On the zeros of $\sum a_i expg_i$

Tuen-Wai Ng and Chung-Chun Yang*

Abstract

Abstract: We consider entire functions of the form $f = \sum a_i e^{g_i}$, where $a_i (\not\equiv 0)$, g_i are entire functions and the orders of all a_i are less than one. If all the zeros of f are real, then $f = e^g \sum a_i e^{h_i}$, where h_i are linear functions. Using this result, we can prove that $f = a_1 e^g$ if all zeros of f are positive, which also generalizes a result obtained by A. Eremenko and L.A. Rubel.

1 Introduction and Main Results.

For $i \geq 1$ and $z \in \mathbf{C}$, let $g_i(z)$ be entire functions. Let $a_i(z)$ be a non-zero entire function with $\rho(a_i) < 1$, where $\rho(g)$ denotes the order of an entire function g. Let B_1 denote the class of entire functions of the form

$$f = \sum_{i=1}^{n} a_i e^{g_i},$$

where $e^{g_i - g_j}$ is non-constant for $i \neq j$.

If all the a_i are polynomials, then such f is said to be in the class B. Clearly, B is a proper subset of B_1.

Let $Z(g)$ be the zero set of an entire function g. In [**?**], by using H.Cartan's theory of holomorphic curves, A. Eremenko and L. A. Rubel proved the following theorem.

Theorem A. Let $f \in B$. If $Z(f)$ is a subset of the positive real axis ,except possibily finitely many points, then $f = pe^g$, where p is a polynomial and g is an entire function.

Therefore, it is natural to ask whether we can say something about the form of f if $f \in B$ and $Z(f)$ is a subset of the real axis. By adapting some of the arguments used in [6] and Nevanlinna value distribution theory for functions meromorphic in a half plane, we can answer this question even for the case $f \in B_1$. In fact, we obtained the following results.

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Theorem 1 . *Let $f \in B_1$. If $Z(f)$ is a subset of the real axis ,except possibly finite points, then $f(z) = e^{g(z)} \sum_{i=1}^{n} a_i(z) e^{b_i z}$, where $b_i \in \mathbf{C}$, g and $a_i (\not\equiv 0)$ are entire functions with $\rho(a_i) < 1$.*

Using theorem 1, we can generalize theorem A to the following theorem .

Theorem 2 . *Let $f \in B_1$. If $Z(f)$ is a subset of the positive real axis, except possibly finite points, then $f = ae^g$, where g, a are entire functions with $\rho(a) < 1$.*

Our basic tool is J.Rossi's half-plane version of Borel theorem. J.Rossi proved this version in [**?**] by using Tsuji's half-plane version of Nevanlinna theory. Therefore, we shall start with the basic notations of Tsuji's theory (c.f. [4],[7]); assuming the readers are familiar with the Nevanlinna Theory and its basic notations (c.f. [3]).

Let $n_u(t, \infty)$ be the number of poles of f in $\{z : |z - \frac{it}{2}| \leq \frac{t}{2}, |z| \geq 1\}$, where f is meromorphic in the open upper half-plane. Define

$$N_u(r, \infty) = N_u(r, f) = \int_1^r \frac{n_u(t, \infty)}{t^2} dt,$$

$$m_u(r, \infty) = m_u(r, f) = \frac{1}{2\pi} \int_{arcsinr^{-1}}^{\pi - arcsinr^{-1}} log^+ |f(rsin\theta e^{i\theta})| \frac{d\theta}{rsin^2\theta},$$

$$N_u(r, a) = N_u(r, \frac{1}{f-a}), \quad m_u(r, a) = m_u(r, \frac{1}{f-a}) \quad (a \neq \infty) \quad and$$

$$T_u(r, f) = m_u(r, f) + N_u(r, f) \quad .$$

Remark 1: We can also define $m_l(r, f), N_l(r, f), T_l(r, f)$ for functions meromorphic in the open lower half-plane in the obvious way.

Lemma 1 *[**?**]. Let f be meromorphic in $Imz > 0 (< 0)$. Define $m_{\alpha,\beta}(r, f) = \frac{1}{2\pi} \int_{\alpha}^{\beta} log^+ |f(re^{i\theta})| d\theta$. Then*

$$\int_r^{\infty} \frac{m_{0,\pi}(t, f)}{t^3} dt \leq \int_r^{\infty} \frac{m_u(t, f)}{t^2} dt \qquad (\int_r^{\infty} \frac{m_{\pi,2\pi}(t, f)}{t^3} dt \leq \int_r^{\infty} \frac{m_l(t, f)}{t^2} dt).$$

Lemma 2 *[**?**]. Let $n \geq 2$, $S = \{f_0, ..., f_n\}$ be a set of meromorphic functions such that any proper subset of S is linearly independent over $\mathbf{C}$. If S is linearly dependent over $\mathbf{C}$, then for all r except possibly on a set of finite measure,*

$$T_u(r) = O\{\sum_{k=0}^{n} [N_u(r, 1/f_k) + N_u(r, f_k)] + logT_u(r) + logr\},$$

where $T_u(r) = max\{T_u(r, f_i/f_j)| \quad 0 \leq i, j \leq n\}$.

Remark 2: If we replace $m_u(r,f)$, $N_u(r,f)$ and $T_u(r,f)$ by the standard Nevanlinna functionals $m(r,f), N(r,f), T(r,f)$ in lemma 2, we shall obtain the orginal full-plane version of Borel theorem.

Lemma 3 *[?]. Let g_i be a transcendental entire function and h be a non-zero entire function such that $T(r,h) = o(T(r,g_i))$ as $r \to \infty$, for $1 \le i \le n$. Suppose $\sum_{i=1}^{n} g_i(z) = h(z)$, then $\sum_{i=1}^{n} \delta(0,g_i) \le n-1$.*

Lemma 4 . *For $n \ge 2$ and each $1 \le i \le n$, let a_i denote a non-zero entire function with $\rho(a_i) < 1$ and b_i be a non-zero complex number . Then, there exists a positive constant A such that for sufficiently large r, $T(r, a_1(z) + \sum_{i=2}^{n} a_i(z)e^{b_i z}) \ge Ar$.*

The proof of lemma 4. It is not difficult to prove for $n = 2$. Assume $n \ge 3$. Let $g(z) = a_1(z) + \sum_{i=2}^{n} a_i(z)e^{b_i z}$ and $G(z) = a_1(z) + \sum_{i=2}^{n-1} a_i(z)e^{b_i z}$. Then $T(r,G) = O(r)$ for large r. From $g = G + a_n e^{b_n z}$ and a simple calculation give

$$(a_n b_n + a_n' - a_n G'/G)e^{b_n z} = g' - gG'/G.$$

It is well-known that (for large r) $T(r, G'/G) = o(T(r,G)$ and $T(r,g') \le AT(Br,g)$,where $A, B \ge 1$. Hence,

$$\frac{1}{\pi}|b_n|r \sim T(r, e^{b_n z}) \le T(r, g' - gG'/G) + T(r, a_n b_n + a_n' - a_n G'/G) + O(1) \le CT(Br,g) + o(r)$$

Therefore, for large r, $T(r,g) \ge Ar$ for some suitable positive constant A.

2 Proofs of Theorems.

The proof of theorem 1. $f \in B_1$ implies that $f = \sum_{i=1}^{n} a_i \exp g_i$, where $a_i (\not\equiv 0)$,g_i are entire functions with $T(r,a_i) = O(r^\epsilon)$ for some fixed positive $\epsilon < 1$.

If n=1, then we are done. For $n \ge 2$, Given that $\exp(g_i - g_j)$ is non-constant for $i \ne j$. From these and using the full-plane version of Borel theorem, we can show that the functions $f_i = a_i \exp g_i$ are linearly independent. Set $f_0 = f$, then the set $\{f_0, ...f_n\}$ will satisifies the independence criteria of Lemma 2.

Given that $Z(f)$ is a subset of the real axis, except possibly finite points. Therefore, $N_u(r, 1/f_0) = O(logr)$. For $1 \le i \le n$, we also have $N_u(r, 1/f_i) = O(r^\epsilon)$, since

$$N_u(r, 1/f_i) = \int_1^r \frac{n_u(t, 1/a_i)}{t^2} dt \le \int_1^r \frac{n(t, 1/a_i)}{t} dt = N(r, 1/a_i) + O(1) = O(r^\epsilon).$$

It follows from lemma 2 that $T_u(r) = O(r^\epsilon)$ and hence $T_u(r, f_i/f_j) = O(r^\epsilon)$ for all i,j. Since $T_u(r, f_i/f_j) = N_u(r, f_i/f_j) + m_u(r, f_i/f_j)$, we also have $m_u(r, f_i/f_j) = O(r^\epsilon)$. Similarly, $m_l(r, f_i/f_j) = O(r^\epsilon)$. Now,

$$T(t, f_i/f_j) = N(t, f_i/f_j) + m(t, f_i/f_j) = O(t^\epsilon) + m_{0,\pi}(t, f_i/f_j) + m_{\pi,2\pi}(t, f_i/f_j).$$

Then by Lemma 1, we have

$$T(r, f_i/f_j)O(1/r^2) \le \int_r^\infty \frac{T(t, f_i/f_j)}{t^3} dt = O(r^{-\epsilon}).$$

Consequently, $T(r, f_i/f_j) = O(r^{2-\epsilon})$. This implies that the order of $\exp(g_i - g_j)$ is less than 2 and hence equal to one.

Now, $f = e^{g_1}(a_1 + \sum_{i=2}^n a_i e^{g_i - g_1})$, where $g_i - g_1$ is linear for $2 \le i \le n$. This also completes the proof.

The proof of theorem 2. Let $f \in B_1$ such that $Z(f)$ is a subset of the positive real axis, except possibly finite points. By Theorem 1, either (i)$f = ae^g$ or (ii) $f(z) = e^{g(z)}(a_1(z) + \sum_{i=2}^n a_i(z) e^{b_i z})$, where $g, a_i (\not\equiv 0)$ are entire functions, $\rho(a_i) < 1$ and the b_i's are non-zero complex numbers. We only need to consider case (ii).

Let $G(z) = a_1(z) + \sum_{i=2}^n a_i(z)e^{b_i z}, h = -a_1, g_1 = -G, g_i(z) = a_i(z)e^{b_i z}$ for $2 \le i \le n$. Then $Z(G) = Z(f)$, $\sum_{i=1}^n g_i(z) = h(z)$ and $T(r, h) = o(T(r, g_i))$ as r tends to infinity for $1 \le i \le n$. By Lemma 3 ,$\sum_{i=1}^n \delta(0, g_i) \le n - 1$. Since $\delta(0, g_i) = 1$ for $i \ge 2$, it follows that $\delta(0, G) = \delta(0, g_1) = 0$.

Hence there exists an unbounded sequence $\{r_i\}$ such that $N(r_i, 0, G) \ge \frac{1}{2}T(r_i, G)$. By Lemma 4,

$$\int_{r_i}^\infty \frac{N(t, 0, G)}{t^2} dt \ge \int_{r_i}^\infty \frac{N(r_i, 0, G)}{t^2} dt \ge \int_{r_i}^\infty \frac{\frac{1}{2}T(r_i, G)}{t^2} dt \ge \int_{r_i}^\infty \frac{\frac{1}{2}Ar_i}{t^2} dt = \frac{1}{2}A > 0.$$

Therefore,$\int_0^\infty \frac{N(t,0,G)}{t^2} dt$ does not converage and hence the genus of G is at least one. Now, G is an entire function of finite order with a genus at least one, which has at most finitely many non-positive zeros. By a result of A.Edrei and W.Fuchs [1], $\delta(0, G) > 0$, which is a contradiction. Hence f must equal to the required form, ae^g.

Remark 3: It is obvious that Theorem A can also be derived from the present arguments by assuming that the coefficients $a_i(z)$s are polynomials in Theorem 2.

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Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong.

E-mail address: mayang@usthk.ust.hk