On the zeros of $\sum a_i e^{g_i}$

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Abstract

Abstract: We consider entire functions of the form $f = \sum a_i e^{g_i}$, where $a_i(\neq 0)$, $g_i$ are entire functions and the orders of all $a_i$ are less than one. If all the zeros of $f$ are real, then $f = e^g \sum a_i e^{h_i}$, where $h_i$ are linear functions. Using this result, we can prove that $f = a_1 e^g$ if all zeros of $f$ are positive, which also generalizes a result obtained by A. Eremenko and L.A. Rubel.

1 Introduction and Main Results.

For $i \geq 1$ and $z \in \mathbb{C}$, let $g_i(z)$ be entire functions. Let $a_i(z)$ be a non-zero entire function with $\rho(a_i) < 1$, where $\rho(g)$ denotes the order of an entire function $g$. Let $B_1$ denote the class of entire functions of the form

$$f = \sum_{i=1}^{n} a_i e^{g_i},$$

where $e^{g_i - g_j}$ is non-constant for $i \neq j$.

If all the $a_i$ are polynomials, then such $f$ is said to be in the class $B$. Clearly, $B$ is a proper subset of $B_1$.

Let $Z(g)$ be the zero set of an entire function $g$. In [2], by using H.Cartan’s theory of holomorphic curves, A. Eremenko and L. A. Rubel proved the following theorem.

Theorem A. Let $f \in B$. If $Z(f)$ is a subset of the positive real axis, except possibly finitely many points, then $f = p e^g$, where $p$ is a polynomial and $g$ is an entire function.

Therefore, it is natural to ask whether we can say something about the form of $f$ if $f \in B$ and $Z(f)$ is a subset of the real axis. By adapting some of the arguments used in [6] and Nevanlinna value distribution theory for functions meromorphic in a half plane, we can answer this question even for the case $f \in B_1$. In fact, we obtained the following results.

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Theorem 1. Let \( f \in B_1 \). If \( Z(f) \) is a subset of the real axis, except possibly finite points, then
\[
f(z) = e^{g(z)} \sum_{i=1}^{n} a_i(z) e^{b_i z},
\]
where \( b_i \in \mathbb{C} \), \( g \) and \( a_i(\neq 0) \) are entire functions with \( \rho(a_i) < 1 \).

Using theorem 1, we can generalize theorem A to the following theorem.

Theorem 2. Let \( f \in B_1 \). If \( Z(f) \) is a subset of the positive real axis, except possibly finite points, then \( f = ag^b \), where \( g, a \) are entire functions with \( \rho(a) < 1 \).

Our basic tool is J. Rossi’s half-plane version of Borel theorem. J. Rossi proved this version in [?] by using Tsuji’s half-plane version of Nevanlinna theory. Therefore, we shall start with the basic notations of Tsuji’s theory (c.f. [4],[7]), assuming the readers are familiar with the Nevanlinna Theory and its basic notations (c.f. [3]).

Let \( n_u(t, \infty) \) be the number of poles of \( f \) in \( \{ z \mid |z - \frac{i \pi}{2}| \leq \frac{t}{2}, |z| \geq 1 \} \), where \( f \) is meromorphic in the open upper half-plane. Define
\[
N_u(r, \infty) = N_u(r, f) = \int_1^r \frac{n_u(t, \infty)}{t^2} dt,
\]
\[
m_u(r, \infty) = m_u(r, f) = \frac{1}{2\pi} \int_{\text{arcsin}^{-1} r}^{\pi} \log^+ |f(r \sin \theta e^{i\theta})| \frac{d\theta}{r \sin^2 \theta},
\]
\[
N_u(r, a) = N_u(r, \frac{1}{f - a}), \quad m_u(r, a) = m_u(r, \frac{1}{f - a}) \quad (a \neq \infty) \quad \text{and}
\]
\[
T_u(r, f) = m_u(r, f) + N_u(r, f).
\]

Remark 1: We can also define \( m_r(r, f), N_r(r, f), T_r(r, f) \) for functions meromorphic in the open lower half-plane in the obvious way.

Lemma 1 [?]. Let \( f \) be meromorphic in \( \text{Im} z > 0 \). Define \( m_{\alpha, \beta}(r, f) = \frac{1}{2\pi} \int_{\alpha}^{\beta} \log^+ |f(re^{i\theta})| d\theta \).

Then
\[
\int_r^\infty \frac{m_{0, \pi}(t, f)}{t^3} dt \leq \int_r^\infty \frac{m_u(t, f)}{t^2} dt \leq \left( \int_r^\infty \frac{m_{\pi, 2\pi}(t, f)}{t^3} dt \right) \leq \int_r^\infty \frac{m_f(t, f)}{t^2} dt.
\]

Lemma 2 [?]. Let \( n \geq 2 \), \( S = \{ f_0, \ldots, f_n \} \) be a set of meromorphic functions such that any proper subset of \( S \) is linearly independent over \( \mathbb{C} \). If \( S \) is linearly dependent over \( \mathbb{C} \), then for all \( r \) except possibly on a set of finite measure,
\[
T_u(r) = O\{ \sum_{k=0}^{n} [N_u(r, 1/f_k) + N_u(r, f_k)] + \log T_u(r) + \log r \},
\]
where \( T_u(r) = \max\{ T_u(r, f_i/f_j) \mid 0 \leq i, j \leq n \} \).
Remark 2: If we replace $m_n(r, f), N_n(r, f)$ and $T_n(r, f)$ by the standard Nevanlinna functionals $m(r, f), N(r, f), T(r, f)$ in lemma 2, we shall obtain the original full-plane version of Borel theorem.

Lemma 3 [?]. Let $g_i$ be a transcendental entire function and $h$ be a non-zero entire function such that $T(r, h) = o(T(r, g_i))$ as $r \to \infty$, for $1 \leq i \leq n$. Suppose $\sum_{i=1}^{n} g_i(z) = h(z)$, then $\sum_{i=1}^{n} \delta(0, g_i) \leq n - 1$.

Lemma 4. For $n \geq 2$ and each $1 \leq i \leq n$, let $a_i$ denote a non-zero entire function with $p(a_i) < 1$ and $b_i$ be a non-zero complex number. Then, there exists a positive constant $A$ such that for sufficiently large $r$, $T(r, a_1(z) + \sum_{i=2}^{n} a_i(z)e^{b_i z}) \geq Ar$.

The proof of lemma 4. It is not difficult to prove for $n = 2$. Assume $n \geq 3$. Let $g(z) = a_1(z) + \sum_{i=2}^{n} a_i(z)e^{b_i z}$ and $G(z) = a_1(z) + \sum_{i=2}^{n} a_i(z)e^{b_i z}$. Then $T(r, G) = O(r)$ for large $r$. From $g = G + a_n e^{b_n z}$ and a simple calculation give

\[(a_n b_n + a_n' - a_n G'/G)e^{b_n z} = g' - gG'/G.\]

It is well-known that (for large $r$) $T(r, G'/G) = o(T(r, G))$ and $T(r, g') \leq ATr Br, g)$, where $A, B \geq 1$. Hence,

\[
\frac{1}{\pi} \ln |r| = \int T(r, G'/G) \leq \int T(r, g' - gG'/G) + T(r, a_n b_n + a_n' - a_n G'/G) + O(1) \leq CT(Br, g) + o(r)
\]

Therefore, for large $r$, $T(r, g) \geq Ar$ for some suitable positive constant $A$.

2 Proofs of Theorems.

The proof of theorem 1. $f \in B_1$ implies that $f = \sum a_i \exp g_i$, where $a_i(\neq 0)$ $g_i$ are entire functions with $T(r, a_i) = O(r^e)$ for some fixed positive $e < 1$.

If $n=1$, then we are done. For $n \geq 2$, Given that $\exp(g_i - g_j)$ is non-constant for $i \neq j$. From these and using the full-plane version of Borel theorem, we can show that the functions $f_i = a_i \exp g_i$ are linearly independent. Set $f_0 = f$, then the set $\{f_0, ..., f_n\}$ will satisfies the independence criteria of Lemma 2.

Given that $Z(f)$ is a subset of the real axis, except possibly finite points. Therefore, $N_n(r, 1/f_0) = O(\log r)$. For $1 \leq i \leq n$, we also have $N_n(r, 1/f_i) = O(r^e)$, since

\[
N_n(r, 1/f_i) = \int_1^r \frac{n(t, 1/a_i)}{t^2} dt \leq \int_1^r \frac{n(t, 1/a_i)}{t} dt = N(r, 1/a_i) + O(1) = O(r^e).
\]

It follows from lemma 2 that $T_n(r) = O(r^e)$ and hence $T_n(r, f_i/f_j) = O(r^e)$ for all $i, j$. Since $T_n(r, f_i/f_j) = N_n(r, f_i/f_j) + m_n(r, f_i/f_j)$, we also have $m_n(r, f_i/f_j) = O(r^e)$. Similarly, $m_q(r, f_i/f_j) = O(r^e)$. Now,

\[
T(t, f_i/f_j) = N(t, f_i/f_j) + m(t, f_i/f_j) = O(t^e) + m_{0,t}(t, f_i/f_j) + m_{2,t}(t, f_i/f_j).
\]
Then by Lemma 1, we have

$$T(r, f_i/f_j)O(1/r^2) \leq \int_r^\infty \frac{T(t, f_i/f_j)}{t^2} dt = O(r^{-\varepsilon}).$$

Consequently, $T(r, f_i/f_j) = O(r^{2-\varepsilon})$. This implies that the order of $\exp(g_k - g_j)$ is less than 2 and hence equal to one.

Now, $f = e^{\theta_1}(a_1 + \sum_{i=2}^n a_i e^{\theta_i - \theta_1})$, where $g_i - g_1$ is linear for $2 \leq i \leq n$. This also completes the proof.

The proof of theorem 2. Let $f \in B_1$ such that $Z(f)$ is a subset of the positive real axis, except possibly finite points. By Theorem 1, either (i) $f = \rho e^g$ or (ii) $f(z) = e^{g(z)}(a_1(z) + \sum_{i=2}^n a_i(z) e^{\theta_i z})$, where $g, a_i(\neq 0)$ are entire functions, $\rho(a_i) < 1$ and the $b_i$’s are non-zero complex numbers. We only need to consider case (ii).

Let $G(z) = a_1(z) + \sum_{i=2}^n a_i(z) e^{b_i z}$, $h = -a_1, g_1 = -G, g_i(z) = a_i(z) e^{b_i z}$ for $2 \leq i \leq n$. Then $Z(G) = Z(f), \sum_{i=1}^n g_i(z) = h(z)$ and $T(r, h) = o(T(r, g_i))$ as $r$ tends to infinity for $1 \leq i \leq n$. By Lemma 3, $\sum_{i=1}^n \delta(0, g_i) \leq n - 1$. Since $\delta(0, G) = 1$ for $i \geq 2$, it follows that $\delta(0, G) = \delta(0, g_1) = 0$.

Hence there exists an unbounded sequence $\{r_i\}$ such that $N(r_i, 0, G) \geq \frac{1}{2} T(r_i, G)$. By Lemma 4,

$$\int_{r_i}^\infty \frac{N(t, 0, G)}{t^2} dt \geq \int_{r_i}^\infty \frac{N(r_i, 0, G)}{t^2} dt \geq \int_{r_i}^\infty \frac{1}{2} T(r_i, G) dt \geq \int_{r_i}^\infty \frac{1}{2} \pi A r_i dt = \frac{1}{2} A > 0.$$

Therefore, $\int_0^\infty \frac{N(t, 0, G)}{t^2} dt$ does not converge and hence the genus of $G$ is at least one. Now, $G$ is an entire function of finite order with a genus at least one, which has at most finitely many non-positive zeros. By a result of A. Edrei and W. Fuchs [1], $\delta(0, G) > 0$, which is a contradiction. Hence $f$ must equal to the required form, $\rho e^g$.

**Remark 3:** It is obvious that Theorem A can also be derived from the present arguments by assuming that the coefficients $a_i(z)$s are polynomials in Theorem 2.

**References**


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