z')\big)$ converges to $s(e^{-i\theta} z; z')$ uniformly on compact subsets of P.

Proof. Write $\gamma_i = \lambda_i \tau_\theta h_i$, where $h_i \in H$ and $\lambda_i \in G$ converges to the identity element $e \in G$. Then for $(z; z') \in P$

$$(s \circ \gamma_i^{-1})\big(\lambda_i(z; z')\big) = s\big(\gamma_i^{-1}(\lambda_i(z; z'))\big)$$
$$= s\big(h_i^{-1} \tau_\theta^{-1} \lambda_i^{-1}\big(\lambda_i(z; z')\big)\big) = s\big(h_i^{-1} \tau_{-\theta}(z; z')\big)$$
$$= s\big(h_i^{-1}(e^{-i\theta} z; z')\big) = s\big(e^{-i\theta} z; \mu_i(z')\big)$$

for some $\mu_i \in \mathrm{Aut}(P')$. Here we make use of the fact that any $h \in H$ preserves P, and that $h|_P$ is necessarily of the form $h(z;z') = \big(z, \nu(z')\big)$, where $\nu \in \mathrm{Aut}(P')$. By Main Proposition, we conclude that

$$(s\circ\gamma_i^{-1})\big(\lambda_i(z;z')\big) = s(e^{-i\theta}z;z') = (s\circ\tau_{-\theta})(z,z')\,.$$

Fix an arbitrary compact subset $Q \subset P$. Then there exists a compact subset $Q' \subset \Omega$ such that $\lambda_i(Q) \subset Q'$ for any i. On the other hand, $s \circ \gamma_i^{-1} : \Omega \to \Delta$, so that by Cauchy estimates

$$\begin{aligned}\Big|(s\circ\gamma_i^{-1})\big(\lambda_i(z;z')\big) - (s\circ\gamma_i^{-1})(z;z')\Big| \\ \le C(Q')\|\lambda_i(z,z') - (z;z')\|\ ,\end{aligned}$$

where $C(Q')$ is a constant depending only on Q' (and independent of i), and $\|\cdot\|$ denotes the Euclidean norm. Since λ_i converges to $e \in G$, we conclude that the right hand side converges to 0. It follows from (2) that

$$\lim_{i\to\infty} \|(s\circ\tau_{-\theta}) - (s\circ\gamma_i^{-1})\|_Q = 0$$

for every compact subset $Q\subset P$, $\|\cdot\|_Q$ being the supremum norm for continuous functions on Q. In other words, $s\circ\gamma_i^{-1}$ converges uniformly on compact subsets of P to $s\circ\tau_{-\theta}$. $\square$

Derivation of injectivity from Main Proposition.

- *Special extremal functions.* Let s be an $e(\mathcal{F})$-extremal function on Ω. For any $\gamma\in\Gamma$, $s\circ\gamma\in\mathcal{F}$. An $e(\mathcal{F})$-extremal function σ will be called special if $\sigma(z_1;z')=\sigma(z_1)=\lambda z_1$ for some $\lambda\neq 0$. Injectivity follows if this can be done for all $P=\Delta\times P'$.
- *Moore's Ergodicity Theorem.* Since $\Gamma\subset G$ is discrete, its left action on G/H is ergodic

for any noncompact closed subgroup $H \subset G$. As a consequence, the orbit under Γ of $\nu H \in G/H$ is dense in G/H, provided that νH lies outside a certain null set $E \subset G/H$.

- Since $s(z_1, z') = s(z_1)$, $s|_P$ is invariant under the group H. Suppose we choose $\gamma_i \in \Gamma$ such that $\gamma_i H$ converges to $\tau_\theta H$. Then, by Lemma, $s \circ \gamma_i^{-1}\big|_P$ converges to $s \circ \tau_{-\theta}\big|_P$, and the S^1-averaging argument applies to produce a special function σ adapted to $(P, P'; \alpha)$.
- *The null set E.* There may in fact be a maximal polydisk P such that its orbit under Γ gives a discrete set of maximal polydisks on Ω. Then, completing P to a triple $(P, P'; \alpha)$, the latter corresponds to an element of G/H whose orbit under Γ is discrete, and the argument above to produce

special functions by S^1-averaging fails.

- However, from the estimate $|s'(0)| > c > 0$ for the $e(\mathcal{F})$-extremal function s the S^1-averaging argument produces a special function σ for which $|\lambda| = |s'(0)|$ is bounded from below independent of (P, P', α), which allows us to take care of the 'exceptional' by taking limits to obtain special functions for *every* triple $(P, P'; \alpha)$. This proves that $F : \Omega \to D$ is injective. $\square$

Theorem on the Extension Problem.
$\Omega \Subset \mathbb{C}^n$ Harish-Chandra realization of a bounded symmetric domain of rank ≥ 2, $\Gamma \subset \mathrm{Aut}(\Omega)$ irreducible lattice, $X := \Omega/\Gamma$.

N quasi-compact, i.e., it is a Zariski-open subset of some compact complex manifold

$f : X \to N$ holomorphic map

$F : \Omega \to \widetilde{N}$ lifting to universal covers

Assume $(X, N; f)$ satisfies the non-degeneracy condition (†). Then,

there exists a bounded vector-valued holomorphic map : $\tilde{N} \to \mathbb{C}^n$ such that

$$R \circ F = id_\Omega \ .$$

The Set-up and Ideas of Proof

Write $\mathcal{H}(\cdot)$ for the space of bounded holomorphic functions, $F^*\mathcal{H}(\widetilde{N}) := \mathcal{F} \subset \mathcal{H}(\Omega)$.

- $s \in \mathcal{F}$ and $\gamma \in \Gamma \Rightarrow \gamma^* s \in \mathcal{F}$
- $s_k \in \mathcal{F}$ and uniformly bounded
  $\Rightarrow s_k$ subconverges to $s \in \mathcal{F}$.

To get $\sigma \in \mathcal{F}$ such that $\sigma(z_1, \ldots, z_n) = \lambda z_1$ on a maximal polydisk P we look first of all for $s \in \mathcal{F}$ such that $s(z_1, \ldots, z_n) = t(z_1) \not\equiv$ constant and then introduce an averaging argument. For a bounded holomorphic function $g \in \mathcal{H}(P)$, $P = \Delta \times P'$, for almost every $\eta \in \text{Shilov}(P') \cong (S^1)^{r-1}$ we have the non-tangential limit
$g^*_\eta(z) = \lim \big\{ g(z,w) : w \mapsto \eta \text{ non-tangentially} \big\}$

Given $g \in \mathcal{F}$, the idea is to recuperate g^*_η as a limit of $\gamma_k^* g$ for a sequence $\gamma_k \in \Gamma$ so that in the limit we get $s = g^*_\eta \circ \pi \in \mathcal{F}$ for some projection $P \xrightarrow{\pi} \Delta$, s depending only on 1 variable.

The basis of the limiting process is given by a following lemma deduced from Moore's Ergodicity Theorem.

Lemma 6. *Let $P \cong \Delta^r$, $P \subset \Omega$ be a maximal polydisk in Ω, which gives canonically the embedding* $\mathrm{Aut}(\Delta)^r \hookrightarrow \mathrm{Aut}_0(\Omega)$*. Let $H_0 \subset$* $\mathrm{Aut}(\Delta)$ *be the* 1*-parameter group of transvections given by $H_0 = \{\psi \in \mathrm{Aut}(\Delta) : \psi(z) = \frac{z+t}{1+tz}$ for some t, $-1 < t < 1\}$, and $H = \{\mathrm{id}_\Delta\} \times$* $\mathrm{diag}(H_0^{r-1})$*, $H \subset \mathrm{Aut}(\Delta)^r \hookrightarrow \mathrm{Aut}_0(\Omega)$. For $\theta \in \mathbb{R}$, $-1 < t < 1$, denote by $\varphi_{t,\theta} \in S^1 \times$* $\mathrm{diag}(H_0^{r-1})$ *the element given by $(e^{i\theta}, \psi_t, \ldots, \psi_t)$. Suppose $\Gamma H := \{\gamma H : \gamma \in \Gamma\} \subset G/H$ is dense in G/H. Then, excepting for $\zeta = e^{i\theta}, \theta \in [0, 2\pi]$ belonging to an at most countable subset $E \subset \partial\Delta$, there always exists a discrete sequence (γ_k), $\gamma_k \in \Gamma$, such that $\gamma_k = \varphi_{t_k,\theta}\delta_k$ for some $\delta_k \in$* $\mathrm{Aut}_0(\Omega)$ *converging to the identity and for some $t_k \in (-1, 1)$ such that $|t_k| \to 1$.*

Scheme of proof.

- When Ω is reducible the analogous theorem holds true provided that Γ is irreducible.
- For $\Omega = \Delta^n$, the Density Lemma allows us to apply the S^1-averaging argument to get special extremal functions. Thus, there are bounded holomorphic functions $h_1, \cdots h_n$ on $\tilde{N}$ such that $\big(h_1(F(z)), \cdots, h_n(F(z)\big) = \big(z_1, \cdots, z_n\big)$.
- Extremal functions are not important in the argument. One can start with any bounded holomorphic function h and compose with $\gamma_i \in \Gamma$, where γ_i converges to the projection onto a boundary disk in a non-tangential way. The limit of functions thus obtained is given by the boundary values of the holomorphic function F^*h. Now choose h with nontrivial boundary values.
- When Ω is irreducible and of rank ≥ 2,

the analogue of the Density Lemma is given by Moore's Ergodicity Theorem. γ_i can be chosen to converge to a projection map π onto a rank-1 boundary component Φ. If F^*h extends continuously to $\overline{\Omega}$, then $\gamma_i^* h$ converges to $\pi^* h$, where h is defined on the face Φ. In general, choose γ_i so that for any point $x \in \Omega$, $\gamma_i(x)$ converges "non-tangentially" to $\pi(x) \in \Phi$.

- The usual Fatou-type results in Harmonic Analysis on bounded symmetric domains are in terms of non-tangential convergence to the Šilov boundary. We need Fatou-type theorems for non-tangential convergence to a boundary component, which is related to the standard result for the Šilov boundary when we express boundary values on a boundary component in terms of Poisson integrals on a subset of the Šilov boundary. Such a Fatou-type result is covered by Koranyi 1976.

- Each face Φ is biholomorphic to a complex unit ball. By the technique of S^1-averaging we can recover the projection map $\pi = \pi_\Phi$.
- Averaging π_Φ over the set of rank-1 boundary components Φ recovers the identity map, giving $R : \widetilde{N} \to \mathbb{C}^n$ such that $R \circ F = \mathrm{id}_\Omega$, i.e., $R(F(x)) = x$.
- In the averaging argument we need to uniformly bound from below constants appearing in first derivative of certain bounded holomorphic functions, which follows from Finsler metric rigidity. It requires actually something weaker, viz. a metric inequality for minimal characteristic vectors.

The Fibration Theorem.

$\Omega \Subset \mathbb{C}^n$ bounded symmetric domain of rank ≥ 2; $X = \Omega/\Gamma$

N quasi-compact, i.e., it is a Zariski-open subset of some compact complex manifold

$f : X \to N$ holomorphic map

Suppose $f_* : \Gamma \cong \pi_1(N)$. Then, (a) $f : X \to N$ is a holomorphic embedding (b) $\exists$ a holomorphic fibration $\rho : N \to X$ with connected fibers such that $\rho \circ f = id_X$.

Argument. Lifting to universal covers we obtain $R \circ F = id_\Omega$. Then, we prove the Γ equivariance of R, i.e., $R \circ \gamma \equiv \gamma \circ R$ for every $\gamma \in \Gamma$, which follows from the Maximum Principle.

Isomorphism Theorem.

$\Omega \Subset \mathbb{C}^n$ bounded symmetric domain of rank ≥ 2; $X = \Omega/\Gamma$

M Stein manifold, $D \Subset M$

$\Gamma' =$ torsion-free discrete group of automorphisms on D; $N := D/\Gamma'$

$\mu =$ Kobayashi-Royden measure

$$\boxed{\begin{array}{c}\text{Suppose } \mu(N) < \infty;\ f_* : \Gamma \cong \Gamma'.\ \text{Then,}\\ f : X \to N \text{ is a biholomorphic map}\end{array}}$$

Scheme of Proof

- $f : X \to N$ lifts to a holomorphic embedding $F : \Omega \to N$.
- To make use of Kähler metrics, embed D into its hull of holomorphy $\widehat{D}$, on which there is a complete Kähler-Einstein metric of negative Ricci curvature.

- f induces $\Gamma \cong \Gamma' = \pi_1(N)$. Γ' acts on $\widehat{D}$, giving $\widehat{N} = \widehat{D}/\Gamma' := \widehat{N}$.
- By an estimate of Kobayashi-Royden volume form, we show that $\widehat{N} - N$ is of zero Lebesgue measure. By the Schwarz Lemma on volume forms we conclude that *Volume* $(\widehat{N}, \omega_{KE}) < \infty$.
- The argument of the Fibration Theorem yields a projection $\rho : \widehat{N} \to X$. Integration by part and Fubini's Theorem yield that the fibers of ρ are 0-dimensional. This relies on

  Lemma. Let (Z, ω) be a complete Kähler manifold of finite volume, and u be a uniformly Lipschitz bounded plurisubharmonic function on Z. Then, u is a constant.

Varieties of Minimal Rational Tangents

X uniruled,

$\mathcal{K}$ = component of Chow space of minimal rational curves

$\mu : \mathcal{U} \to X$; $\rho : \mathcal{U} \to \mathcal{K}$ universal family

$x \in X$ generic; $\mathcal{U}_x$ smooth

The tangent map $\tau : \mathcal{U}_x \to \mathbb{P}T_x(X)$ *is given by*

$$\tau([C]) = [T_x(C)] \ ;$$

for C *smooth at* $x \in X$.

τ is rational, generically finite,

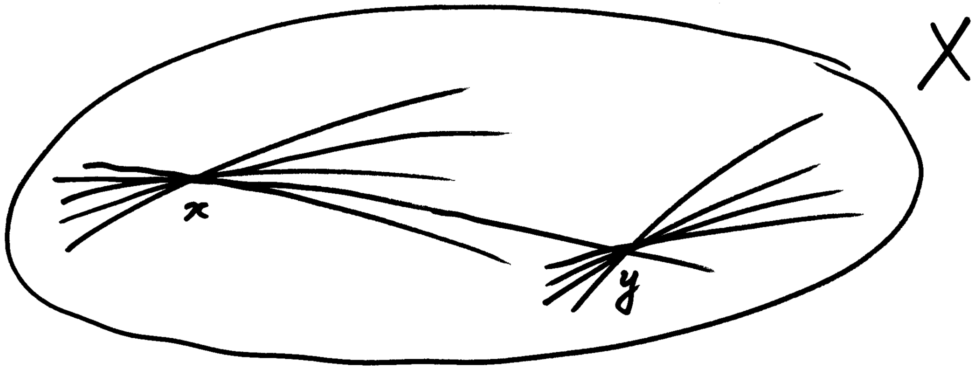
a priori **undefined for** C *singular* **at** x.

We call the strict transform

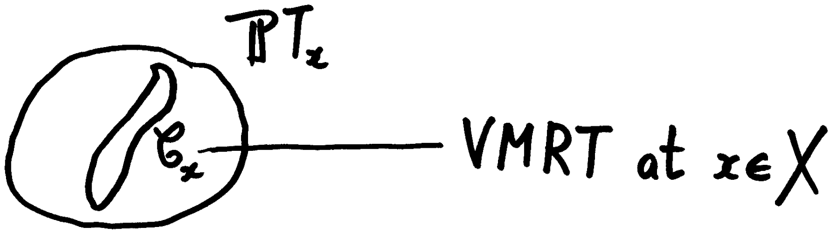
$$\tau(\mathcal{U}_x) = \mathcal{C}_x \subset \mathbb{P}T_x(X)$$

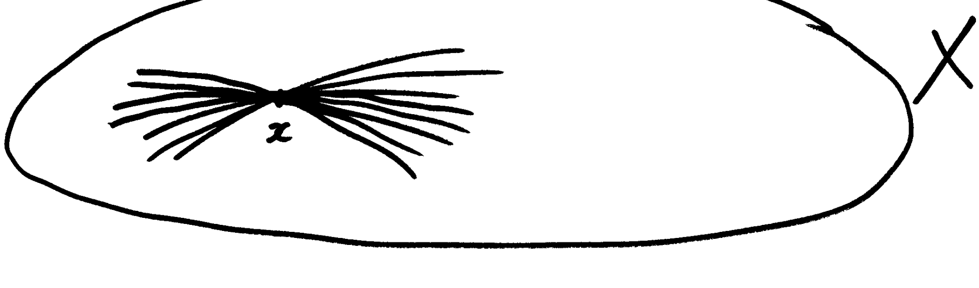
variety of minimal rational tangents.

Minimal Rational Curves

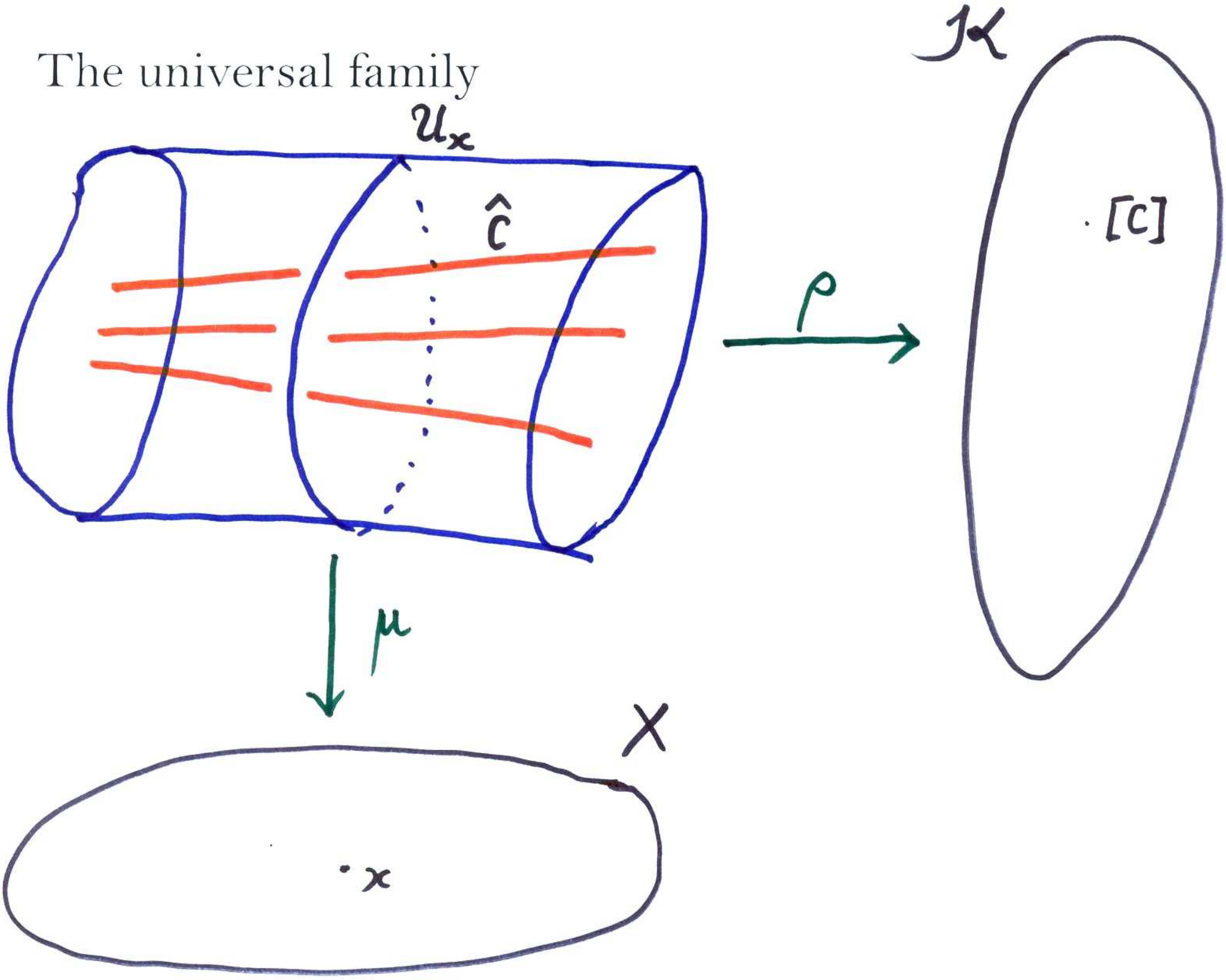


Variety of Minimal Rational Tangents (VMRT)

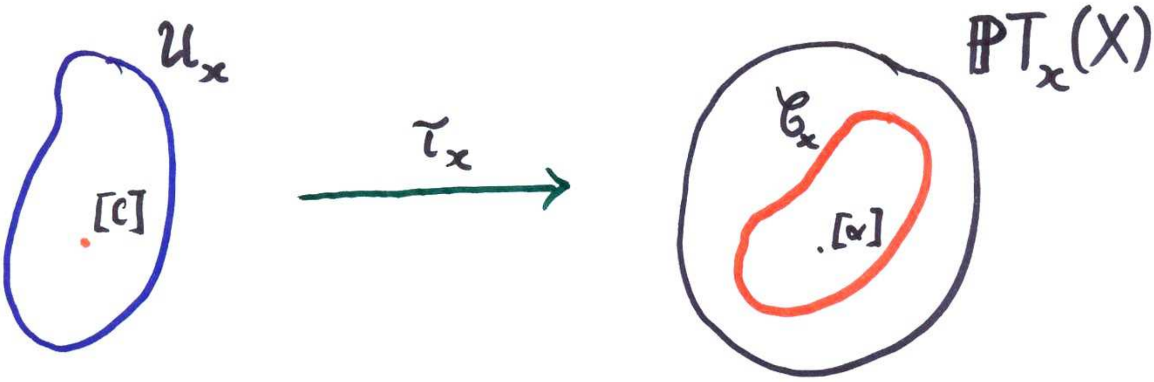


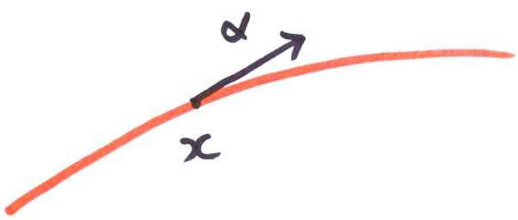


The universal family



The tangent map





$\alpha \in T_x(C)$

Theorem (Kebekus 2002, JAG).

The tangent map

$$\tau_x : \mathcal{U}_x \to \mathbb{P}T_x(X)$$

is a morphism at a generic point $x \in X$.

Theorem (Hwang-Mok 2004, AJM).

The tangent map

$$\tau_x : \mathcal{U}_x \to \mathcal{C}_x \subset \mathbb{P}T_x(X)$$

is a birational morphism at a generic point $x \in X$.

Examples of VMRTs

Fermat hypersurface $1 \le d \le n-1$

$$X = \{Z_0^d + Z_1^d + \cdots + Z_n^d = 0\}$$

$x = [z_0, z_1, \ldots, z_n] \in X$.

FIND all $(w_0, w_r, \ldots, w_n)$ such that $\forall\ t \in \mathbb{C}$.

$$\begin{gathered}
[z_0 + tw_0, z_1 + tw_1, \ldots, z_n + tw_n] \in X \\
(z_0 + tw_0)^d + \cdots + (z_n + tw_n)^d = 0 \\
0 = (z_0^d + \cdots + z_n^d) \\
+t(z_0^{d-1} w_0 + \cdots + z_n^{d-1} w_n) \cdot d \\
+t^2 (z_0^{d-2} w_0^2 + \cdots + z_n^{d-2} w_n^2) \cdot \frac{d(d-1)}{2} \\
+ \cdots + t^d (w_0^d + \cdots + w_n^d)\ .
\end{gathered}$$

When $(z_0, z_1, \ldots, z_n)$ is fixed, we get $d+1$ equations.

If $d \le n-1$, $\dim(\mathcal{C}_x) = (n+1) - (d+1) - 1 = n - d - 1 \ge 0$.

Examples of VMRT

X	(generic) VMRT $\mathcal{C}_x$
$\mathbb{P}^n$	$\mathbb{P}^{n-1}$
Q^n	Q^{n-2}
cubic in $\mathbb{P}^{n+1}$	codim $2 \subset \mathbb{P}^{n-1}$ $=$ quadric $\cap$ cubic, deg. 6
$X_3^3 \subset \mathbb{P}^4$	6 points
$X_3^4 \subset \mathbb{P}^5$	deg. 6 curve of genus 4
$X_3^5 \subset \mathbb{P}^6$	$K3$ – surfaces
$X_d^n \subset \mathbb{P}^{n+1}$, $d < n$	complete intersection $\subset \mathbb{P}^n$ of degrees $1, 2, \ldots, d$

In these examples,

$$\{\text{mrc}\} = \{\text{lines in } \mathbb{P}^n \text{ contained in } X\} .$$

Type	G	K	$G/K = S$	$\mathcal{C}_o$	Embedding
I	$SU(p+q)$	$S(U(p) \times U(q))$	$G(p,q)$	$\mathbb{P}^{p-1} \times \mathbb{P}^{q-1}$	Segre
II	$SO(2n)$	$U(n)$	$G^{II}(n,n)$	$G(2,n-2)$	Plücker
III	$Sp(n)$	$U(n)$	$G^{III}(n,n)$	$\mathbb{P}^{n-1}$	Veronese
IV	$SO(n+2)$	$SO(n) \times SO(2)$	Q^n	Q^{n-2}	by $\mathcal{O}(1)$
V	E_6	$\mathrm{Spin}(10) \times U(1)$	$\mathbb{P}^2(\mathbb{O}) \otimes_{\mathbb{R}} \mathbb{C}$	$G^{II}(5,5)$	by $\mathcal{O}(1)$
VI	E_7	$E_6 \times U(1)$	exceptional	$\mathbb{P}^2(\mathbb{O}) \otimes_{\mathbb{R}} \mathbb{C}$	Severi

<u>Theorem</u> (Hwang-Mok, AJM 2004)
X projective uniruled, $b_2(X) = 1$,
$\mathcal{K}$ minimal rational component on X.
Assume

(†) The VMRT $\mathcal{C}_x$ at a general point x is not a finite union of linear subspaces. Then,

> For any Fano manifold X' of Picard number 1 equipped with a minimal rational component $\mathcal{K}'$, any local VMRT-preserving holomorphic map $f : (U, \mathcal{K}\big|_U) \to (V, \mathcal{K}'\big|_{V'})$ extends to a biholomorphic map $f : (X, \mathcal{K}) \cong (X'\mathcal{K}')$.

We say that $(X, \mathcal{K})$ has the Cartan-Fubini Extension Property. Examples include

(1) $X = G/P \neq \mathbb{P}^N$, G simple, P maximal.

(2) $X \subset \mathbb{P}^N$ smooth complete intersection, Fano with $\dim(X) \geq 3$, $c_1(X) \geq 3$.

Proper holomorphic maps and related problems

Problems:

- To characterize convex realizations of an irreducible bounded symmetric domain D of rank ≥ 2.
- To characterize proper holomorphic mappings from D into a bounded symmetric domain D'.

Convex realizations

Background:
Every bounded symmetric domain D admits a convex realization as an open subset of some $\mathbb{C}^N$ by means of the Harish-Chandra embedding (E. Cartan's realizations in the classical case). They also admit unbounded realizations via Cayley transforms.

Results

- Mok-Tsai (J. reine angew. Math. 1992) proves that every bounded convex realization of a bounded symmetric domain of rank ≥ 2 must be the Harish-Chandra realization up to an affine transformation.

- In the same paper, it was proven that unbounded realizations of D must come from Cayley transforms up to affine linear transformations. E.g.

  $$\mathcal{H}_n = \{\tau \in M(n,n;\mathbb{C}) : \tau^t = \tau, Im(\tau) > 0\}$$

  is the Siegel upper half-plane, which is a Cayley transform of a Type-III bounded symmetric domain.

- Generalizations to the cases of reducible bounded symmetric domains of rank ≥ 2 were obtained by Taishun Liu and Guangbin Ren (J. reine angew. Math. 1998).

Proper holomorphic mappings

Link with rigidity problems for compact quotients

Suppose $X = \Gamma \backslash G/K$ is compact, $D' = G'/K'$, $\Gamma' \subset G'$ is discrete, and $F : D \to D'$ is the lifting of a holomorphic mapping $f : X \to X'$ such that the induced map $f_* : \Gamma \to \Gamma'$ is injective, then $f : D \to D'$ is a proper holomorphic map.

In 1989, I made a conjecture on proper holomorphic mappings under some conditions on the ranks of the domain and target manifolds. This was later established by Tsai.

Theorem (Tsai, JDG 1993).

$F : D \to D'$ proper holomorphic, $\mathrm{rank}(D) \geq 2$, $\mathrm{rank}(D') \leq \mathrm{rank}(D)$. Then, $\mathrm{rank}(D) = \mathrm{rank}(D')$, and f is totally geodesic.

The Set-up for proper holomorphic maps between bounded symmetric domains

$\Omega \Subset \mathbb{C}^n$ irreducible of rank $r \geq 2$
There exists a totally geodesic subspace $\Omega_0 \times \Delta \subset \Omega$ such that Ω is a bounded symmetric domain of rank $r-1$.

Given a proper mapping $F : \Omega \to \Omega'$. For *almost every* $\zeta \in \partial\Delta$ by Fatou's Lemma

$$\lim \left\{ F(w,z) : z \to \zeta \ \textit{non-tangentially} \right\} :=$$
$$F^*(w,\zeta) := F^*_\zeta(w)$$

exists as a vector-valued bounded holomorphic function on Ω_0.
Properness of $F : \Omega \to \Omega'$ implies that
$F^*_\zeta : \Omega_0 \to \partial\Omega'$. By Fatou's Lemma

$$F(w,z) = \int_{\partial D} \frac{F^*(w,\zeta)d\zeta}{\zeta - z}$$

forces restrictions on images of $F_z : \Omega_0 \to \Omega'$, where $F_z(w) = F(w, z)$.
Thus, non-tangential limits + integrals of boundary values $\Rightarrow$ algebraic constraints on images of totally geodesic subpaces congruent to $\Omega_0 \hookrightarrow \Omega$

The Set-up for bounded convex realizations

$F : \Omega \to D$ a biholomorphism, where D is a bounded convex domain.

The same set-up gives $F^*_\zeta(\Omega_0) \subset \partial D$.
There is no structure of boundary components on ∂D, but convexity implies that $\zeta(\Omega_0)$ must lie on some proper affine linear subspace.

Cauchy's Integral Formula forces each "interior face" Ω_0 to be mapped into a proper affine linear subspace.

Relevant geometric ideas for convex realizations

- There is a class of complex submanifolds of a given bounded symmetric domain D which are totally geodesic submanifolds and which correspond to affine-linear sections of D with respect to the Harish-Chandra embedding. We call these the characteristic subdomains. They are open subsets of certain Hermitian symmetric submanifolds S' of the compact dual S of D.

- By taking nontangent limits on product subdomains of D, we obtain a holomorphic map $F^\sharp$ defined on some connected open subset $\mathcal{U}$ of a moduli space $\mathcal{M}$ of characteristic subdomains into some Grassmann manifold of affine linear subspaces, by the assumption of convexity of the embedding.

- $\mathcal{U}$ consists of those S' which intersect D. It is a 'big' open subset complex-analytically. In fact, it is pseudoconcave, which implies a meromorphic extension of $f^\sharp$ from $\mathcal{U}$ to $\mathcal{M}$.

- Employing the idea of duality in projective geometry, an extension of $F^\sharp$ yields an extension of F, by interpreting a point x on S simply as the intersections of members of $\mathcal{M}$ containing x.

Scheme of Proof of Tsai's result

- There is the notion of rank of a (holomorphic) tangent vector. The hypothesis on the ranks of the domain and the target manifold, together with the idea of taking non-tangential limits of product subdomains, implies that a generic tangent vector of rank 1 is mapped to a tangent vector of rank 1. A tangent vector of rank 1 is nothing other than a minimal characteristic vector. [For the first 3 classical series, the notion of rank of a tangent vector agrees with that of a matrix.]

- After this step, the rest involves local differential-geometric computations and Lie Theory.

Other results

- Zhenhan Tu (Proc. AMS 2002) established that any equi-dimensional proper holomorphic map from an irreducible bounded symmetric domain of rank ≥ 2 to a bounded symmetric domain is a biholomorphism.
- For the non-equidimensional case he established (Math. Zeit. 2002) examples where $rank(D') = rank(D) + 1$ for which still rigidity for proper holomorphic maps hold.
- Given any integer $\ell > 0$, Tu's method can be expanded to give examples of pairs of irreducible bounded symmetric domains D and D', such that $rank(D') - rank(D) = \ell$ and such that there are *no* proper holomorphic mapping from D to D'.

Proper holomorphic mappings from the perspective of geometric structures:

- It is desirable to incorporate the study of proper holomorphic maps into the study of germs of holomorphic embeddings preserving some form of geometric structures.
- Properness should be used solely to verify a condition on the preservation of geometric structures. After that, the problem involves projective geometry of subvarieties of the projectivized tangent space at a general point
- An irreducible BSD is dual to an irreducible HSS of the compact type, which is a Fano manifold of Picard number 1. A general theory for *variable* geometric structures have been developed for such manifolds X.

Proposition A.
Let $p, q \geq 2$. Suppose $p \leq p', q \leq q'$. Let $U \subset G(p, q)$ be a connected open subset. Suppose $f : U \to G(p', q')$ is a local holomorphic embedding such that (*) for every rank-1 vector α), $df(\alpha)$ is also a rank-1 vector, Then, f extends to a holomorphic embedding of of $G(p, q)$ into $G(p', q')$ congruent to the standard embedding up to automorphisms of $G(p, q)$ and $G(p', q')$.

The proposition was established by Yu. A. Neretin (AMS translation of Sbornik, 1999). A stronger result was established by J. Hong (Trans. AMS 2006). I will sketch a proof of the Proposition involving a non-equidimensional Cartan-Fubini extension principle. The proof can be extended to the general context of Fano manifolds of Picard number 1.

The basic difficulty of the argument in the Cartan-Fubini extension principle comes from the fact that the distribution defined on the submanifold need not *a priori* extend locally to the ambient manifold in a way that corresponds to families of local holomorphic curves.

The non-equidimensional analogue of Ochiai's Theorem.

Proposition.
Let Ω_1 and Ω_2 be two irreducible bounded symmetric domains in their Harish-Chandra realizations. Let $U \subset \Omega_1$ be an nbd. of 0, and $f : U \to \Omega$ be a holomorphic map such that $f(0) = 0$ and $df_x\big(\widetilde{\mathcal{S}}_x(\Omega_1)\big) \subset \widetilde{\mathcal{S}}_{f(x)}(\Omega_2)$ for ev-

ery $x \in U$. For $y \in \Omega_2$, $\beta \in \widetilde{\mathcal{S}}_y(\Omega_2)$, write

$$\sigma_\beta : T_\beta\big(\widetilde{\mathcal{S}}_0(\Omega_2)\big) \times T_\beta\big(\widetilde{\mathcal{S}}_0(\Omega_2)\big) \to T_\beta\big(T_y(\Omega_2)\big)/T_\beta\big(\widetilde{\mathcal{S}}_0(\Omega_2)\big)$$

for the second fundamental form with respect to the Euclidean flat connection ∇ on $T_y(\Omega_2)$. For any subspace $V \subset T_\beta\big(\widetilde{\mathcal{S}}_0(\Omega_2)\big)$, define

$$\operatorname{Ker} \sigma_\beta(V, \cdot) := \Big\{\delta \in T_\beta\big(\widetilde{\mathcal{S}}_0(\Omega_2)\big) : \sigma_\beta(\delta, \gamma) = 0 \ , \quad \forall\, \gamma \in V\Big\} \ .$$

For any $x \in U$, and $\alpha \in \widetilde{\mathcal{S}}_x(\Omega_1)$, denote by $\widetilde{\alpha}$ the constant vector field on Ω_1 which is α at x and identify $T_{df(\alpha)}\big(T_0(\Omega_2)\big)$ with $T_0(\Omega_2)$. Then,

$$\nabla_{df(\alpha)} df(\widetilde{\alpha}) \in \operatorname{Ker} \sigma_{df(\alpha)}\Big(T_{df(\alpha)}\big(df(\widetilde{\mathcal{S}}_x(\Omega_1))\big), \cdot\Big) \ .$$

Scheme of proof of Proposition A

- A computation of second fundamental forms shows that f maps lines into lines.
- Analytic continuation as in Cartan-Fubini applies to get a meromorphic extension.
- Comparison of the image with a sub-Grassmannian yields total geodesy of the mapping. This relies on an argument on parallel transport of tangents to VMRTs along a minimal rational curve.

Theorem (Hong-Mok 2007)
Many examples of pairs of rational homogeneous manifolds of $G/P \hookrightarrow G'/P'$ of Picard number 1 are found such that they exhibit rigidity of holomorphic maps as in Theorem A. For such pairs the moduli space of deformations of G/P in G'/P' is <u>compact</u>.

Open problems:

- Characterize convex realizations of bounded homogeneous domains

Bounded homogeneous domains were studied by Piatetski-Shapiro, who produced the first examples of such domains which are not biholomorphic to bounded symmetric domains, starting with 4 dimensions.

Piatetski-Shapiro proved that any bounded homogeneous domain is biholomorphic to a Siegel domain of the second kind, which is a convex domain. So far, there are no bounded convex realizations in the non-symmetric case. Gindikin raised the question whether bounded symmetric domains are characterized among bounded homogeneous domains by the existence of bounded convex realizations.

- Characterize proper holomorphic mappings between bounded homogeneous domains under some rank conditions

This problem gives a motivation for developing a theory of geometric structures at least for certain bounded homogeneous domains, and to place the problem within the framework of local holomorphic embeddings preserving such geometric structures.