Games and the Mathematical Mind

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What shall we do in this workshop ?

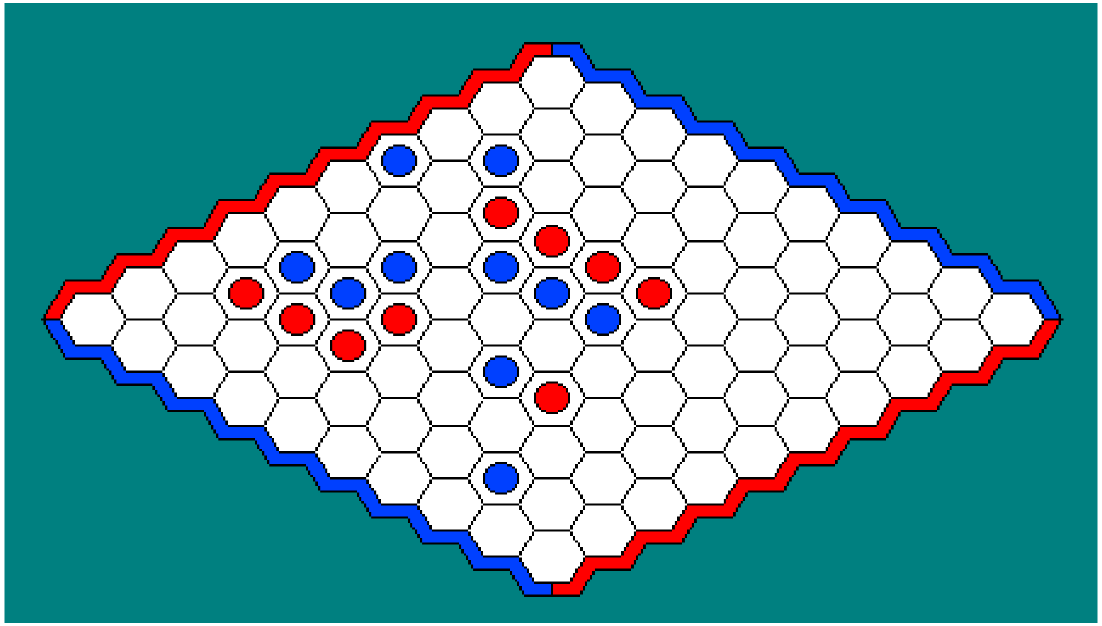
- Play some interesting games like HEX.
- For these games, consider the following questions:
  - **Is it possible for the game to have a draw?**
  - **Is it possible for one of the players to have a winning strategy? If it is possible, who should have a winning strategy and what should be a winning strategy?**
- We shall apply Zermelo's Theorem to answer these questions.

What shall we do in this workshop?

- Learn some problem solving techniques based on the great mathematician **George Pólya 's method (please refer to "How to solve it").**

- Finally, there will be a group competition on solving some mathematics problems.

HEX



You can play Hex with your friends at http://hkumath.hku.hk/~wkc/MathModel/index.php

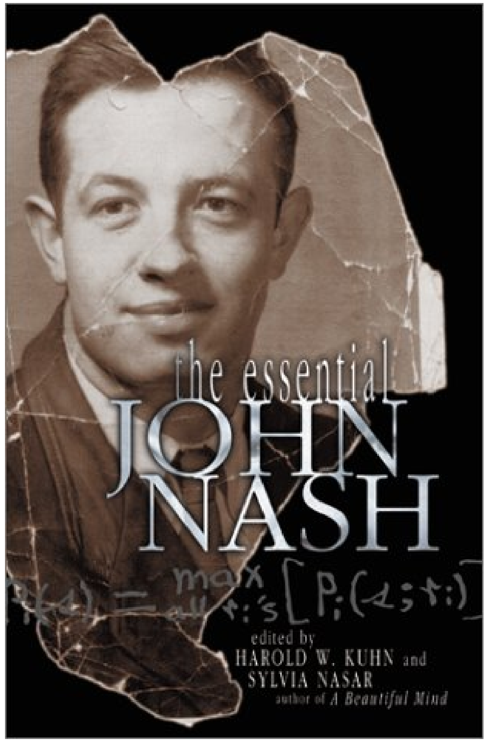
- Hex or Nash is a two-player game played on a rhombic board with hexagonal cells.

- It was invented by a Danish mathematician Piet Hein in 1942, and became popular under the name of Hex.

The game was re-discovered in 1948 by John Nash, when he was a PhD student at Princeton.

At that time, the game was commonly called Nash.

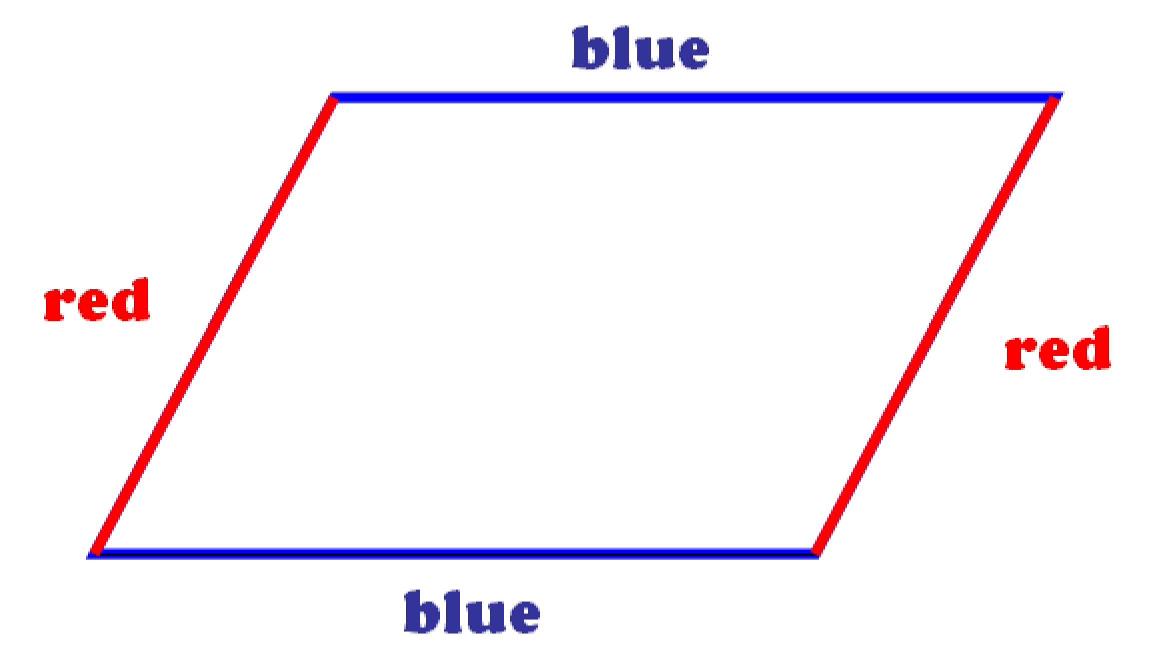
A Beautiful Mind

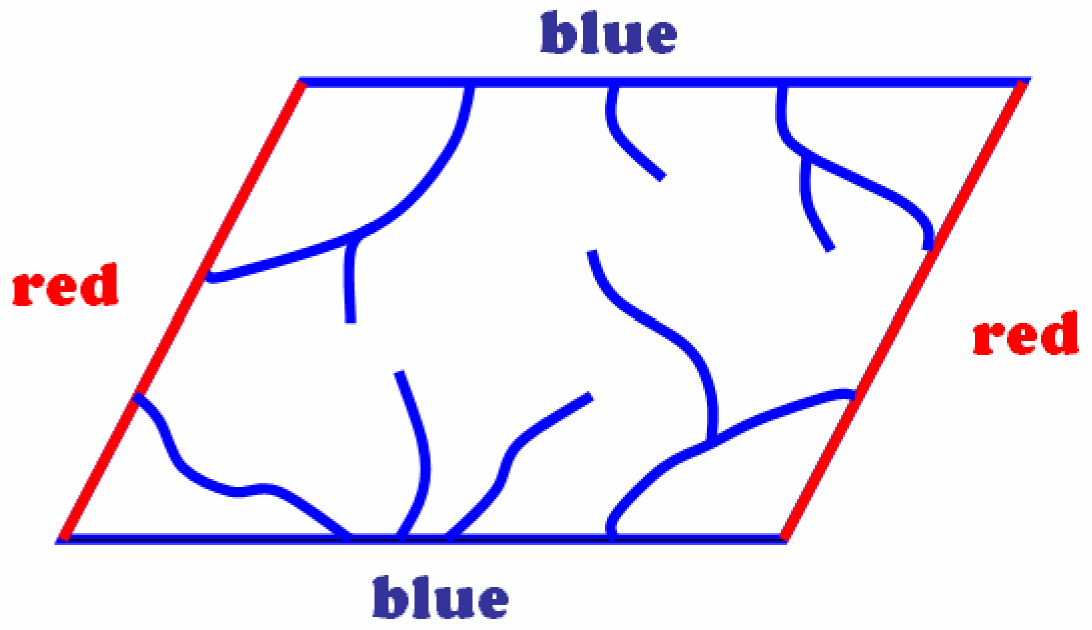


Questions one may ask…

- Is it possible to end in a draw ?
- Is there a winning strategy for one of the players ?

We first rotate the board so that it has the following orientation.




blue
red
red
blue

Further questions

- We may then ask the following questions:
  - **Is it possible for one of the players to have a winning strategy?**
  - **If it is possible, who should have a winning strategy?**
  - **What do we mean by a winning strategy ? Drawing strategy ?**

Winning strategy of a game

- A winning strategy for a player is a strategy that enables the player to win no matter what moves his or her opponent makes.
- If a game can end in a draw, then we can also speak of drawing strategy.
- A drawing strategy is a strategy which does not guarantee a win for a particular player, but guarantees that he or she does not lose.

Zermelo's Theorem

- We have just proven that Hex must have a winner.
- To show that one of the players has a winning strategy, we shall apply the so-called Zermelo's Theorem which is an important result about finite two-person game of perfect information.

Finite and Perfect Information Game

- A finite game is one that must necessarily terminate in a finite number of moves.

- A perfect information game is a game in which the players are aware at all times of all aspects of the structure of the games.

Finite and Perfect Information Game

- In a perfect information game, each player knows, at any point in the game, what moves have been made prior to that point as well as what moves the opponent will be able to make in response to any possible move.

Zermelo’s Theorem

- Zermelo’s Theorem says that in any finite two-person game of perfect information in which the players move alternatively and in which chance does not affect the decision making process,

if the game cannot end in a draw, then one of the two players must have a winning strategy.

Zermelo's Theorem

- To understand why Zermelo's theorem is true, let the two players be A and B.
- Note that since the game is finite, it must end in a win for one player or in a draw.
- Suppose one of the players (say A) does not have a winning strategy, then whatever play A makes, B must have a counter play to prevent A from winning; thus B has at least a drawing strategy.

Zermelo's Theorem

- This drawing strategy guarantees that B will not lose.
- Since the game cannot end in a draw, the drawing strategy of B actually guarantees B a win.

- Therefore, B has a winning strategy.
- So we can conclude that one of the players has a winning strategy.

One of players must have a winning strategy

- Since Hex is finite, cannot end in a draw, and the players move alternatively with complete information,

  Zermelo's Theorem then asserts that one of the two players of Hex must have a winning strategy.

One of players must have a winning strategy

- Apply Zermelo’s Theorem

John Nash proved that the first player has a winning strategy

Nash at HKU,2003

John Nash proved that the first player has a winning strategy

- Nash's argument is known as the strategy stealing argument.

However, explicit winning strategies are only known for boards of sizes up to 9x9. More can be found from the followings:

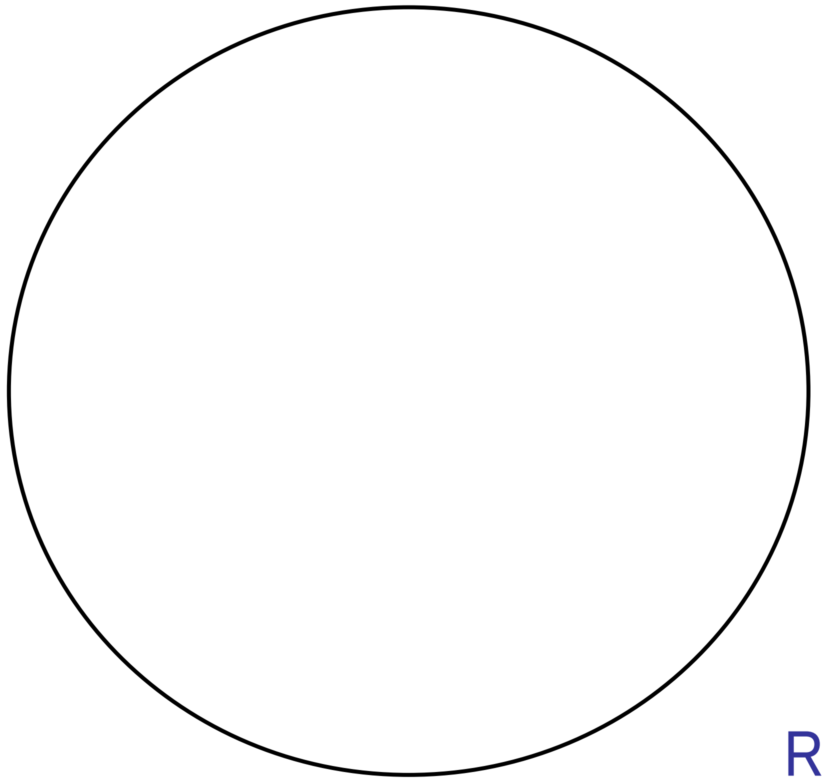
- Anshelevich, Vadim V, The game of Hex: the hierarchical approach. More games of no chance (Berkeley, CA, 2000), 151--165, Cambridge Univ. Press, Cambridge, 2002.
- http://home.earthlink.net/~vanshel/
- http://www.cs.ualberta.ca/~javhar/hex/

You can play Hex with your friends at

http://hkumath.hku.hk/~wkc/MathModel/index.php

A coin game

Radius = r



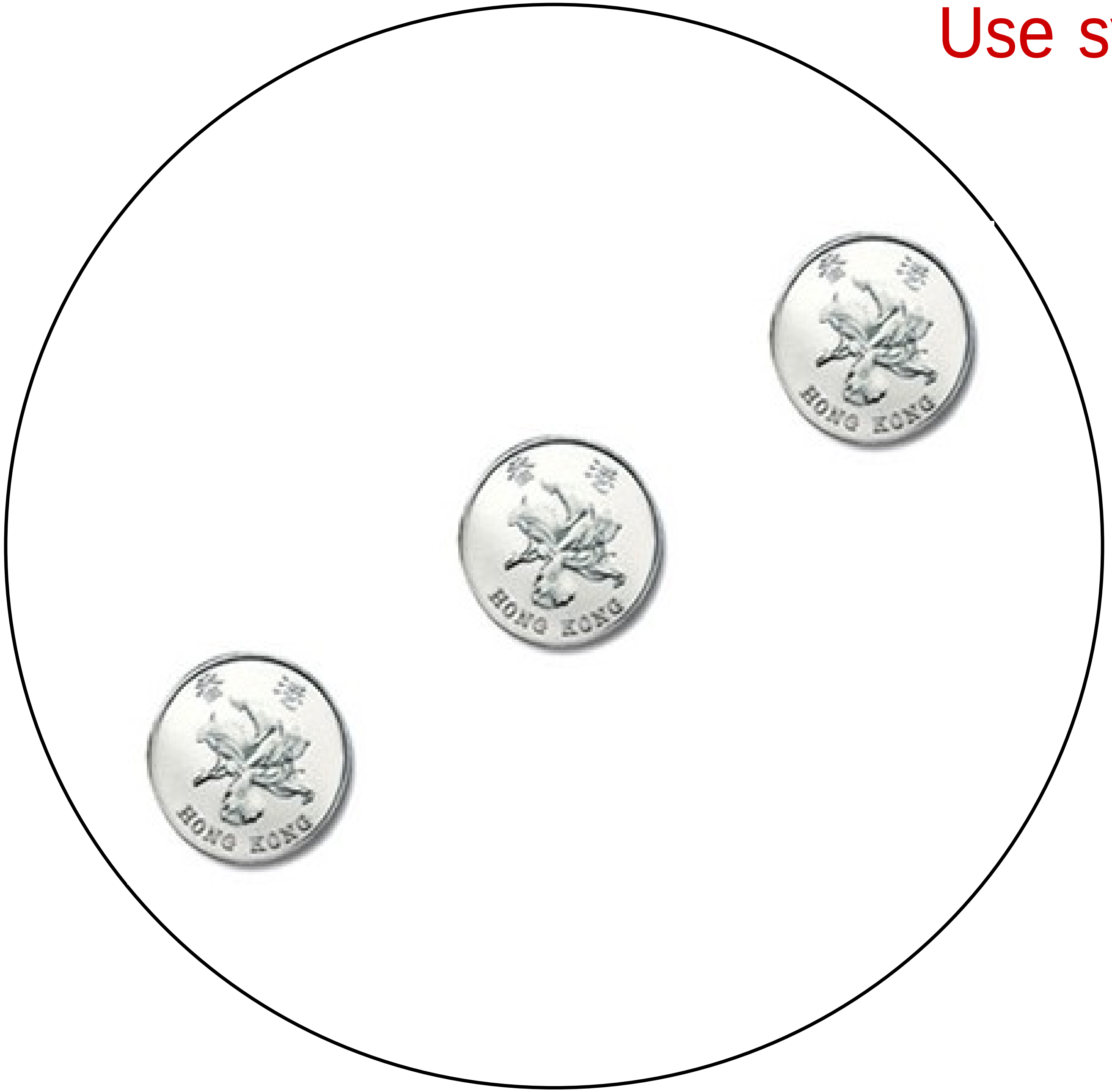
Radius = R

One of the players has a winning strategy

- Apply Zermelo's Theorem

- Who has the winning strategy and what is the winning strategy?

Use symmetry !

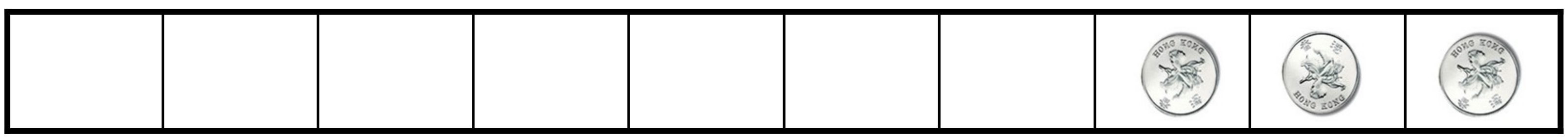


Group competition

Finish two questions in 30 minutes

First question

- This is a coin game.
- The rules are as follows:
- There is a 1 x n (n > 3) rectangular region which consists of n unit squares. We start by putting three coins on the last three squares on the right.
- Each player then takes turns to move any one of the coins to its left hand side. The coins cannot overlap.

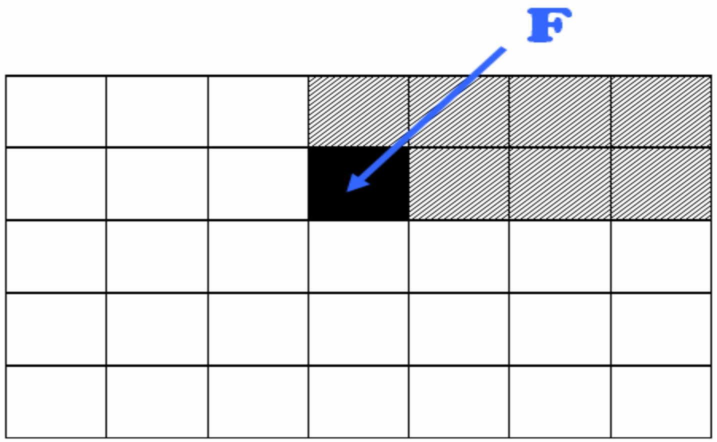


First question (10 points)

- The first one who cannot move a coin loses.
- Does any one of the players has a winning strategy? If so, what is it? Your solution should work for all possible values of $n > 3$.

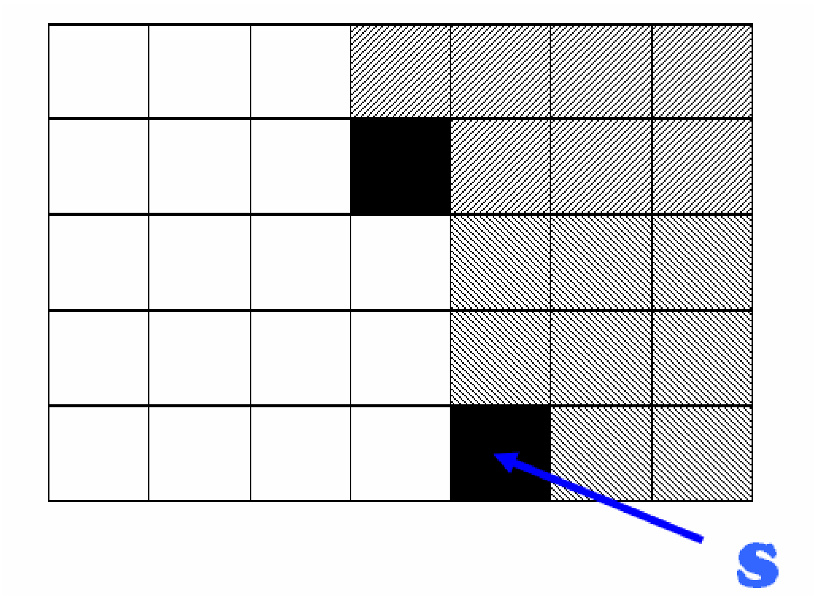
Second question

- The Game of Chomp is played with a rectangular *m* by *n* grid. Once a player selects a certain square on what are left on the grid, all squares above and to the right of that square will be removed.
- The winner is the one who forces the opponent to take the square in the bottom leftmost corner.



Second question (15 points)

- Is there a winning strategy? If so, for which player?
- When $m = n$, find a winning strategy.



Solution to the first question

- The first player has a winning strategy.

							A	B	C

- When n is even, the winning strategy is first move the last coin (coin C) to the leftmost position. Then there will be an EVEN “gap number” (including 0) of unoccupied positions in between C and A.

C							A	B	

Solution to the first question

C			A					B	

- What the first player needs to do in the remaining part of the game is try to make sure that A and B stay next to each other and keep an EVEN "gap number" (including 0) of unoccupied positions in between C and AB or C and BA.

C			A	B					

Solution to the first question

- Therefore, as long as the second player can make a move, the first player can also make a move.
- Hence, the first player will eventually win the game.
- In some sense, the winning strategy is to steal the other player's strategy.
- When *n* is odd, the first player should first move coin A to the leftmost position and then follow the previous strategy.

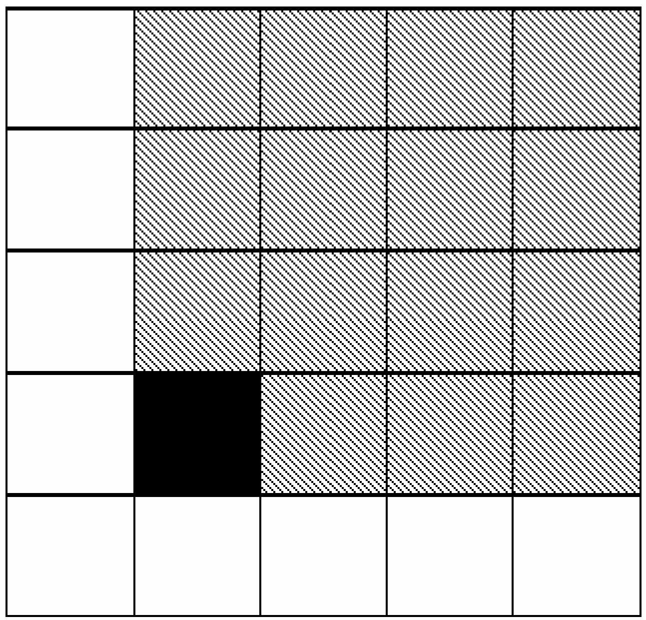
Solution to the second question

- Since the Game of Chomp is a finite game which cannot end in a draw, and the players move alternatively with perfect information, Zermelo's Theorem then asserts that one of the two players must have a winning strategy.
- We claim that the second player cannot have a winning strategy.
- Assume to the contrary that the second player has a winning strategy.

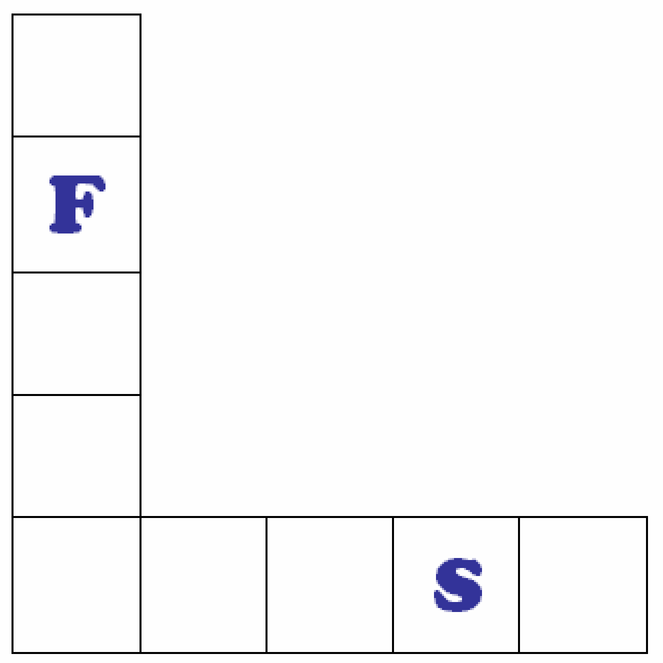
- Let the first player take the upper-rightmost square S1.
- Then the second player will make a move according to his winning strategy.
- Note that no matter which square the second player has chosen, the square S1 will always be one of the squares to be removed and the remaining part of the board is always a L shape.

- Therefore, the first player can simply pretend he was the second player and apply the winning strategy for the second player.
- Hence, both the players are going to win which is impossible.
- Thus, the hypothesis that the second player has a winning strategy leads to a contradiction.
- On the other hand, according to the Zermelo's Theorem, one of the players has a winning strategy. Therefore, the first player must have a winning strategy.

- When $m = n$, the winning strategy for the first player is as follow.
- The first player first selects the following black box and then removes it as well as those grey boxes.

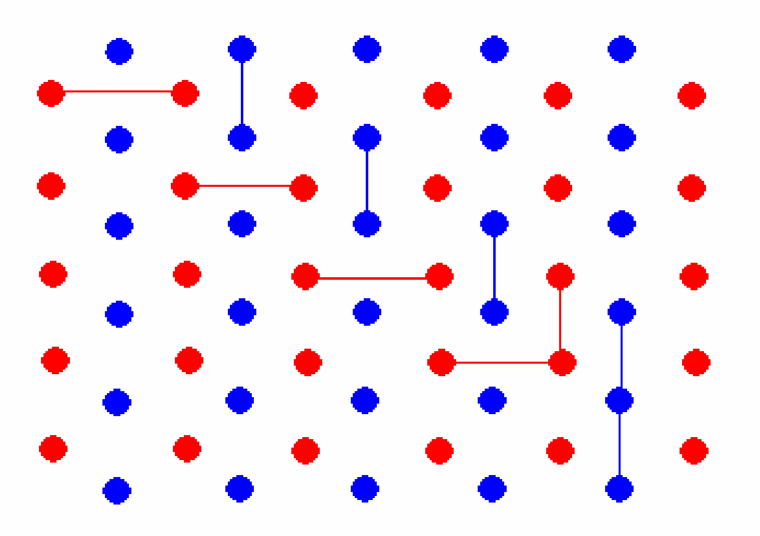


- Then what the first player needs to do is to “steal” the second player’s strategy by using the symmetry of the L shaped board.



Homework

- Mathematician David Gale invented the game Bridg-it in 1958.
- A board is drawn up on squared paper as shown, colouring the dots on alternate rows in two colours. The players decide which colour to be, and only join up dots of their own colour.



Homework

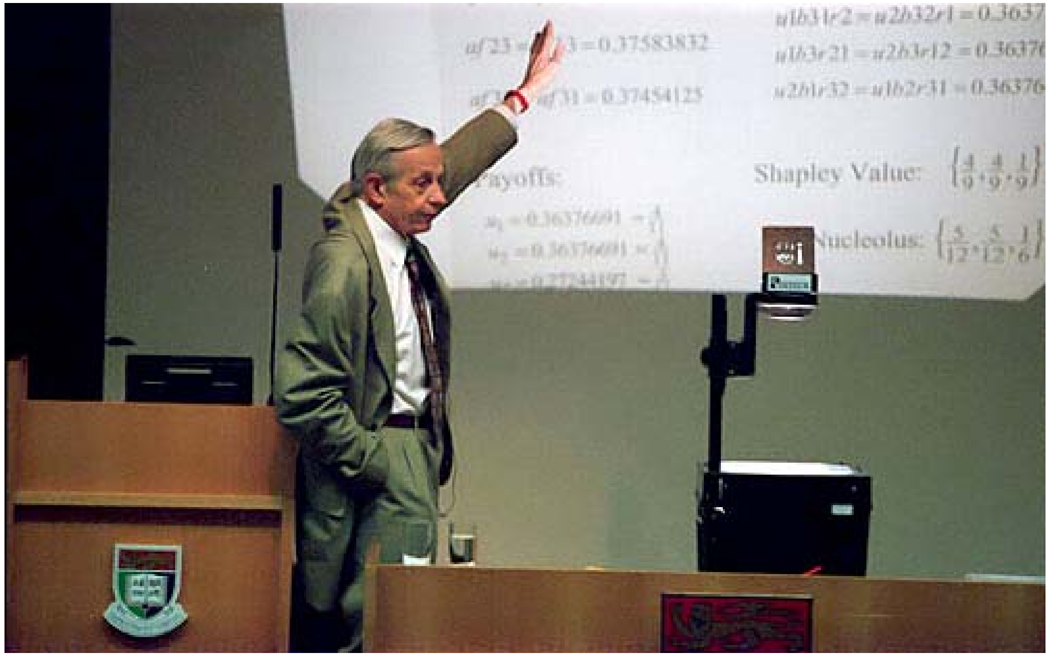
- Playing alternately, each player is allowed to draw a line connecting two dots which must be next to each other. Lines are not allowed to cross other lines. The winner is the person who connects up a line of his or her colour dots going from one side of the board to the opposite side.
- The board can be any size but it must have an equal number of dots of each colour. Use squared paper to help you draw up the grid.
- You can play Bridg-it online at the following website.

http://www.studyworksonline.com/cda/content/applet/0,1033,NAV3-15_SAP28,00.html

Homework

- For this game, consider the following questions:
  - **Is it possible for the game to have a draw?**
  - **Is it possible for one of the players to have a winning strategy? If it is possible, who should have a winning strategy and what should be a winning strategy?**

This ends this workshop and I hope you enjoy it. You can send your solution to me. My e-mail address is ntw@maths.hku.hk.

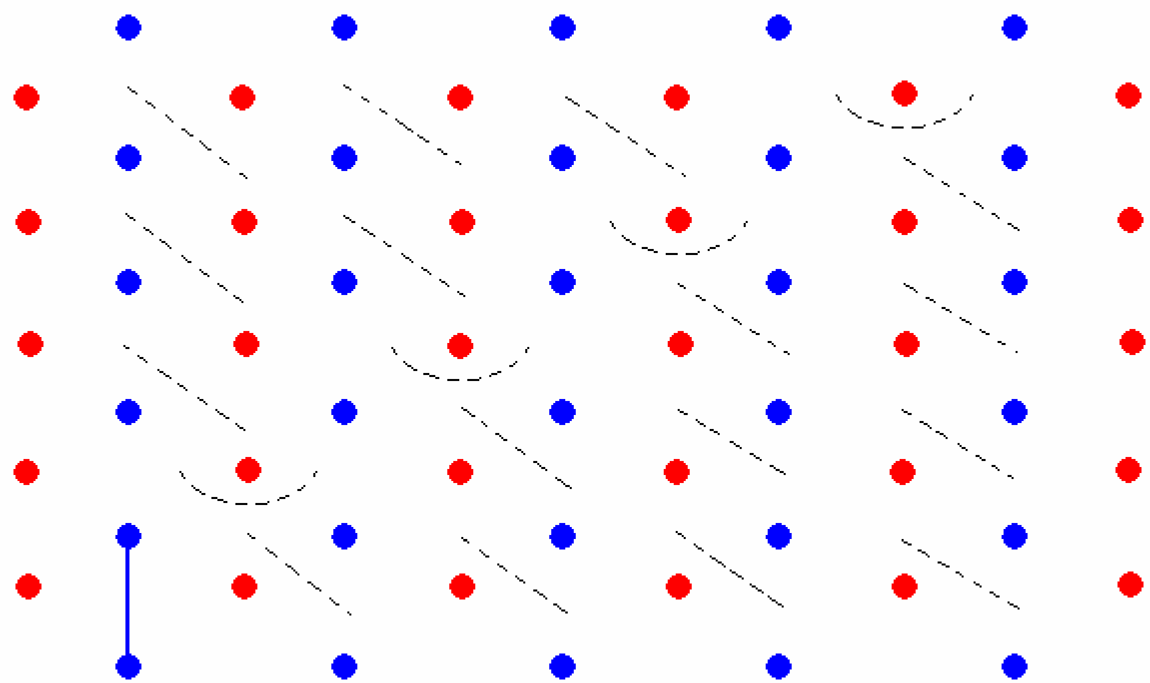


Thank You and Keep in Touch !

A brief solution to the homework question

- First of all, by using the argument similar to the one we used to prove that Hex cannot end in a draw, we can show that Bridg-it also cannot end in a draw.
- Since Bridg-it is a finite game which cannot end in a draw, and the players move alternatively with perfect information, Zermelo's Theorem then asserts that one of the two players must have a winning strategy.
- We claim that the first player has a winning strategy.

In fact, the first player can simply make the first move shown in the lower left corner, and then whenever the second player makes a move that touches the end of a dotted line, the first player makes a move that touches the other end.



- It is not difficult to see that this is a drawing strategy for the first player. Since this game cannot end in a draw, this drawing strategy is in fact a winning strategy for the first player.