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Remarks on lines and minimal rational curves

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Abstract We determine all of lines in the moduli space M of stable bundles for arbitrary rank and degree. A further application of minimal rational curves is also given in last section.

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1 Introduction

Let C be a smooth projective curve of genus $g \geq 2$ and $\mathcal{L}$ be a line bundle of degree d on C. Let $M := \mathcal{S}U_C(r, \mathcal{L})$ be the moduli space of stable vector bundles on C of rank r and with fixed determinant $\mathcal{L}$, which is a smooth quasi-projective Fano variety with $Pic(M) = \mathbb{Z} \cdot \Theta$ and $-K_M = 2(r,d)\Theta$, where Θ is an ample divisor. The second author[10] proved that any rational curve $\phi : \mathbb{P}^1 \to M$ is defined by a vector bundle E on $C \times \mathbb{P}^1$ and gave a formula of its $(-K_M)$-degree in terms of splitting type of E on the general fiber of $f : X = C \times \mathbb{P}^1 \to C$. This formula implies immediately that a rational curve through a general point of M has $(-K_M)$-degree at least $2r$ and it has degree $2r$ if and only if it is a Hecke curve. In particular, rational curves of $(-K_M)$-degree smaller than $2r$, which we call *small rational curves*, must fall in a proper closed subvariety of M. In fact, the formula contains the following information about points of small rational curves: There exist, for any small rational curve, a sequence of fixed bundles F_1, F_2, ..., F_n on C such that bundles corresponding points of the small rational curve are obtained by extensions of F_1, F_2, ..., F_n. We should remark here that the bundles F_1, F_2, ..., F_n are independent of points of the small rational curve, and sometime only depend on the degree of the small rational curves.

In this paper, we study the rational curves of degree 1 with respect to Θ for arbitrary r and d, which we call lines of M. The geometry of M at the case when $(r,d) < r$ is different from the case when $(r,d) = r$. When $(r,d) < r$, the lines of M fill up a proper closed subvariety. However, when $(r,d) = r$, M is generally covered by lines. In Section 2, we recall firstly two constructions of lines, then, in Theorem 2.7, we prove that all lines in M are obtained by the two constructions. In Section 3, we determine the variety $\mathbf{Hom}_1(\mathbb{P}^1, M)$ of degree 1 morphisms $\phi : \mathbb{P}^1 \to M$ (Theorem 3.1) and the variety $\mathscr{L}(M)$ of lines in M (as subvarieties of Chow variety

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of M) in Corollary 3.3. In Section 4, we present some partial results on geometry of lines in M. The proof of main theorem[10] has some implications about the properties of the bundle E on $C \times \mathbb{P}^1$ (which defines the minimal rational curve). In Section 5, we write down first of all these implications (Theorem 5.1). Then, as an application of it, we give an alternate proof of some known results (Theorem 5.2).

2 The constructions of lines

Let C be a smooth projective curve of genus $g \geq 2$ and $\mathcal{L}$ a line bundle on C of degree d. Let $M = \mathcal{SU}_C(r, \mathcal{L})^s$ be the moduli spaces of stable bundles on C of rank r, with fixed determinant $\mathcal{L}$. It is well-known that $Pic(M) = \mathbb{Z} \cdot \Theta$, where Θ is an ample divisor.

Definition 2.1. For any rational curve $\phi : \mathbb{P}^1 \to M$, its degree is defined to be $\deg(\phi^*(\Theta))$. The images $\phi(\mathbb{P}^1) \subset M$ of degree 1 rational curves $\phi : \mathbb{P}^1 \to M$ are called lines in M.

In this section, we give the constructions of all lines in M. Before stating the first construction, we need the following lemma, which is a generalization of Lemma 3.1[10].

Lemma 2.2. *Let $0 \to V_1 \to V \to V_2 \to 0$ be a nontrivial extension of vector bundles on C. Let $r_i = \mathrm{rk}(V_i)$, $d_i = \deg(V_i)$ $(i = 1, 2)$, $r = \mathrm{rk}(V)$, $d = \deg(V)$ be the rank and degree respectively. Then, when $r_1 d - d_1 r = (r, d)$, V is stable if and only if V_1 and V_2 are stable.*

Proof. It is clear that r_i, d_i, r, d satisfy $d_2 r - r_2 d = (r, d)$, and

$$\mu(V_1) = \mu(V) - \frac{(r,d)}{r_1 r}, \quad \mu(V_2) = \mu(V) + \frac{(r,d)}{r_2 r}.$$

Writing

$$r_1 \frac{d}{(r,d)} - d_1 \frac{r}{(r,d)} = 1, \quad d_2 \frac{r}{(r,d)} - r_2 \frac{d}{(r,d)} = 1,$$

we observe that $(r_i, d_i) = 1$ $(i = 1, 2)$.

Assuming that V is stable, we are going to prove the stability of V_1 and V_2. For any subbundle $V_1' \subset V_1$ of rank r_1' and degree d_1', we have, by stability of V,

$$r_1' r(\mu(V) - \mu(V_1')) = r_1' d - r d_1' \geq (r, d).$$

Thus $\mu(V_1') \leq \mu(V) - \frac{(r,d)}{r_1' r} = \mu(V_1) + \frac{(r,d)}{r_1 r} - \frac{(r,d)}{r_1' r} < \mu(V_1)$, i.e., V_1 is stable. For any subbundle $V_2' \subset V_2$ of rank r_2', define the subsheaf $V' \subset V$ by $0 \to V_1 \to V' \to V_2' \to 0$. Then

$$\mu(V') \leq \mu(V) - \frac{(r,d)}{r(r_1 + r_2')}$$

and stability of V_2 can be seen as follows.

$$\begin{aligned}
\mu(V_2') &= \mu(V') \frac{r_1 + r_2'}{r_2'} - \mu(V_1) \frac{r_1}{r_2'} \\
&\leq \mu(V) \frac{r_1 + r_2'}{r_2'} - \frac{(r,d)}{r_2' r} - \mu(V_1) \frac{r_1}{r_2'} \\
&= \mu(V) < \mu(V_2).
\end{aligned}$$

Conversely, assuming that V_1 and V_2 are stable, we are going to prove the stability of V. For any nontrivial subbundle $V' \subset V$ of rank r', let $V_2' \subset V_2$ be the image of V' and $V_1' \subset V_1$ such that

$$0 \to V_1' \to V' \to V_2' \to 0$$

is exact. When $V_2' = 0$, it is clear that $\mu(V') < \mu(V)$ since V_1 is stable and $\mu(V_1) < \mu(V)$. If $V_1' = 0$, then V_2' is a proper subsheaf of V_2 since the extension is nontrivial. Thus

$$\mu(V_2') - \mu(V_2) = -\frac{\deg(V_2'^* \otimes V_2)}{r_2 r_2'} < 0$$

since V_2 is stable. Let r_2', d_2' be the rank and degree of V_2'. Then

$$\begin{aligned} \mu(V') = \mu(V_2') &= \mu(V) + \frac{(r,d)}{r_2 r} - \frac{\deg(V_2'^* \otimes V_2)}{r_2 r_2'} \\ &= \mu(V) + \frac{(r,d)}{r_2 r_2' r}\left(r_2' - \frac{r}{(r,d)}\deg(V_2'^* \otimes V_2)\right) < \mu(V). \end{aligned}$$

The last inequality holds because $r_2' - \frac{r}{(r,d)}\deg(V_2'^* \otimes V_2) < r_2$ and is divisible by r_2, thus it must be negative. It is divisible by r_2 since

$$d_2\left(r_2' - \frac{r}{(r,d)}\deg(V_2'^* \otimes V_2)\right) = r_2\left(d_2' - \frac{d}{(r,d)}\deg(V_2'^* \otimes V_2)\right)$$

and $(r_2, d_2) = 1$. If V_1', V_2' are nontrivial of rank r_1', r_2' and degree d_1', d_2', then

$$\begin{aligned} \mu(V') = \mu(V_1')\frac{r_1'}{r'} + \mu(V_2')\frac{r_2'}{r'} &\leq \mu(V_1)\frac{r_1'}{r'} + \mu(V_2')\frac{r_2'}{r'} \\ &< \mu(V)\frac{r_1'}{r'} + \mu(V_2)\frac{r_2'}{r'} - \frac{\deg(V_2'^* \otimes V_2)}{r_2 r'} \\ &= \mu(V) + \frac{(r,d)}{r_2 r' r}\left(r_2' - \frac{r}{(r,d)}\deg(V_2'^* \otimes V_2)\right) < \mu(V). \end{aligned}$$

Thus V is a stable vector bundle, as desired. ☐

Now we can describe the first construction of lines. For any given r and d, let r_1, r_2 be positive integers and d_1, d_2 be integers that satisfy the equalities $r_1 + r_2 = r$, $d_1 + d_2 = d$ and

$$r_1\frac{d}{(r,d)} - d_1\frac{r}{(r,d)} = 1, \quad d_2\frac{r}{(r,d)} - r_2\frac{d}{(r,d)} = 1.$$

Let $\mathcal{U}_C(r_1, d_1)$ (resp. $\mathcal{U}_C(r_2, d_2)$) be the moduli space of stable vector bundles of rank r_1 (resp. r_2) and degree d_1 (resp. d_2). Then, since $(r_1, d_1) = 1$ and $(r_2, d_2) = 1$, they are smooth projective varieties and there are universal vector bundles $\mathcal{V}_1$, $\mathcal{V}_2$ on $C \times \mathcal{U}_C(r_1, d_1)$ and $C \times \mathcal{U}_C(r_2, d_2)$ respectively. Consider the morphism

$$\mathcal{U}_C(r_1, d_1) \times \mathcal{U}_C(r_2, d_2) \xrightarrow{\det(\bullet)\times\det(\bullet)} J_C^{d_1} \times J_C^{d_2} \xrightarrow{(\bullet)\otimes(\bullet)} J_C^d$$

and let $\mathcal{R}(r_1, d_1)$ be its fiber at $[\mathcal{L}] \in J_C^d$. We still use $\mathcal{V}_1$, $\mathcal{V}_2$ to denote the pullback on $C \times \mathcal{R}(r_1, d_1)$ by the projection $C \times \mathcal{R}(r_1, d_1) \to C \times \mathcal{U}_C(r_i, d_i)$ $(i = 1, 2)$ respectively. Let $p : C \times \mathcal{R}(r_1, d_1) \to \mathcal{R}(r_1, d_1)$ and $\mathcal{G} = R^1 p_*(\mathcal{V}_2^\vee \otimes \mathcal{V}_1)$. Then, since $\mathrm{Hom}(V_2, V_1) = 0$, $\mathcal{G}$ is a vector bundle of rank $r_1 r_2(g-1) + (r,d)$. Let $q : P(r_1, d_1) = \mathbb{P}(\mathcal{G}) \to \mathcal{R}(r_1, d_1)$ be the projective bundle parametrzing 1-dimensional subspaces of $\mathcal{G}_t$ $(t \in \mathcal{R}(r_1, d_1))$ and $f : C \times P(r_1, d_1) \to C$, $\pi : C \times P(r_1, d_1) \to P(r_1, d_1)$ be the projections. Then there is a universal extension

$$0 \to (id \times q)^*\mathcal{V}_1 \otimes \pi^*\mathcal{O}_{P(r_1,d_1)}(1) \to \mathcal{E} \to (id \times q)^*\mathcal{V}_2 \to 0 \tag{2.1}$$

on $C \times P(r_1, d_1)$ such that for any $x = ([V_1], [V_2], [e]) \in P(r_1, d_1)$, where $[V_i] \in \mathcal{U}_C(r_i, d_i)$ with $\det(V_1) \otimes \det(V_2) = \mathcal{L}$ and $[e] \subset \mathrm{H}^1(C, V_2^\vee \otimes V_1)$ being a line through the origin, the bundle $\mathcal{E}|_{C\times\{x\}}$ is the isomorphic class of vector bundles E given by extensions

$$0 \to V_1 \to E \to V_2 \to 0$$

that defined by vectors on the line $[e] \subset \mathrm{H}^1(C, V_2^\vee \otimes V_1)$.

To see the existence of the universal extension, recall Lemma 2.4[9]: For two families $(E_s)_{s\in S}$, $(F_t)_{t\in T}$ of bundles on $C\times S$, $C\times T$, there exists a universal extension if (1) $\dim \mathrm{H}^1(C, \mathcal{H}om(F_t, E_s))$ is independent of $(s, t) \in S \times T$, (2) $\mathrm{H}^i(S \times T, p_{S\times T*}\mathcal{H}om(F, E) \otimes V^*) = 0$ $(i = 1, 2)$, where V is the vector bundle on $S \times T$ with fibers $\mathrm{H}^1(C, \mathcal{H}om(F_t, E_s))$ at $(s, t) \in S \times T$. In our case, $E = \mathcal{V}_1$, $F = \mathcal{V}_2$, and the above conditions are satisfied since $\mathrm{Hom}(V_2, V_1) = 0$ for any $[V_i] \in \mathcal{U}_C(r_i, d_i)$ $(i = 1, 2)$. By Lemma 2.2, the universal extension

$$0 \to (id \times q)^*\mathcal{V}_1 \otimes \pi^*\mathcal{O}_{P(r_1,d_1)}(1) \to \mathcal{E} \to (id \times q)^*\mathcal{V}_2 \to 0$$

on $C \times P(r_1, d_1)$ defines a morphism

$$\Phi : P(r_1, d_1) \to \mathcal{SU}_C(r, \mathcal{L})^s = M. \tag{2.2}$$

Construction 2.3. The images (under Φ) of lines in the fibres of

$$q : P(r_1, d_1) = \mathbb{P}(\mathcal{G}) \to \mathcal{R}(r_1, d_1)$$

are lines of M.

Lemma 2.4. *On each fiber $P(r_1, d_1)_\xi := q^{-1}(\xi)$ at $\xi \in \mathcal{R}(r_1, d_1)$,*

$$\Phi_\xi := \Phi|_{P(r_1,d_1)_\xi} : P(r_1, d_1)_\xi \to M$$

*is the normalization of its image. The rational curves constructed in **Construction 2.3** are lines in M.*

Proof. Write $\mathcal{P} = P(r_1, d_1) = \mathbb{P}(\mathcal{G})$, $\mathcal{R} = \mathcal{R}(r_1, d_1)$, $\mathcal{U}_i = \mathcal{U}_C(r_i, d_i)$ $(i = 1, 2)$, recall

$$\mathcal{G} = R^1p_*(\mathcal{V}_2^\vee \otimes \mathcal{V}_1).$$

Let $\omega_C = \mathcal{O}_C(\sum_{i=1}^{2g-2} y_i)$, $\omega_{\mathcal{U}_1}$ and $\omega_{\mathcal{U}_2}$ be the canonical line bundles of C, $\mathcal{U}_1$ and $\mathcal{U}_2$. It is not difficult, using (2.1), to compute

$$\Phi^*(\omega_M^{-1}) = q^*\left(\omega_{\mathcal{U}_1}^{-1} \otimes \omega_{\mathcal{U}_2}^{-1} \otimes \det(\mathcal{G})^{\otimes 2} \otimes \bigotimes_{i=1}^{2g-2} \det(\mathcal{V}_1^\vee \otimes \mathcal{V}_2)_{y_i}\right) \otimes \mathcal{O}_{\mathcal{P}}(2(r, d)).$$

Thus, for any $\xi \in \mathcal{R}$, $\Phi_\xi^*(\Theta) = \mathcal{O}_{\mathcal{P}_\xi}(1)$ which implies that Φ_ξ is birational, therefore it is the normalization of $\Phi_\xi(\mathcal{P}_\xi)$. In particular, for any line $\ell \subset \mathcal{P}_\xi$, $\Phi_\xi(\ell) \subset M$ is a line and ℓ is the normalization of $\Phi_\xi(\ell)$. □

Now we recall the construction of Hecke curves which are also lines in M when $(r,d)=r$. Let $\mathcal{U}_C(r,d-1)$ be the moduli space of stable bundles of rank r and degree $d-1$. Let $\mathfrak{O}\subset\mathcal{U}_C(r,d-1)$ be the open set of $(1,0)$-semistable bundles in the following sense

Definition 2.5. A vector bundle V on C is called (k,ℓ)-semistable (resp. (k,ℓ)-stable) if for any proper subbundle $W\subset V$, we have

$$\frac{\deg(W)+k}{\mathrm{rk}(W)}\le\ (\text{resp. }<)\frac{\deg(V)+k-\ell}{\mathrm{rk}(V)}.$$

Let $C\times\mathfrak{O}\xrightarrow{\psi}J^d(C)$ be defined as $\psi(x,V)=\mathcal{O}_C(x)\otimes\det(V)$ and let $\mathscr{R}_C:=\psi^{-1}(\mathcal{L})\subset C\times\mathfrak{O}$. There is a fibration $\mathscr{R}_C\to C$ with fibres $\mathfrak{O}\cap\mathcal{SU}_C(r,\mathcal{L}(-x))$ at $x\in C$. Let $\mathscr{V}$ be the universal bundle on $\mathscr{R}_C$, let $p:\mathbb{P}(\mathscr{V}^\vee)\to\mathscr{R}_C$ be the projective bundle and

$$p^*(\mathscr{V}^\vee)\to\mathcal{O}_{\mathbb{P}(\mathscr{V}^\vee)}(1)\to 0$$

be the universal quotient. Let $C\times\mathbb{P}(\mathscr{V}^\vee)\xrightarrow{\pi}\mathbb{P}(\mathscr{V}^\vee)$ be the projection and $\Gamma\subset C\times\mathbb{P}(\mathscr{V}^\vee)$ be the graph of $\mathbb{P}(\mathscr{V}^\vee)\xrightarrow{p}\mathscr{R}_C\to C$. We have

$$0\to\mathscr{E}^\vee\to\pi^*p^*(\mathscr{V}^\vee)\to\mathcal{O}_\Gamma\otimes\pi^*\mathcal{O}_{\mathbb{P}(\mathscr{V}^\vee)}(1)\to 0$$

where $\mathscr{E}^\vee$ is defined to the kernel of the surjection. Taking duals, we have

$$0\to\pi^*p^*\mathscr{V}\to\mathscr{E}\to\mathcal{O}_\Gamma(\Gamma)\otimes\pi^*\mathcal{O}_{\mathbb{P}(\mathscr{V}^\vee)}(-1)\to 0,\tag{2.3}$$

which, at any point $\xi=(x,V,V_x^\vee\twoheadrightarrow\Lambda)\in\mathbb{P}(\mathscr{V}^\vee)$, gives exact sequence

$$0\to V\xrightarrow{\iota}\mathscr{E}_\xi\to\mathcal{O}_x\to 0$$

on C such that $\ker(\iota_x)=\Lambda^\vee\subset V_x$. That V being $(1,0)$-semistable (resp. stable) implies semistability (resp. stability) of $\mathscr{E}_\xi$. Observe that $\mathscr{E}$ determines a morphism

$$\Psi:\mathbb{P}(\mathscr{V}^\vee)\to\mathcal{SU}_C(r,\mathcal{L})\supseteq\mathcal{SU}_C(r,\mathcal{L})^s=M.$$

Let $\mathscr{P}^0:=\Psi^{-1}(M)\subset\mathbb{P}(\mathscr{V}^\vee)$, $\mathscr{R}^0_C:=p(\mathscr{P}^0)\subset\mathscr{R}_C$ and

$$p:\mathscr{P}^0\to\mathscr{R}^0_C\,,\quad\Psi:\mathscr{P}^0\to M\,.\tag{2.4}$$

Construction 2.6. The images (under Ψ) of lines in the fibres of

$$p:\mathscr{P}^0\to\mathscr{R}^0_C$$

are so called Hecke curves in M, which are lines if and only if $(r,d)=r$ by Theorem 1[10].

Theorem 2.7. *(i) If $(r,d)\neq r$, all lines in M are obtained by performing **Construction 2.3** for all pairs $\{r_1,d_1\}$ satisfying*

$$0<r_1<r,\quad r_1d-d_1r=(r,d).$$

*(ii) If $(r,d)=r$, perform **Construction 2.3** as in (i) and the **Construction 2.6**, we obtain all lines in M.*

Proof. It was shown[10] that any rational curve $\phi : \mathbb{P}^1 \to M$ is defined by a vector bundle E on $X = C \times \mathbb{P}^1$. A degree formula[10] was also proved. To recall it, let $f : X \to C$ and $\pi : X \to \mathbb{P}^1$ be the projections. On a general fiber $f^{-1}(\xi) = X_\xi$, E has the form

$$E|_{X_\xi} = \bigoplus_{i=1}^{n} \mathcal{O}_{X_\xi}(\alpha_i)^{\oplus r_i}, \quad \alpha_1 > \cdots > \alpha_n.$$

The $\alpha = (\alpha_1^{\oplus r_1}, ..., \alpha_n^{\oplus r_n})$ is called the generic splitting type of E. Tensoring E by $\pi^*\mathcal{O}_{\mathbb{P}^1}(-\alpha_n)$, we can assume without loss of generality that $\alpha_n = 0$. Any such E admits a relative Harder-Narasimhan filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

in which the quotient sheaves $F_i = E_i/E_{i-1}$ are torsion-free with generic splitting type $(\alpha_i^{\oplus r_i})$ respectively. Let $F_i' = F_i \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-\alpha_i)$ $(i = 1, ..., n)$, thus they have generic splitting type $(0^{\oplus r_i})$ respectively. Without risk of confusion, we denote the degree of F_i (resp. E_i) on the general fiber of π by $\deg(F_i)$ (resp. $\deg(E_i)$). Accordingly, $\mu(E_i)$ (resp. $\mu(E)$) denotes the slope of the restriction of E_i (resp. E) to the general fiber of π respectively. Let $\mathrm{rk}(E_i)$ denote the rank of E_i. Then we have the formula[10]

$$\deg(\phi^*(\Theta)) = \frac{r}{(r,d)} \left(\sum_{i=1}^{n} c_2(F_i') + \sum_{i=1}^{n-1} (\mu(E) - \mu(E_i))(\alpha_i - \alpha_{i+1})\mathrm{rk}(E_i) \right).$$

When $(r, d) \neq r$, we have $c_2(F_i') = 0$ and $n = 2$. Thus there are bundles V_1, V_2 of rank r_1, r_2 and degree d_1, d_2 on C such that

$$0 \to f^*V_1 \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(1) \to E \to f^*V_2 \to 0$$

where r_1, r_2, d_1, d_2 satisfy $r_1 + r_2 = r$, $d_1 + d_2 = d$ and

$$r_1 \frac{d}{(r,d)} - d_1 \frac{r}{(r,d)} = 1, \quad d_2 \frac{r}{(r,d)} - r_2 \frac{d}{(r,d)} = 1.$$

By Lemma 2.2, V_1 and V_2 must be stable and $det(V_1) \otimes det(V_2) = \mathcal{L}$. Thus ϕ factors through $\mathbb{P}^1 \xrightarrow{\sigma} \mathcal{P}_\xi \xrightarrow{\Phi_\xi} M$, where $\xi = (V_1, V_2) \in \mathcal{R}$ and $\sigma^*\mathcal{O}_{\mathcal{P}_\xi}(1) = \mathcal{O}_{\mathbb{P}^1}(1)$ (so that σ is an embedding and $\sigma(\mathbb{P}^1)$ is a line of $\mathcal{P}_\xi$). This proves (i).

When $(r, d) = r$, we have either $c_2(F_i') = 0$ and $n = 2$ or $c_2(E) = 1$ and $n = 1$. Thus the line is either obtaining by **Construction 2.3** or defined by a vector bundle E on $X = C \times \mathbb{P}^1$ satisfying

$$0 \to f^*V \to E \to \mathcal{O}_{\{p\}\times\mathbb{P}^1}(-1) \to 0$$

where $f : X = C \times \mathbb{P}^1 \to C$ and $\pi : X = C \times \mathbb{P}^1 \to \mathbb{P}^1$ are projections, V is a vector bundle on C. The stability of $E_t = E|_{C\times\{t\}}$ $(\forall\, t \in \mathbb{P}^1)$ implies immediately that V is $(1, 0)$-semistable. Thus, in this case, the line is obtained by **Construction 2.6**. □

3 The variety of lines

By the variety of lines, we mean the quotient $\mathbf{Hom}_1(\mathbb{P}^1, M)/\mathrm{Aut}(\mathbb{P}^1)$ which can be defined by means of the Chow variety. To determine $\mathbf{Hom}_1(\mathbb{P}^1, M)$, recall from **Construction 2.3** and

Construction2.6, we have

$$q : P(r_1, d_1) = \mathbb{P}(\mathcal{G}) \to \mathcal{R}(r_1, d_1), \quad p : \mathscr{P}^0 \to \mathscr{R}^0_C.$$

Let $\mathbb{P}^1_{\mathcal{R}(r_1,d_1)} = \mathbb{P}^1 \times \mathcal{R}(r_1, d_1)$ and $\mathbf{Hom}_1\left(\mathbb{P}^1_{\mathcal{R}(r_1,d_1)}\,,\ \mathbb{P}(\mathcal{G})/\mathcal{R}(r_1, d_1)\right)$ be the scheme such that for any scheme T over $\mathcal{R}(r_1, d_1)$

$$\mathbf{Hom}_1\left(\mathbb{P}^1_{\mathcal{R}(r_1,d_1)}\,,\ \mathbb{P}(\mathcal{G})/\mathcal{R}(r_1, d_1)\right)(T)$$

is the set of T-morphisms $\mathbb{P}^1_{\mathcal{R}(r_1,d_1)} \times_{\mathcal{R}(r_1,d_1)} T \to P(\mathcal{G}) \times_{\mathcal{R}(r_1,d_1)} T$ of degree 1 with respect to $\mathcal{O}_{P(\mathcal{G})}(1)$. It is the variety of degree 1 maps

$$\mathbb{P}^1 \to P(r_1, d_1) = \mathbb{P}(\mathcal{G})$$

with images in the fibers of $q : P(r_1, d_1) = \mathbb{P}(\mathcal{G}) \to \mathcal{R}(r_1, d_1)$. Similarly, recall that

$$p : \mathscr{P}^0 \to \mathscr{R}^0_C$$

is an open set of the projective bundle $p : \mathbb{P}(\mathscr{V}^\vee) \to \mathscr{R}_C$, we can define the variety

$$\mathbf{Hom}^r_1(\mathbb{P}^1, \mathscr{P}^0) := \mathbf{Hom}_1\left(\mathbb{P}^1_{\mathscr{R}^0_C}\,,\ \mathscr{P}^0/\mathscr{R}^0_C\right)$$

of degree 1 maps $\mathbb{P}^1 \to \mathscr{P}^0$ with images in the fibers of $p : \mathscr{P}^0 \to \mathscr{R}^0_C$ (we use $\mathbf{Hom}^r$ to denote relative maps). Let

$$\mathbf{Hom}^r_1(\mathbb{P}^1, \mathbb{P}) := \bigsqcup_{\{r_1,d_1\}} \mathbf{Hom}_1\left(\mathbb{P}^1_{\mathcal{R}(r_1,d_1)}\,,\ \mathbb{P}(\mathcal{G})/\mathcal{R}(r_1, d_1)\right)$$

be the disjoint union, where $\{r_1, d_1\}$ runs through the pairs satisfying:

$$0 < r_1 < r, \quad r_1 d - d_1 r = (r, d).$$

Theorem 3.1. *Let $\mathbf{Hom}_1(\mathbb{P}^1, M)$ be the variety of degree 1 morphisms $\mathbb{P}^1 \to M$ (respect to Θ). Then*

$$\mathbf{Hom}_1(\mathbb{P}^1, M) \cong \begin{cases} \mathbf{Hom}^r_1(\mathbb{P}^1, \mathbb{P}) & \text{if} \quad (r, d) \neq r \\ \mathbf{Hom}^r_1(\mathbb{P}^1, \mathbb{P}) \bigsqcup \mathbf{Hom}^r_1(\mathbb{P}^1, \mathscr{P}^0) & \text{if} \quad (r, d) = r. \end{cases}$$

Proof. By sending a T-morphism

$$\mathbb{P}^1 \times T \cong \mathbb{P}^1_{\mathcal{R}(r_1,d_1)} \times_{\mathcal{R}(r_1,d_1)} T \xrightarrow{\varphi_T} P(\mathcal{G}) \times_{\mathcal{R}(r_1,d_1)} T$$

to a T-morphism

$$\mathbb{P}^1 \times T \xrightarrow{\varphi_T} P(\mathcal{G}) \times_{\mathcal{R}(r_1,d_1)} T \to P(\mathcal{G}) \times T \xrightarrow{\Phi \times id_T} M \times T,$$

we have the canonical morphism

$$\mathbf{Hom}^r_1(\mathbb{P}^1, \mathbb{P}) \to \mathbf{Hom}_1(\mathbb{P}^1, M)$$

which is surjective when $(r, d) \neq r$ by Theorem 2.7. To show it is also injective, let ξ_1, $\xi_2 \in \mathbf{Hom}^r_1(\mathbb{P}^1, \mathbb{P})$ defined by the exact sequences

$$0 \to f^* V_1 \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(1) \to \mathcal{E}_1 \to f^* V_2 \to 0,$$

$$0 \to f^*W_1 \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(1) \to \mathcal{E}_2 \to f^*W_2 \to 0$$

on $C \times \mathbb{P}^1$, where $f : X = C \times \mathbb{P}^1 \to C$ and $\pi : X = C \times \mathbb{P}^1 \to \mathbb{P}^1$ are the projections. If ξ_1, ξ_2 have the same image in $\mathbf{Hom}_1(\mathbb{P}^1, M)$, then there is a line bundle $\mathcal{N}$ on $\mathbb{P}^1$ such that $\mathcal{E}_1 \cong \mathcal{E}_2 \otimes \pi^*\mathcal{N}$. If $\deg(\mathcal{N}) \leq 0$, then $\mathrm{Hom}(f^*V_1 \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(1), f^*W_2 \otimes \pi^*\mathcal{N}) = 0$ and $\mathcal{E}_1 \cong \mathcal{E}_2 \otimes \pi^*\mathcal{N}$ induces $f^*V_1 \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(1) \hookrightarrow f^*W_1 \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes \pi^*\mathcal{N}$, which implies $V_1 \hookrightarrow W_1 \otimes \mathrm{H}^0(\mathcal{N})$. Thus $\mathcal{N} = \mathcal{O}_{\mathbb{P}^1}$, $V_1 \cong W_1$ and $V_2 \cong W_2$, which implies $\xi_1 = \xi_2$. If $\deg(\mathcal{N}) \geq 0$, using $\mathcal{E}_2 \cong \mathcal{E}_1 \otimes \pi^*\mathcal{N}^{-1}$, we get $\xi_1 = \xi_2$ by the same arguments. Thus $\mathbf{Hom}_1^r(\mathbb{P}^1, \mathbb{P}) \to \mathbf{Hom}_1(\mathbb{P}^1, M)$ is bijective when $(r, d) \neq r$.

Similarly, when $(r, d) = r$, we have a surjective morphism

$$\mathbf{Hom}_1^r(\mathbb{P}^1, \mathbb{P}) \bigsqcup \mathbf{Hom}_1^r(\mathbb{P}^1, \mathscr{P}^0) \to \mathbf{Hom}_1(\mathbb{P}^1, M)$$

by Theorem 2.7. To see the injectivity, we only need to consider ξ_1, $\xi_2 \in \mathbf{Hom}_1^r(\mathbb{P}^1, \mathscr{P}^0)$ defined by the following two exact sequences on $C \times \mathbb{P}^1$

$$0 \to f^*V \to \mathcal{E}_1 \to \mathcal{O}_{\{x_1\} \times \mathbb{P}^1}(-1) \to 0,$$

$$0 \to f^*W \to \mathcal{E}_2 \to \mathcal{O}_{\{x_2\} \times \mathbb{P}^1}(-1) \to 0$$

where V, W are stable vector bundles on C of rank r and degree $d-1$, x_1, $x_2 \in C$ are two points. If ξ_1, ξ_2 have the same image in $\mathbf{Hom}_1(\mathbb{P}^1, M)$, then there is a line bundle $\mathcal{N}$ on $\mathbb{P}^1$ such that

$$\mathcal{E}_1 \cong \mathcal{E}_2 \otimes \pi^*\mathcal{N}, \quad x_1 = x_2.$$

If $\deg(\mathcal{N}) \leq 0$, then $\mathrm{Hom}(f^*V, \mathcal{O}_{\{x_2\} \times \mathbb{P}^1}(-1) \otimes \pi^*\mathcal{N}) = 0$. The isomorphism $\mathcal{E}_1 \to \mathcal{E}_2 \otimes \mathcal{N}$ induces an injection $f^*V \hookrightarrow f^*W \otimes \pi^*\mathcal{N}$, which implies that $\mathcal{N} = \mathcal{O}_{\mathbb{P}^1}$ and $V \cong W$, thus $\xi_1 = \xi_2$. If $\deg(\mathcal{N}) \geq 0$, using $\mathcal{E}_2 \cong \mathcal{E}_1 \otimes \pi^*\mathcal{N}^{-1}$, we have $\xi_1 = \xi_2$ by the same arguments.

To show the isomorphism, it is enough to show that $\mathbf{Hom}_1(\mathbb{P}^1, M)$ is smooth. To see the smoothness of $\mathbf{Hom}_1(\mathbb{P}^1, M)$, let $\varphi : \mathbb{P}^1 \to M$ be a point of $\mathbf{Hom}_1(\mathbb{P}^1, M)$, which, by Lemma 2.1[10], is defined by a vector bundle E on $C \times \mathbb{P}^1$ such that $\varphi^*T_M = R^1\pi_*Ad(E)$. Then E must satisfy either $0 \to f^*V_1 \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(1) \to E \to f^*V_2 \to 0$ or

$$0 \to f^*V \to E \to \mathcal{O}_{\{x\} \times \mathbb{P}^1}(-1) \to 0.$$

Using these exact sequences, we can show

$$\mathrm{H}^1(\varphi^*T_M) = \mathrm{H}^1(R^1\pi_*Ad(E)) = 0.$$

Thus $\mathbf{Hom}_1(\mathbb{P}^1, M)$ is smooth. □

By Theorem 3.21[6], there is a semi-normal variety $\mathrm{Chow}_{1,1}(M)$ parametrizing effective cycles of dimension 1 and degree 1 (respect to Θ) with a universal cycle

$$\mathrm{Univ}_{1,1}(M) \to \mathrm{Chow}_{1,1}(M).$$

Since $\mathbf{Hom}_1(\mathbb{P}^1, M)$ is smooth, there is an $\mathrm{Aut}(\mathbb{P}^1)$-invariant morphism

$$\mathbf{Hom}_1(\mathbb{P}^1, M) \to \mathrm{Chow}_{1,1}(M).$$

Let $\mathscr{L}(M) \subset \mathrm{Chow}_{1,1}(M)$ be the image, which is precisely the locus of $\mathrm{Chow}_{1,1}(M)$ parametrizing the cycles with rational components. Then, by Proposition 2.2[6], $\mathscr{L}(M) \subset \mathrm{Chow}_{1,1}(M)$ is a closed subset.

Definition 3.2. The closed subset $\mathscr{L}(M) \subset \mathrm{Chow}_{1,1}(M)$ with the reduced scheme structure is called the **variety of lines in** M. The induced universal cycle $\mathbb{L} \subset M \times \mathscr{L}(M)$ defined by

$$\mathbb{L} := \mathrm{Univ}_{1,1}(M) \times_{\mathrm{Chow}_{1,1}(M)} \mathscr{L}(M) \to \mathscr{L}(M)$$

is called the **universal line in** M.

Let $G(r_1,d_1) \to \mathcal{R}(r_1,d_1)$ (resp. $\mathscr{H} \to \mathscr{R}^0_C$) be the relative Grassmannian bundles of lines in $P(r_1,d_1)$ (resp. $\mathscr{P}^0$), and let

$$\begin{tikzcd}
\mathfrak{L}(r_1,d_1) \arrow[r, hook] \arrow[dr] & P(r_1,d_1) \times_{\mathcal{R}(r_1,d_1)} G(r_1,d_1) \arrow[d] \\
 & G(r_1,d_1)
\end{tikzcd} , \quad
\begin{tikzcd}
\mathfrak{L}(h) \arrow[r, hook] \arrow[dr] & \mathscr{P}^0 \times_{\mathscr{R}^0_C} \mathscr{H} \arrow[d] \\
 & \mathscr{H}
\end{tikzcd}$$

be the universal lines. Recall the morphisms

$$\Phi : P(r_1,d_1) \to M\,, \quad \Psi : \mathscr{P}^0 \to M \tag{3.1}$$

in (2.2) and (2.4), which induce

$$\begin{tikzcd}
\mathfrak{L}(r_1,d_1) \arrow[r, "\Phi\times\mathrm{id}"] \arrow[dr] & M \times G(r_1,d_1) \arrow[d] \\
 & G(r_1,d_1)
\end{tikzcd} , \quad
\begin{tikzcd}
\mathfrak{L}(h) \arrow[r, "\Psi\times\mathrm{id}"] \arrow[dr] & M \times \mathscr{H} \arrow[d] \\
 & \mathscr{H}
\end{tikzcd} \tag{3.2}$$

Then the families $\mathrm{Im}(\Phi \times \mathrm{id}) \subset M \times G(r_1,d_1)$ and $\mathrm{Im}(\Psi \times \mathrm{id}) \subset M \times \mathscr{H}$ of lines define the morphisms

$$G(r_1,d_1) \xrightarrow{\Upsilon_{r_1,d_1}} \mathscr{L}(M) \quad \text{and} \quad \mathscr{H} \xrightarrow{\theta} \mathscr{L}(M) \ \text{ if } (r,d) = r. \tag{3.3}$$

Let $\mathscr{L}(M)_{r_1,d_1} := \mathrm{Im}(\Upsilon_{r_1,d_1})$, $\mathscr{H}_\theta := \mathrm{Im}(\theta)$, and let

$$G(M) := \bigsqcup_{\{r_1,d_1\}} G(r_1,d_1)\,, \quad \mathscr{S}(M) := \bigsqcup_{\{r_1,d_1\}} \mathscr{L}(M)_{r_1,d_1}$$

be the disjoint unions of varieties, where $\{r_1,d_1\}$ runs through the pairs $\{r_1,d_1\}$ satisfying: $0 < r_1 < r, \quad r_1 d - d_1 r = (r,d)$.

Corollary 3.3. *$G(r_1,d_1) \xrightarrow{\Upsilon_{r_1,d_1}} \mathscr{L}(M)_{r_1,d_1}$ are the normalizations and θ induces $\mathscr{H} \cong \mathscr{H}_\theta \subset \mathscr{L}(M)$ when $(r,d) = r$. Moreover,*

$$\mathscr{L}(M) = \begin{cases} \mathscr{S}(M) & \text{if } (r,d) \neq r \\ \mathscr{S}(M) \sqcup \mathscr{H}_\theta & \text{if } (r,d) = r \end{cases}$$

and $G(M) \to \mathscr{L}(M)$ is an injective morphism.

Proof. $\mathscr{H} \cong \mathscr{H}_\theta$ follows from the study of Hecke cycles[8]. Since all $G(r_1,d_1)$ are smooth projective varieties, to show the other statements, it is enough to show that $G(M) \to \mathscr{L}(M)$ (resp. $G(M) \sqcup \mathscr{H} \to \mathscr{L}(M)$) is bijective if $(r,d) \neq r$ (resp. $(r,d) = r$). Theorem 2.7 implies surjectivity. The same arguments in the proof Theorem 3.1 imply injectivity. $\square$

4 The geometry of lines

The morphism $\Psi : \mathscr{P}^0 \to M$ was well studied[8] for arbitrary rank. In particular, for any $\xi \in \mathscr{R}^0_C$, the morphism

$$\Psi_\xi := \Psi|_{\mathscr{P}^0_\xi} : \mathscr{P}^0_\xi = p^{-1}(\xi) \to M$$

is a closed embedding. In this section, we study the morphism

$$\Phi : \mathcal{P} := P(r_1, d_1) \to M$$

for arbitrary rank. In general, we are not able to show that

$$\Phi_\xi := \Phi|_{\mathcal{P}_\xi} : \mathcal{P}_\xi = q^{-1}(\xi) \to M$$

is a closed embedding for each $\xi \in \mathcal{R} := \mathcal{R}(r_1, d_1)$. Consequently, we are not able to show that every line in M is smooth for arbitrary rank case (it is true in rank two case). However, we will show that Φ_ξ is a closed embedding for $\xi \in \mathcal{R} \setminus \mathcal{D}$, where

$$\mathcal{D} = \{\xi = (V_1, V_2) \in \mathcal{R} \,|\, \mathrm{Hom}(V_1, V_2) \neq 0\}.$$

Observe that $\mathcal{D}$ can be realized as the degenercy locus of a morphism between two vector bundles on $\mathcal{R}$. Thus, if $\mathcal{D} \neq \emptyset$, it has

$$\mathrm{Codim}(\mathcal{D}) \leq r_1 r_2 (g-1) + 1 - (r, d)$$

and $\mathcal{D}$ is Cohen-Macaulay if the equality holds. To prove a lower bound of the codimension, we start it with an elementary lemma.

Lemma 4.1. *Let V_1, V_2, V be stable vector bundles on C of rank r_1, r_2, $r = r_1 + r_2$ and degree d_1, d_2, $d = d_1 + d_2$. Then, when $r_1 d - d_1 r = (r, d)$, we have*

(1) Any nontrivial morphism $V_1 \to V$ must be an injective morphism of bundles, and any nontrivial morphism $V \to V_2$ must be surjective.

(2) For any nontrivial morphism $f : V_1 \to V_2$, if $\mu(f(V_1)) \neq \mu(V)$, then it must be injective when $r_1 \leq r_2$ and surjective when $r_1 > r_2$. If $\mu(f(V_1)) = \mu(V)$ and $(r, d) \neq r$, then $f(V_1)$ is semistable and $V_2/f(V_1)$ is torsion-free.

Proof. Let $V_1' \subset V$ be the image of $V_1 \to V$ with $\mathrm{rk}(V_1') = r_1'$, $\deg(V_1') = d_1'$. Then

$$\frac{(r,d)}{r_1 r} = \mu(V) - \mu(V_1') + \mu(V_1') - \mu(V_1) > \mu(V) - \mu(V_1') = \frac{r_1' d - r d_1'}{r_1' r} > 0$$

if $r_1' \neq r_1$, which is impossible since $r_1' d - r d_1' \geq (r, d)$. It also shows that V_1' must be a subbundle of V. The surjectivity of any nontrivial morphism $V \to V_2$ can be proved similarly. To prove (2), let $f(V_1)$ be of rank r_1' and degree d_1', then

$$\begin{aligned}\frac{(r,d)}{r_1 r_2} &= \mu(V_2) - \mu(f(V_1)) + \mu(f(V_1)) - \mu(V_1) \\ &= \frac{r_1' d_2 - r_2 d_1'}{r_1' r_2} + \frac{r_1 d_1' - r_1' d_1}{r_1' r_1}.\end{aligned}$$

When $r_1 \leq r_2$, if $V_1 \to V_2$ is not injective, then both $\deg(V_2 \otimes f(V_1)^*) = r'_1d_2 - r_2d'_1$ and $\deg(f(V_1) \otimes V_1^*) = r_1d'_1 - r'_1d_1$ are positive. Their difference

$$(r'_1d_2 - r_2d'_1) - (r_1d'_1 - r'_1d_1) = r'_1d - d'_1r = r'_1r(\mu(V) - \mu(f(V_1))) \neq 0$$

is a nonzero integer divisible by (r,d), thus one of them is bigger than (r,d), which contradicts the above equality (4). When $r_1 > r_2$, then $\deg(f(V_1) \otimes V_1^*) > 0$ and $\deg(V_2 \otimes f(V_1)^*) \geq 0$. The same argument shows that $\deg(V_2 \otimes f(V_1)^*) = r_2r'_1(\mu(V_2) - \mu(f(V_1)))$ must be zero. Thus $f(V_1) = V_2$ by the stability of V_2.

When $\mu(f(V_1)) = \mu(V)$, we show first of all that $V_2/f(V_1)$ is torsion-free. Let $f(V_1) \subset W \subset V_2$ such that V_2/W is torsion-free, $\mathrm{rk}(W) = r'_1$, $\deg(W) = \tilde{d}'_1$. Then

$$(r_1\tilde{d}'_1 - r'_1d_1) - (r'_1d_2 - r_2\tilde{d}'_1) = r(\tilde{d}'_1 - d'_1)$$

and

$$\begin{aligned}\frac{(r,d)}{r_1r_2} &= \mu(V_2) - \mu(W) + \mu(W) - \mu(V_1)\\ &= \frac{r'_1d_2 - r_2\tilde{d}'_1}{r'_1r_2} + \frac{r_1\tilde{d}'_1 - r'_1d_1}{r'_1r_1} \geq \frac{r(\tilde{d}'_1 - d'_1)}{r_1r'_1}.\end{aligned}$$

Thus, if $\tilde{d}'_1 - d'_1 > 0$, we get $r \leq (r,d)$, which contradicts the assumption $r \neq (r,d)$. To see that $f(V_1)$ is semistable, let $V_0 \subset f(V_1)$ be a proper subbundle of rank r_0 and degree d_0. If $\mu(V_0) > \mu(f(V_1)) = \mu(V)$, then $\mu(V_1) < \mu(V_0) < \mu(V_2)$, which is impossible by the above arguments. Thus $f(V_1)$ is semistable. ☐

Lemma 4.2. *Let* $\mathcal{D} = \{(V_1, V_2) \in \mathcal{U}_C(r_1, d_1) \times \mathcal{U}_C(r_2, d_2) | \mathrm{Hom}(V_1, V_2) \neq 0\}$ *and* $\mathcal{R} := \mathcal{R}(r_1, d_1)$. *Then, when* $\min\{r_1, r_2\} > \frac{r}{(r,d)}$, *we have*

$$\operatorname{codim}(\mathcal{D} \cap \mathcal{R}) \geq \frac{r}{(r,d)}(r - \frac{r}{(r,d)})(g-1) - 1, \tag{4.1}$$

and when $\min\{r_1, r_2\} \leq \frac{r}{(r,d)}$, *we have*

$$\operatorname{codim}(\mathcal{D} \cap \mathcal{R}) \geq r_1r_2(g-1) + 1 - (r,d) \tag{4.2}$$

The same inequalities also hold for the codimension of $\mathcal{D}$.

Proof. Since taking dual of vector bundles induces an isomorphism between moduli spaces, we can assume $r_1 \geq r_2$ without loss of generality. Let $h : \mathcal{H} \to \mathcal{D}$ be the total space of morphisms $V_1 \to V_2$, let $\mathcal{H}_1 \subset \mathcal{H}$ be the union of irreducible components whose general points are not surjective morphisms $V_1 \to V_2$, and $\mathcal{H}_2 := \mathcal{H} \setminus \mathcal{H}_1$. Then there is an open dense subset $\mathcal{H}_2^0 \subset \mathcal{H}_2$ and an exact sequence $0 \to \mathcal{V}' \to \mathcal{V}_1 \to \mathcal{V}_2 \to 0$ on $C \times \mathcal{H}_2^0$, where $\mathcal{V}'$ is a flat family of vector bundles of rank $r_1 - r_2$ and degree $d_1 - d_2$ any subbundle of which has slope less than d_1/r_1 (so that the set of such bundles is bounded). Let $Q \subset \mathrm{Quot}(\mathcal{O}_C(-m)^{p(m)})$ be the open set consisting of locally free quotients $\mathcal{O}_C(-m)^{p(m)} \to V' \to 0$ of rank $r_1 - r_2$ and degree $d_1 - d_2$ such that $V'(m)$ is generated by global sections, $\mathrm{H}^1(V'(m)) = 0$ and the quotient map induces $\mathbb{C}^{p(m)} \cong \mathrm{H}^0(V'(m))$. Let $\mathcal{F} \to \mathcal{H}_2^0$ be the frame bundle of $\pi_*(\mathcal{V}'(m))$, where $\pi : C \times \mathcal{H}_2^0 \to \mathcal{H}_2^0$, then the pullback of the exact sequence gives a morphism from $\mathcal{F}$ to the projective bundle over $Q \times \mathcal{U}_2$ that parametrizes nontrivial extensions. The fiber of this morphism has dimension at most 1 since V_1 is a stable bundle. Note that the irreducible component of Q containing stable bundles has maximal dimension and sending any extension $(0 \to V' \to V_1 \to V_2 \to 0)$ to $\det(V_2)^2 \otimes \det(V')$ defines a surjective morphism to $J^d(C)$. Thus

$$\dim(\mathcal{H}_2) \leq (r_1 - r_2)^2(g-1) + 1 + r_2^2(g-1) + 1 + (r,d) + r_2(r_1 - r_2)(g-1) - g$$

and codimension of $h(\mathcal{H}_2) \subset \mathcal{R}$ is at least $r_1 r_2(g-1) + 1 - (r,d)$.

To estimate $h(\mathcal{H}_1)$, by Lemma 4.1 (2), there are two cases: (1) $r_1 = r_2$, $(V_1, V_2) \in h(\mathcal{H}_1)$ satisfy $0 \to V_1 \to V_2 \to {}_{x_{n_1}}\mathbb{C}^{n_1} \oplus \cdots \oplus {}_{x_{n_k}}\mathbb{C}^{n_k} \to 0$ for some $x_{n_i} \in C$ and $\sum n_i = d_2 - d_1$ (the locus of these points has codimension at least $r_1^2(g-1) + 1 - (r,d)$), or (2) $\min\{r_1, r_2\} > \frac{r}{(r,d)}$, $(V_1, V_2) \in h(\mathcal{H}_1)$ where V_1, V_2 are nontrivial extensions $0 \to V_1' \to V_1 \to V_{k-1} \to 0$, $0 \to V_{k-1} \to V_2 \to V_2' \to 0$ such that V_{k-1} is a bundle of rank $r_{k-1} = k\frac{r}{(r,d)}$ and degree $d_{k-1} = k\frac{d}{(r,d)}$, where $1 \leq k < \min\{\frac{r_1(r,d)}{r}, \frac{r_2(r,d)}{r}\}$. The locus of such points has codimension at least $r_{k-1}(r - r_{k-1})(g-1) + 1 - \frac{2r_{k-1}}{r}(r,d)$. Note that the function

$$f(x) = x(r-x)(g-1) + 1 - \frac{2x}{r}(r,d)$$

is an increase function for $x \leq \frac{r}{2} - \frac{(r,d)}{r(g-1)}$, $r_0 := \frac{r}{(r,d)} \leq r_{k-1} \leq \frac{r}{2} - \frac{(r,d)}{r(g-1)}$, and $f(r_1) \leq r_1(r - r_1)(g-1) + 1 - (r,d)$, we get (4.1) when $\min\{r_1, r_2\} > \frac{r}{(r,d)}$. If $\min\{r_1, r_2\} \leq \frac{r}{(r,d)}$, any morphism $V_1 \to V_2$ must be surjective when $r_1 > r_2$ and injective when $r_1 = r_2$. Thus we get the inequality (4.2). The same estimates also hold clearly for $\mathcal{D}$. □

Corollary 4.3. *If $(r,d) \leq 2$ and $\mathcal{D} \cap \mathcal{R} \neq \emptyset$, then $\mathcal{D} \cap \mathcal{R}$, $\mathcal{D}$ are Cohen-Macaulay closed subschemes of codimension $r_1 r_2(g-1) + 1 - (r,d)$.*

Proof. By Lemma 4.2, when $(r,d) \leq 2$, $\mathcal{D}$ and $\mathcal{D} \cap \mathcal{R}$ have codimension at least $r_1 r_2(g-1) + 1 - (r,d)$. On the other hand, it is standard to realize $\mathcal{D}$ (resp. $\mathcal{D} \cap \mathcal{R}$) as the degenercy locus of a morphism between two vector bundles. Then the general theory implies that the codimension of $\mathcal{D}$ (resp. $\mathcal{D} \cap \mathcal{R}$) is at most $r_1 r_2(g-1) + 1 - (r,d)$ and $\mathcal{D}$ (resp. $\mathcal{D} \cap \mathcal{R}$) Cohen-Macaulay if the bound is reached. □

Write $\mathcal{P} = P(r_1, d_1)$, $\mathcal{R} = \mathcal{R}(r_1, d_1)$. Recall that we have

$$\begin{tikzcd}
C \times \mathcal{P} \arrow[r, "\pi"] \arrow[d, "1\times q"'] & \mathcal{P} \arrow[d, "q"'] \\
C \times \mathcal{R} \arrow[r, "\pi"] & \mathcal{R}
\end{tikzcd}$$

and the exact sequence

$$0 \to (1 \times q)^*\mathcal{V}_1 \otimes \pi^*\mathcal{O}_{\mathcal{P}}(1) \to \mathcal{E} \to (1 \times q)^*\mathcal{V}_2 \to 0$$

which induces the morphism

$$\Phi : \mathcal{P} \to M.$$

Let $Ad(\mathcal{E})$ denote the sheaf of trace free endomorphisms of $\mathcal{E}$ and $\Delta(\mathcal{E}) \subset Ad(\mathcal{E})$ the subsheaf of endomorphisms that preserve the above exact sequence. Then

$$0 \to \Delta(\mathcal{E}) \to Ad(\mathcal{E}) \to (1 \times q)^*(\mathcal{V}_1^\vee \otimes \mathcal{V}_2) \otimes \pi^*\mathcal{O}_{\mathcal{P}}(-1) \to 0. \tag{4.3}$$

By Lemma 4.2, $\pi_*(\mathcal{V}_1^\vee \otimes \mathcal{V}_2) = 0$, thus the sequence (4.3) induces

$$0 \to R^1\pi_*\Delta(\mathcal{E}) \to R^1\pi_*Ad(\mathcal{E}) \to q^*R^1\pi_*(\mathcal{V}_1^\vee \otimes \mathcal{V}_2) \otimes \mathcal{O}_{\mathcal{P}}(-1) \to 0. \tag{4.4}$$

Lemma 4.4. *The infinitesimal deformation map* $T_{\mathcal{P}} \to R^1\pi_*\Delta(\mathcal{E})$ *induces an isomorphism. Under this identification, the sequence* (4.4) *induces*

$$0 \to T_{\mathcal{P}} \xrightarrow{d\Phi} \Phi^*T_M \to q^*R^1\pi_*(\mathcal{V}_1^\vee \otimes \mathcal{V}_2) \otimes \mathcal{O}_{\mathcal{P}}(-1) \to 0. \tag{4.5}$$

Proof. Let $\mathcal{E}nd^0 = \ker(\mathcal{E}nd(\mathcal{V}_1) \oplus \mathcal{E}nd(\mathcal{V}_2) \xrightarrow{tr(\cdot)+tr(\cdot)} \mathcal{O}_{C\times\mathcal{R}})$, then

$$0 \to (1\times q)^*(\mathcal{V}_1 \otimes \mathcal{V}_2^\vee) \otimes \pi^*\mathcal{O}_{\mathcal{P}}(1) \to \Delta(\mathcal{E}) \to (1\times q)^*\mathcal{E}nd^0 \to 0.$$

Now the proof is a straightforward generalization of Lemma 6.6[8] since we have here $T_{\mathcal{R}} = R^1\pi_*\mathcal{E}nd^0$, thus we omit it. $\square$

For any $\xi = (V_1, V_2) \in \mathcal{R}$, in order to study differential $d\Phi_\xi$ of the morphism

$$\Phi_\xi := \Phi|_{\mathcal{P}_\xi} : \mathcal{P}_\xi = q^{-1}(\xi) \to M,$$

let $[e] \in \mathcal{P}_\xi$ be represented by a nontrivial extension

$$0 \to V_1 \xrightarrow{i} V \xrightarrow{j} V_2 \to 0$$

and

$$K_{[e]} := \{(f,g) \in \mathrm{Hom}(V_1, V) \times \mathrm{Hom}(V, V_2) \,|\, g\cdot i + j \cdot f = 0\}.$$

Lemma 4.5. *The kernel of* $(d\Phi_\xi)_{[e]} : T_{\mathcal{P}_\xi,[e]} \to T_{M,\Phi([e])}$ *has dimension*

$$dim(K_{[e]}) - 1.$$

In particular, when $rk(V) = 2$, $d\Phi_\xi$ *is injective at every point* $[e] \in \mathcal{P}_\xi$.

Proof. The $k[\epsilon]$-value points of $\mathcal{P}_\xi$ over $[e] \in \mathcal{P}_\xi$, which lie in kernel of $(d\Phi_\xi)_{[e]}$, are precisely represented by the extensions ($\epsilon^2 = 0$)

$$0 \to V_1 \otimes_k k[\epsilon] \xrightarrow{i_\epsilon} V \otimes_k k[\epsilon] \xrightarrow{j_\epsilon} V_2 \otimes_k k[\epsilon] \to 0$$

with $i_\epsilon = i \otimes 1 + \epsilon f \otimes 1$ and $j_\epsilon = j \otimes 1 + \epsilon g \otimes 1$ where $(f,g) \in K_{[e]}$. Thus the kernel of $(d\Phi_\xi)_{[e]}$ has dimension $dim(K_{[e]}) - 1$. When $rk(V) = 2$, using Lemma 4.1 (1), we can show $\mathrm{Hom}(V_1, V)$ has dimension 1, which implies the injectivity of $(d\Phi_\xi)_{[e]}$ (which also implies that Φ_ξ is an embedding in the case of rank two). $\square$

Proposition 4.6. *For any* $\xi \in \mathcal{R} \setminus \mathcal{R} \cap \mathcal{D}$, *the morphism* $\Phi_\xi : \mathcal{P}_\xi \to M$ *is an embedding. For any two different points* ξ_1, $\xi_2 \in \mathcal{R}$, *the intersection of* $\Phi_{\xi_1}(\mathcal{P}_{\xi_1})$ *and* $\Phi_{\xi_2}(\mathcal{P}_{\xi_2})$ *has dimension zero, i.e., a finite set.*

Proof. $\xi = (V_1, V_2) \notin \mathcal{R} \cap \mathcal{D}$ means $\mathrm{Hom}(V_1, V_2) = 0$, which implies that both Φ_ξ and $d\Phi_\xi$ are injective, thus Φ_ξ is an embedding.

Let $\xi_1 = (V_1, V_2) \in \mathcal{R}$, $\xi_2 = (W_1, W_2) \in \mathcal{R}$ be any two different points. Fix isomorphisms $\mathbb{P} \cong \mathcal{P}_{\xi_1} \cong \mathcal{P}_{\xi_2}$ and pull back the universal extensions to $C \times \mathbb{P}$

$$0 \to p_1^*V_1 \otimes \pi^*\mathcal{O}_{\mathbb{P}}(1) \to \mathcal{E}_1 \to p_1^*V_2 \to 0,$$

$$0 \to p_1^* W_1 \otimes \pi^* \mathcal{O}_{\mathbb{P}}(1) \to \mathcal{E}_2 \to p_1^* W_2 \to 0$$

where $p_1 : C \times \mathbb{P} \to C$, $\pi : C \times \mathbb{P} \to \mathbb{P}$ are the projections. If the intersection $\Phi_{\xi_1}(\mathcal{P}_{\xi_1}) \cap \Phi_{\xi_2}(\mathcal{P}_{\xi_2})$ has positive dimension, then there is a nonsingular projective curve $Y \to \mathbb{P}$ such that on $C \times Y$ the pullback of above exact sequences

$$0 \to p_1^* V_1 \otimes \pi^* \mathcal{O}_Y(1) \to \mathcal{E}_1 \to p_1^* V_2 \to 0,$$

$$0 \to p_1^* W_1 \otimes \pi^* \mathcal{O}_Y(1) \to \mathcal{E}_2 \to p_1^* W_2 \to 0$$

define the same morphism $Y \to M$. Thus there is a line bundle $\mathcal{N}$ on Y such that $\mathcal{E}_1 \cong \mathcal{E}_2 \otimes \pi^* \mathcal{N}$. If $\deg(\mathcal{N}) \leq 0$, then

$$\mathrm{Hom}(p_1^* V_1 \otimes \pi^* \mathcal{O}_Y(1), p_1^* W_2 \otimes \pi^* \mathcal{N}) = 0$$

and $\mathcal{E}_1 \cong \mathcal{E}_2 \otimes \pi^* \mathcal{N}$ induces an injection

$$p_1^* V_1 \otimes \pi^* \mathcal{O}_Y(1) \to p_1^* W_1 \otimes \pi^* \mathcal{O}_Y(1) \otimes \pi^* \mathcal{N},$$

which implies an injection $V_1 \to W_1 \otimes \mathrm{H}^0(\mathcal{N})$. Thus $\mathcal{N} = \mathcal{O}_Y$, $V_1 \cong W_1$ and $V_2 \cong W_2$, which contradicts with $\xi_1 \neq \xi_2$. If $\deg(\mathcal{N}) \geq 0$, using $\mathcal{E}_2 \cong \mathcal{E}_1 \otimes \pi^* \mathcal{N}^{-1}$, we get contradiction by the same arguments. Hence the intersection $\Phi_{\xi_1}(\mathcal{P}_{\xi_1}) \cap \Phi_{\xi_2}(\mathcal{P}_{\xi_2})$ has dimension zero. $\square$

It would be interesting to have a formula of the intersection number of $\Phi_{\xi_1}(\mathcal{P}_{\xi_1})$ and $\Phi_{\xi_2}(\mathcal{P}_{\xi_2})$. We end this section with a question.

Question 4.7. Is it true that any two lines on M has at most one intersection point ? It is interesting to describe the configurations of lines on M and on subvarieties (such as the Brill-Noether locus) of M.

5 Remarks on minimal rational curves on M

Let M be the moduli space of stable bundles of rank r and degree d with fixed determinant $\mathcal{L}$ on a nonsingular projective C of genus $g \geq 3$. We assume $(r, d) = 1$ in this section. Then M is a smooth projective Fano variety and there is an universal bundle $\mathcal{E}$ on $C \times M$. The universal bundle $\mathcal{E}$ is unique up to tensoring the pullback of a line bundle on M. Since $\mathrm{Pic}(M) \cong \mathbb{Z}$, according to Remark 2.9[9], there is a unique universal bundle $\mathcal{E}$ on $C \times M$ such that $\det(\mathcal{E}|_{\{x\} \times M}) = \Theta_M^{\alpha}$ for any $x \in C$, where Θ_M is the ample generator of $\mathrm{Pic}(M)$ and α is the smallest positive integer such that $\alpha d \equiv 1 \bmod (r)$. We will denote this canonical universal bundle by $\mathcal{E}$ in this section.

For any rational curve $\phi : \mathbb{P}^1 \to M$ through a general point of M, denote by $f : X = C \times \mathbb{P}^1 \to C$ and $\pi : X = C \times \mathbb{P}^1 \to \mathbb{P}^1$ the projections, the proof of Theorem 1[10] implies in fact the following

Theorem 5.1. *If $\phi : \mathbb{P}^1 \to M$ is a minimal rational curve through a general point, then $\deg(\phi^* \Theta_M) = r$ and $E := (1 \times \phi)^* \mathcal{E}$ is a stable bundle on $C \times \mathbb{P}^1$ with respect to any polarization. Moreover, there is a point $x_\phi \in C$ such that $E|_{\{x\} \times \mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(\alpha)^{\oplus r}$ for $x \neq x_\phi$ and*

$$E|_{\{x_\phi\} \times \mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(\alpha + 1) \oplus \mathcal{O}_{\mathbb{P}^1}(\alpha)^{\oplus (r-2)} \oplus \mathcal{O}_{\mathbb{P}^1}(\alpha - 1).$$

There is a stable vector bundle V on C such that

$$0 \to f^*V \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(\alpha) \to E \to \mathcal{O}_{\{x_\phi\}\times\mathbb{P}^1}(\alpha-1) \to 0 \tag{5.1}$$

is an exact sequence.

For any general point $[W] \in M$, let Ω_W be the relative cotangent bundle of $\mathbb{P}(W^\vee) \to C$. Then Theorem 1[10] also implies that the variety of all minimal rational curves passing through $[W] \in M$ is naturally isomorphic to the (double)projective bundle

$$\mathbb{P}(\Omega_W) \xrightarrow{p} C.$$

Thus, for any $x_0 \in C$, the set of minimal rational curves $\phi : \mathbb{P}^1 \to M$ with $x_\phi \neq x_0$ is the dense open set $p^{-1}(C \setminus \{x_0\})$ of the variety of minimal rational curves passing through $[W] \in M$. Let $\pi : C \times M \to M$ be the projection and $d > 2r(g-1)$. Then the direct image $\pi_*\mathcal{E}$ is a vector bundle on M (the so called Picard bundle). By using Theorem 1[10], we give a simple proof of some known results[1], [2], [7].

Theorem 5.2. *The bundles $\mathcal{E}_x := \mathcal{E}|_{\{x\}\times M}$ $(\forall x \in C)$, $\mathcal{E}$ and the Picard bundle $\pi_*\mathcal{E}$ are stable with respect to any polarization on $C \times M$ and M. Moreover, for any $x \neq y$, we have $\mathcal{E}_x \ncong \mathcal{E}_y$.*

Proof. By Proposition 3.7 in Chapter II[6] (cf. also (4.3), proof of Proposition 12[4]), for any closed subset $S \subset M$ of codimension at least two, there is a minimal rational curve $\phi : \mathbb{P}^1 \to M$ such that $\phi(\mathbb{P}^1) \cap S = \emptyset$. If $\mathcal{F} \subset \mathcal{E}_x$ is a subsheaf with $\mu(\mathcal{F}) \geq \mu(\mathcal{E}_x)$, we may assume that the singular locus $S \subset M$ of $\mathcal{F}$ has codimension at least two. Then there is a minimal rational curve $\phi : \mathbb{P}^1 \to M$ with $x_\phi \neq x$ such that $\phi(\mathbb{P}^1) \cap S = \emptyset$ and $\phi(\mathbb{P}^1)$ is not contained in the singular locus of $\mathcal{E}_x/\mathcal{F}$. By Theorem 5.1, $E_x = \phi^*\mathcal{E}_x = \mathcal{O}_{\mathbb{P}^1}(\alpha)^{\oplus r}$, thus $r \cdot a_{\mathcal{F}} = \deg(\phi^* c_1(\mathcal{F})) \leq \mathrm{rk}(\mathcal{F}) \cdot \alpha$ where $a_{\mathcal{F}} \in \mathbb{Z}$ such that $c_1(\mathcal{F}) = a_{\mathcal{F}} c_1(\Theta_M)$ and $c_1(\mathcal{E}_x) = \alpha c_1(\Theta_M)$, which implies $\mu(\mathcal{F}) = \mu(\mathcal{E}_x)$, a contradiction since $\alpha d \equiv 1 \bmod (r)$. Thus $\mathcal{E}_x$ $(\forall x \in C)$ are stable bundles.

To show stability of $\mathcal{E}$ with respect to any polarization $H = a\, f^{-1}(x) + b\, \Theta_M$, for any subsheaf $\mathcal{F} \subset \mathcal{E}$, let $c_1(\mathcal{F}) = d_1\, f^{-1}(x) + \beta\, \Theta_M$ and $c_1(\mathcal{E}) = d\, f^{-1}(x) + \alpha\, \Theta_M$, we have

$$\begin{aligned} c_1(\mathcal{F}) \cdot H^n &= (d_1 b + \beta\, a) b^{n-1} f^{-1}(x) \cdot \Theta_M^n \\ c_1(\mathcal{E}) \cdot H^n &= (d\, b + \alpha\, a) b^{n-1} f^{-1}(x) \cdot \Theta_M^n \end{aligned}$$

where $n = \dim(M)$. Thus it is enough to show

$$\frac{d_1 b + \beta\, a}{\mathrm{rk}(\mathcal{F})} < \frac{d\, b + \alpha\, a}{\mathrm{rk}(\mathcal{E})}. \tag{5.2}$$

We can assume that singular loci $S \subset C \times M$ of $\mathcal{F}$ has codimension at least two. If $f(S) \subsetneqq C$, then stability of $\mathcal{E}_x$ $(x \notin f(S))$ and $\mathcal{E}|_{C\times\{y\}}$ $(y \notin \pi(S))$ implies the inequality (5.2). If $f(S) = C$, for generic $x \in C$, the locus $S_x = S \cap f^{-1}(x) \subset \{x\} \times M$ has codimension at least two. Thus there is a minimal rational curve $\phi : \mathbb{P}^1 \to M$ such that $\phi(\mathbb{P}^1) \cap \pi(S_x) = \emptyset$ and $x_\phi \neq x$. Then $\{x\} \times \phi(\mathbb{P}^1) \subset C \times M$ is disjoint with S and $\mathcal{E}|_{\{x\}\times\phi(\mathbb{P}^1)}$ is semi-stable, which implies $\beta/\mathrm{rk}(\mathcal{F}) \leq \alpha/\mathrm{rk}(\mathcal{E})$. The stability of $\mathcal{E}|_{C\times\{y\}}$ $(y \notin \pi(S))$ implies $d_1/\mathrm{rk}(\mathcal{F}) < d/\mathrm{rk}(\mathcal{E})$. All together, we have the inequality (5.2).

To show stability of $\pi_*\mathcal{E}$, for any subsheaf $\mathcal{F} \subset \pi_*\mathcal{E}$, it is enough to find a $\phi : \mathbb{P}^1 \to M$ disjoint with the singular locus S of $\mathcal{F}$ such that the restrictions $F = \phi^*\mathcal{F}$ and $\pi_*E = \pi^*(\pi_*\mathcal{E})$

satisfy $\mu(F) < \mu(\pi_* E)$, where $E = (1 \times \phi)^* \mathcal{E}$. Let $\mathcal{F}(W) \subset \mathrm{H}^0(W) = \pi_*(\mathcal{E})|_{[W]}$ be the fibre of $\mathcal{F}$ at a general point $[W] \in M$. Let $Z \subset C$ be the set of common zero points of sections of $\mathcal{F}(W)$ and $x \in C \setminus Z$ a general point. Let $\mathcal{F}(W)_x = \{s_x \in W_x \mid s \in \mathcal{F}(W)\}$ and $\zeta \in \mathbb{P}(W_x^\vee)$ a general point such that $\mathcal{F}(W)_x \nsubseteq \zeta^\perp \subset W_x$. Define a vector bundle W^ζ, which is the Hecke modification of W along $\zeta^\perp \subset W_x$, by

$$0 \to W^\zeta \xrightarrow{\iota} W \to (W_x/\zeta^\perp) \otimes \mathcal{O}_x \to 0$$

where $\zeta^\perp$ denotes the hyperplane in W_x annihilated by ζ. The 1-dimensional subspace $\ker(\iota_x) \subset W_x^\zeta$ defines a point $[\ker(\iota_x)] \in \mathbb{P}(W_x^\zeta)$. Then a general line $\ell \subset \mathbb{P}(W_x^\zeta)$ passing through $[\ker(\iota_x)] \in \mathbb{P}(W_x^\zeta)$ defines a minimal rational curve $\phi : \mathbb{P}^1 \to M$ passing through $[W] \in M$ disjoint with S such that $x_\phi = x$. By (5.1), we have

$$0 \to \mathrm{H}^0(V) \otimes \mathcal{O}_{\mathbb{P}^1}(\alpha) \to \pi_* E \to \mathcal{O}_{\{x_\phi\} \times \mathbb{P}^1}(\alpha - 1) \to 0. \tag{5.3}$$

Since $\mathcal{F}(W)_x \nsubseteq \zeta^\perp \subset W_x$ and $\phi(\mathbb{P}^1)$ passes through $[W] \in M$, the image of $F \subset \pi_* E$ under the surjection $\pi_* E \to \mathcal{O}_{\{x_\phi\} \times \mathbb{P}^1}(\alpha - 1)$ is non-trivial. Thus

$$\mu(F) \le \alpha - \frac{1}{\mathrm{rk}(F)} < \alpha - \frac{1}{\mathrm{rk}(\pi_* E)} = \mu(\pi_* E).$$

To show $\mathcal{E}_x \ncong \mathcal{E}_y$ when $x \ne y$, we choose a minimal rational curve $\phi : \mathbb{P}^1 \to M$ with $x_\phi = x$. Then, by Theorem 5.1, we have

$$\phi^* \mathcal{E}_y = \mathcal{O}_{\mathbb{P}^1}(\alpha)^{\oplus r} \ne \mathcal{O}_{\mathbb{P}^1}(\alpha + 1) \oplus \mathcal{O}_{\mathbb{P}^1}(\alpha)^{\oplus (r-2)} \oplus \mathcal{O}_{\mathbb{P}^1}(\alpha - 1) = \phi^* \mathcal{E}_x.$$

Thus $\mathcal{E}_x \ne \mathcal{E}_y$, we finish the proof of theorem. $\square$

Remark 5.3. As far as we know, the semi-stability of $\mathcal{E}_x$ appears firstly as Proposition 1.4[1], its stability is Proposition 2.1[7]. The stability of $\mathcal{E}$ is Theorem 1.5[1]. The stability of $\pi_* \mathcal{E}$ and the fact that $\mathcal{E}_x \ncong \mathcal{E}_y$ $(x \ne y)$ are the main theorems[2], [7].

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