

Meromorphic solutions of higher order Briot–Bouquet differential equations

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Abstract

For differential equations $P(y^{(k)}, y) = 0$, where P is a polynomial, we prove that all meromorphic solutions having at least one pole are elliptic functions, possibly degenerate.

1. Introduction

According to a theorem of Weierstrass, meromorphic functions y in the complex plane \mathbb{C} that satisfy an algebraic addition theorem

$$Q(y(z + \zeta), y(z), y(\zeta)) \equiv 0, \quad \text{where } Q \neq 0 \text{ is a polynomial,} \quad (1.1)$$

are elliptic functions, possibly degenerate [17, 1].

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More precisely, let us denote by W the class of meromorphic functions in \mathbb{C} that consists of doubly periodic functions, rational functions and functions of the form $R(e^{az})$ where R is rational and $a \in \mathbb{C}$. Then each function $y \in W$ satisfies an identity of the form (1.1), and conversely, every meromorphic function¹ that satisfies such an identity belongs to W .

One way to prove this result is to differentiate (1.1) with respect to ζ and then set $\zeta = 0$. Then we obtain a Briot–Bouquet differential equation

$$P(y', y) = 0.$$

The fact that every meromorphic solution of such an equation belongs to W was known to Abel and Liouville, but probably it was stated for the first time in the work of Briot and Bouquet [5, 6].

Here we consider meromorphic solutions of higher order Briot–Bouquet equations

$$P(y^{(k)}, y) = 0, \quad \text{where } P \text{ is a polynomial.} \quad (1.2)$$

Picard [18] proved that for $k = 2$, all meromorphic solutions belong to the class W . This work was one of the first applications of the famous Picard’s theorems on omitted values.

In the end of 1970-s Hille [12, 13, 14, 15] considered meromorphic solutions of (1.2) for arbitrary k . The result of Picard was already forgotten, and Hille stated it as a conjecture. Then Bank and Kaufman [4] gave another proof of Picard’s theorem.

These investigations were continued in [8]. To state the main results from [8] we assume without loss of generality that the polynomial P in (1.2) is irreducible. Let F denote the compact Riemann surface defined by the equation

$$P(p, q) = 0. \quad (1.3)$$

Then every meromorphic solution y of (1.2) defines a holomorphic map $f : \mathbb{C} \rightarrow F$. According to another theorem of Picard, a Riemann surface which admits a non-constant holomorphic map from \mathbb{C} has to be of genus 0 or 1, ([19], see also [2]). The following theorems were proved in [8]:

THEOREM A. *If F is of genus 1, then every meromorphic solution of (1.2) is an elliptic function.*

THEOREM B. *If k is odd, then every meromorphic solution of (1.2) having at least one pole, belongs to the class W .*

The main result of the present paper is the extension of Theorem B to the case of even k .

THEOREM 1. *If y is a meromorphic solution of an equation (1.2) and y has at least one pole, then $y \in W$.*

This can be restated in the following way. *Let y be a meromorphic function in the plane which is not entire and does not belong to W . Then y and $y^{(k)}$ are algebraically independent.*

It is easy to see that for every function y of class W and every natural integer k there exists an equation of the form (1.2) which y satisfies.

¹ A “meromorphic function” in this paper means a function meromorphic in the complex plane, unless some other domain is specified. See [17, 20] for discussion of the equation (1.1) in more general classes of functions.

It is not true that all meromorphic solutions of higher order Briot–Bouquet equations belong to W , a simple counterexample is $y''' = y$. We don't know whether non-linear irreducible counterexamples exist.

In the process of proving of Theorem 1 we will establish an estimate of the degrees of possible meromorphic solutions in terms of the polynomials P . Here by degree of a function of class W we mean the degree of a rational function y , or the degree of R in $y(z) = R(e^{az})$, or the number of poles in the fundamental parallelogram of an elliptic function y . Thus our result permits in principle the determination of all meromorphic solutions having at least one pole of a given equation (1.2).

Our method of proof is based on the so-called “finiteness property” of certain autonomous differential equations: there are only finitely many formal Laurent series with a pole at zero that satisfy these equations. The idea seems to occur for the first time in [12, p. 274] but the argument given there contains a mistake. This mistake was corrected in [8]. Later the same method was applied in [7] and [10] to study meromorphic solutions of other differential equations.

2. Preliminaries

We will use the following refined version of Wiman–Valiron theory which is due to Bergweiler, Rippon and Stallard.

Let y be a meromorphic function and G a component of the set $\{z : |y(z)| > M\}$ which contains no poles (so G is unbounded). Set

$$M(r) = M(r, G, y) = \max\{|y(z)| : |z| = r, z \in G\},$$

and

$$a(r) = d \log M(r) / d \log r = r M'(r) / M(r). \quad (2.1)$$

This derivative exists for all r except possibly a discrete set. According to a theorem of Fuchs [11],

$$a(r) \rightarrow \infty, \quad r \rightarrow \infty,$$

unless the singularity of y at ∞ is a pole. For every $r > r_0 = \inf\{|z| : z \in G\}$ we choose a point z_r with the properties $|z| = r$, $|y(z_r)| = M(r)$.

THEOREM C. *For every $\tau > 1/2$, there exists a set $E \subset [r_0, +\infty)$ of finite logarithmic measure, such that for $r \in [r_0, \infty) \setminus E$, the disk*

$$D_r = \{z : |z - z_r| < r a^{-\tau}(r)\}$$

is contained in G and we have

$$y^{(k)}(z) = \left(\frac{a(r)}{z}\right)^k \left(\frac{z}{z_r}\right)^{a(r)} y(z)(1 + o(1)), \quad r \rightarrow \infty, \quad z \in D_r. \quad (2.2)$$

When y is entire, this is a classical theorem of Wiman. Wiman's proof used power series, so it cannot be extended to the situation when y is not entire. A more flexible proof, not using power series is due to Macintyre [16]; it applies, for example to functions analytic and unbounded in $|z| > r_0$. The final result stated above was recently established in [3].

3. Proof of Theorem 1

In what follows, we always assume that the polynomial P in (1.2) is irreducible.

To state a result of [8] which we will need, we introduce the following notation. Let A be the field of meromorphic functions on F . The elements of A can be represented as rational functions $R(p, q)$ whose denominators are co-prime with P . In particular, p and q in (1.3) are elements of A . For $\alpha \in A$ and a point $x \in F$, we denote by $\text{ord}_x \alpha$ the order of α at the point x . Thus if $\alpha(x) = 0$ then $\text{ord}_x \alpha$ is the multiplicity of the zero x of α , if $\alpha(x) = \infty$ then $-\text{ord}_x \alpha$ is the multiplicity of the pole, and $\text{ord}_x \alpha = 0$ at all other points $x \in F$.

Let $I \subset F$ be the set of poles of q . For $x \in I$ we set $\kappa(x) = \text{ord}_x p / \text{ord}_x q$.

THEOREM D. *Suppose that an irreducible equation (1.2) has a transcendental meromorphic solution y . Let $f : \mathbb{C} \rightarrow F$ be the holomorphic map defined by $z \mapsto (y^{(k)}(z), y(z))$. Then:*

- a) *Every pole of p belongs to I .*
- b) *For every $x \in I$, the number $\kappa(x)$ is either 1 or $1 + k/n$, where n is a positive integer.*
- c) *If $\kappa(x) = 1 + k/n$ for some $x \in I$, then the equation $f(z) = x$ has infinitely many solutions, and all these solutions are poles of order n of y .*
- d) *If $\kappa(x) = 1$ for some $x \in I$, then the equation $f(z) = x$ has no solutions.*

Picard's theorem on omitted values implies that $\kappa(x) = 1$ can happen for at most two points $x \in I$. For the convenience of the reader we include a proof of Theorem D in the Appendix.

The numbers $\kappa(x)$ can be easily determined from the Newton polygon of P . Thus Theorem D gives several effective necessary conditions for the equation (1.2) to have meromorphic or entire solutions.

Remark. The proof of Theorem D in [8] uses Theorem C which was stated in [8] but not proved. One can also give an alternative proof of Theorem D, using Nevanlinna theory instead of Theorem C, by the arguments similar to those in [9].

LEMMA 1. *Suppose that y is a meromorphic solution of (1.2). If $\kappa(x) = 1$ for some $x \in I$ then y has order one, normal type.*

Proof. In view of Theorem A and Theorem D, d), we conclude that the genus of F is zero. Therefore, we can find $t = R(p, q)$ in A which has a single simple pole at x . Then $w = R(y^{(k)}, y)$ is an entire function by Theorem D, d). As t has a simple pole at x , the element $1/t \in A$ is a local parameter at x , and in a neighborhood of x we have

$$q = at^m + \dots \quad \text{and} \quad p = bt^m + \dots,$$

where $-m = \text{ord}_x p = \text{ord}_x q$ as $\kappa(x) = 1$, and the dots stand for the terms of degree smaller than m . Substituting $p = y^{(k)}$ and $q = y$ and differentiating the first equation k times we obtain for w a differential equation of the form

$$\frac{d^k}{dz^k} w^m + \dots = (b/a)w^m, \tag{3.1}$$

where the dots stand for the terms of degree smaller than m . Now we use a standard argument of Wiman–Valiron theory. Applying Theorem C to the entire function w^m , with $G = \mathbb{C}$ and $z = z_r$, we compare the asymptotic relations (2.2) and (3.1) to conclude

that $a(r) \sim cr$, where $c \neq 0$ is a constant. This implies $\log M(r) \sim cr$, which means that w is of order 1, normal type. So y is also of order 1, normal type, because w and y satisfy a polynomial relation of the form $P(y, w) = 0$, where P is a polynomial with constant coefficients.

LEMMA 2. *Suppose that y is a meromorphic solution of (1.2). If $\kappa(x_1) = \kappa(x_2) = 1$ for two different points x_1 and x_2 in I , then y is a rational function of e^{az} , where $a \in \mathbb{C}$.*

Proof. As in the previous lemma, the genus of F is zero. Let $t = R(p, q)$ be a function in A with a single simple pole at x_1 and a single simple zero at x_2 . Then $w = R(y^{(k)}, y)$ is an entire function of order 1, normal type (by Lemma 1) omitting 0 and ∞ (by Theorem D, d). So $w(z) = e^{az}$ for some $a \in \mathbb{C}$. Since t is a generator of A , by Lüroth's theorem, both p and q are rational functions of t and the lemma follows.

LEMMA 3. *Suppose that k is even, the Riemann surface F is of genus zero, y is a non-constant meromorphic solution of (1.2), and $\kappa(x) = 1$ for at most one point $x \in I$. Then the Abelian differential pdq is exact, that is $pdq = ds$ for some $s \in A$.*

Proof. It is sufficient to show that under the assumptions of Lemma 3, the integral of pdq over every closed path in F is zero. As F is of genus zero, we only have to consider residues of pdq . By Theorem D, a), all poles of our differential belong to the set I .

Consider first a point $x \in I$ with $\kappa(x) = 1 + k/n$. By Theorem D, c), we have a meromorphic solution y with a pole of order n at zero, such that the corresponding function f has the property $f(0) = x$. In a neighborhood of x we have a Puiseux expansion

$$pdq = \sum_{j=J}^{\infty} c_j q^{-j/m} dq$$

with some positive integer m . We substitute $p = y^{(k)}$, $q = y$ and obtain

$$y^{(k)}y' = \sum_{j \neq -m} c_j y^{-j/m} y' + r y^{-1} y', \quad (3.2)$$

where $r = c_m$ is the residue of pdq at x . Now we notice that for even k ,

$$y^{(k)}y' = \frac{d}{dz} \left\{ y^{(k-1)}y' - y^{(k-2)}y'' + \dots \pm \frac{1}{2}(y^{(k/2)})^2 \right\}. \quad (3.3)$$

Using this, we integrate (3.2) over a small circle around 0 in the z -plane, described m times anticlockwise. We obtain that $2\pi i m r = 0$, so $r = 0$.

Now we consider a point $x \in I$ with $\kappa(x) = 1$. By the assumptions of the lemma, there is at most one such point. Then the residue of pdq at x is zero because the sum of all residues of a differential on a compact Riemann surface is zero. This proves the lemma.

Using (3.3) and Lemma 3, if the assumptions of Lemma 3 are satisfied, we can rewrite our differential equation

$$y^{(k)} = p(y) \quad (3.4)$$

as

$$y^{(k-1)}y' - y^{(k-2)}y'' + \dots \pm \frac{1}{2}(y^{(k/2)})^2 = s(y) + c, \quad (3.5)$$

where $s \in A$ is an integral of the exact differential pdq , and c is a constant that depends on the particular solution y . We have the relation $p(y) = ds/dy$.

LEMMA 4. *For a given differential equation of the form (3.5), there are only finitely many formal Laurent series with a pole at zero that satisfy the equation.*

Proof. By making a linear change of the independent variable, we may assume that

$$s(y) = y^{2+k/n} + \dots$$

Then

$$p(y) = (2 + k/n)y^{1+k/n} + \dots$$

Now we substitute a Laurent series with undetermined coefficients

$$y(z) = \sum_{j=0}^{\infty} c_j z^{-n+j} \quad (3.6)$$

to the equation (3.4), which is a consequence of (3.5). With even k we have:

$$\begin{aligned} y^{(k)}(z) &= \frac{(k+n-1)!}{(n-1)!} c_0 z^{-n-k} + \frac{(k+n-2)!}{(n-2)!} c_1 z^{-n-k-1} \\ &+ \dots + k! c_{n-1} z^{-k-1} \\ &+ k! c_{n+k} + \frac{(k+1)!}{1!} c_{n+k+1} z + \frac{(k+1)!}{2!} c_{n+k+2} z^2 + \dots; \end{aligned}$$

and

$$\begin{aligned} y^{1+k/n}(z) &= z^{-k-n} \left[c_0^{1+k/n} + \left((1+k/n)c_0^{k/n} c_1 + (\dots)_1 \right) z \right. \\ &+ \left((1+k/n)c_0^{k/n} c_2 + (\dots)_2 \right) z^2 + \dots \\ &\left. + \left((1+k/n)c_0^{k/n} c_j + (\dots)_j \right) z^j + \dots \right]. \end{aligned}$$

In the last formula, the symbol $(\dots)_j$ stands for a finite sum of products of the coefficients of the series (3.6) which contain no coefficients c_i with $i \geq j$. Substituting to (3.4) and comparing the coefficients at z^{-k-n} we obtain

$$\frac{(k+n-1)!}{(n-1)!} c_0 = (2+k/n)c_0^{1+k/n}.$$

This equation has finitely many non-zero roots c_0 . We have

$$(2+k/n)c_0^{k/n} = \frac{(k+n-1)!}{(n-1)!}. \quad (3.7)$$

Further we obtain

$$\frac{(k+n-2)!}{(n-1)!} c_1 = (2+k/n)c_0^{k/n} (1+k/n)c_1 + (\dots)_1. \quad (3.8)$$

Substituting here the value of $(2+k/n)c_0^{k/n}$ from (3.7), we see that the coefficient at c_1 is different from zero, because

$$\frac{(k+n-2)!}{(n-2)!} \neq \frac{(k+n-1)!}{(n-1)!} \frac{k+n}{n}.$$

Thus c_1 is uniquely determined from (3.8). The situation is analogous for all coefficients

c_j with $j < n + k$. These coefficients are uniquely determined from the equation (3.4) once c_0 is chosen.

Now we consider the coefficients c_{n+k+j} with $j \geq 0$. We have

$$\frac{(k+j)!}{j!} c_{n+k+j} = (2+k/n)c_0^{k/n} \frac{n+k}{n} c_{n+k+j} + (\dots)_{n+k+j}.$$

Again we substitute the value of $(2+k/n)c_0^{k/n}$ from (3.7) and conclude that the coefficient at c_{n+k+j} equals

$$\frac{(k+j)!}{j!} - \frac{(k+n)!}{n!}.$$

This coefficient is zero for a single value of j , namely $j = n$. Thus c_{2n+k} cannot be determined from the equation (3.4), but once c_0 and c_{2n+k} are chosen, the rest of the coefficients of the series (3.6) are determined uniquely.

To determine c_{2n+k} we invoke the equation (3.5):

$$y^{(k-1)}y' - y^{(k-2)}y'' + \dots \pm \frac{1}{2}(y^{(k/2)})^2 = y^{2+k/n} + \dots, \quad (3.9)$$

where the dots stand for the terms of lower degrees. We have

$$\begin{aligned} y'(z) &= -nc_0z^{-n-1} + \dots + c_{2n+k}(n+k)z^{n+k-1} + \dots, \\ y'' &= n(n+1)c_0z^{-n-2} + \dots + c_{2n+k}(n+k)(n+k-1)z^{n+k-2} + \dots, \\ &\dots \dots, \\ y^{(k-1)} &= -n(n+1)\dots(n+k-2)c_0z^{-n-k+1} + \dots \\ &\quad + c_{2n+k}(n+k)(n+k-1)\dots(n+2)z^{n+1} + \dots \end{aligned}$$

Substituting this to our equation (3.9) we write the condition that the constant terms in both sides of (3.9) are equal. This condition is a polynomial equation in c, c_0, \dots, c_{2n+k} (it is linear with respect to c_{2n+k}) and the coefficient at c_{2n+k} in this equation equals

$$c_0 \sum_{m=0}^{k-1} \frac{(n+m)!(n+k)!}{(n+m+1)!(n-1)!}.$$

This expression is not zero because each term of the sum is positive. Thus c_{2n+k} is determined uniquely, and this completes the proof of the lemma.

Remark. It follows from this proof that the only meromorphic solutions of the differential equations

$$y^{(k)} = y^m$$

are exponential polynomials when $m = 1$ and functions $c(z - z_0)^{-n}$ where $m = 1 + k/n$, $z_0 \in \mathbb{C}$ and c is an appropriate constant.

The rest of the proof of Theorem 1 is a repetition of the argument from [8].

By Theorems A and B, we may assume that F is of genus zero, and k is even. In view of Lemmas 2 and 3, it is enough to consider the case that the differential pdq is exact. Then every solution of (1.2) also satisfies (3.5) with some constant c .

Assume that y is a transcendental meromorphic solution of (3.5), having at least one pole. By Theorem D, d), c), y has infinitely many poles z_j , $j = 1, 2, 3, \dots$. The functions $y(z - z_j)$ satisfy the assumptions of Lemma 4, therefore some of them are equal. We

conclude that y is a periodic function. By making a linear change of the independent variable we may assume that the smallest period is $2\pi i$.

Consider the strip $D = \{z : 0 \leq \Im z < 2\pi\}$.

Case 1. y has infinitely many poles in D . Applying Lemma 4 again, we conclude that y has a period in D , so y is doubly periodic.

Case 2. y is bounded in $D \cap \{z : |\Re z| > C\}$ for some $C > 0$. Since y is $2\pi i$ -periodic, we have $y(z) = R(e^z)$ where R is meromorphic in \mathbb{C}^* . As R is bounded in some neighborhoods of 0 and ∞ , we conclude that R is rational.

Case 3. y has finitely many poles in D and is unbounded in $D \cap \{z : |\Re z| > C\}$ for every $C > 0$. As y is $2\pi i$ -periodic, we write $y = R(e^z)$ where R is meromorphic in \mathbb{C}^* . Now R has finitely many poles and is unbounded either in a neighborhood of 0 or in a neighborhood of ∞ . Suppose that it is unbounded in a neighborhood of ∞ . Then the set $\{z : |R(z)| > M\}$, where M is large enough has an unbounded component G containing no poles of R . On this component G , the function R satisfies a differential equation

$$\sum_{m=1}^k \binom{k}{m} w^m \frac{d^m R}{dw^m} = (c + o(1))R^\kappa,$$

where c is some constant and $\kappa = 1$ or κ is one of the numbers $1 + k/n$ from Theorem D. Applying Theorem C in G as we did in the proof of Lemma 1, we obtain that $\kappa = 1$ and that R has a pole at infinity. Similar argument works for the singularity at 0, so R is rational, and this completes the proof.

4. Appendix

Proof of Theorem D. Statement a) is a special case of [9, Th. 10], but we give a simple independent proof using Theorem C. Proving it by contradiction, suppose that p has a pole at a point $x \in F$ such that $q(x) = b \in \mathbb{C}$. Let $D_\epsilon \subset \mathbb{C}$ be a disk of radius ϵ centered at b , and $V_\epsilon \subset F$ a component of $q^{-1}(D_\epsilon)$ containing x . We assume that the disk D_ϵ is so small that V_ϵ contains no other poles of p , except the pole at x . Let y be a meromorphic solution of our equation (1.2) and consider the map $f : \mathbb{C} \rightarrow F$ given by $f(z) = (y^{(k)}(z), y(z))$. The image of this map is dense in F and the point x is evidently omitted by f . Let $G_\epsilon \subset \mathbb{C}$ be a component of the preimage $f^{-1}(D_\epsilon)$. Consider the meromorphic function $w = 1/(y - a)$. It is holomorphic and unbounded in G_ϵ , and $|w(z)| = 1/\epsilon$ for $z \in \partial G_\epsilon$. We conclude that G_ϵ is unbounded. Now we apply Theorem C to w in G_ϵ .

Set $M(r) = \max\{|w(z)| : |z| = r, z \in G_\epsilon\}$ and let $a(r)$ be defined as in (2.1). For any $r > r_0 = \inf\{|z| : z \in G_\epsilon\}$, we choose a point z_r with $|z| = r$ and $|w(z_r)| = M(r)$. By Theorem C, we have

$$|w^{(j)}(z_r)| = \left(\frac{a(r)}{r}\right)^j |w(z_r)|(1 + o(1)) = \frac{a(r)^j}{r^j} M(r)(1 + o(1)) \quad (4.1)$$

where $r \rightarrow \infty$ outside a set of finite logarithmic measure.

From Lemma 6.10 of [3], we have for every $\beta > 0$,

$$(a(r))^\beta = o(M(r)), \quad (4.2)$$

as $r \rightarrow \infty$ outside a set of finite logarithmic measure.

Differentiating the equation $y = 1/w + a$ we obtain

$$y^{(k)} = \frac{1}{w} Q \left(\frac{w'}{w}, \frac{w''}{w}, \dots, \frac{w^{(k)}}{w} \right), \quad (4.3)$$

where Q is a polynomial. On the other hand, from the Puiseux expansion at the point x we obtain

$$y^{(k)} = (c + o(1))w^\alpha, \quad w \rightarrow \infty, \quad (4.4)$$

where $c \neq 0$ is a constant and $\alpha > 0$. Combining (4.3) and (4.4) we obtain

$$Q \left(\frac{w'}{w}, \dots, \frac{w^{(k)}}{w} \right) = (c + o(1))w^{1+\alpha}.$$

Inserting to this asymptotic relation $z = z_r$ and using (4.1) and (4.2) we obtain a contradiction which proves a).

Consider now a point $x \in I$. From the Puiseux expansion we obtain

$$y^{(k)} = (c + o(1))y^{\kappa(x)}, \quad y \rightarrow \infty. \quad (4.5)$$

If x has a preimage under the map f , then this preimage is a pole z_0 of y . If this pole is of order n we have $y(z) \sim c_1(z - z_0)^{-n}$ and $y^{(k)}(z) \sim c_2(z - z_0)^{-n-k}$ as $z \rightarrow z_0$. Substituting to (4.5) we conclude that $\kappa(x) = 1 + k/n$. Thus if x has at least one preimage under f then $\kappa(x) = 1 + k/n$ with a positive integer n , and every preimage of x is a pole of order n of y . This implies d).

Now suppose that a point $x \in I$ has only finitely many preimages. Let $U_\epsilon = \{z \in \mathbb{C} : |z| > 1/\epsilon\}$ be a neighborhood of infinity, and $V_\epsilon \subset F$ a component of the preimage $q^{-1}(U_\epsilon)$. We may assume that $\epsilon > 0$ is so small that V_ϵ does not contain other poles of q except x . Let G_ϵ be a component of the preimage $f^{-1}(V_\epsilon)$. If G_ϵ is bounded then $f : G_\epsilon \rightarrow U_\epsilon$ is a ramified covering of a finite degree, and f takes the value x somewhere in G . As we assume that f is transcendental but x has only finitely many preimages, there should exist an unbounded component G_ϵ . Choosing a smaller ϵ if necessary, we achieve that this unbounded component G_ϵ contains no f -preimages of x . Then y is a holomorphic function in G_ϵ , $|y(z)| = 1/\epsilon$, $z \in \partial G_\epsilon$, and y is unbounded in G_ϵ . Applying Theorem C to the function y in G_ϵ we obtain the asymptotic relation (2.2). Putting $z = z_r$ in this relation, taking (4.2) into account, and comparing with (4.5) we conclude that $\kappa = 1$ in (4.5). This implies c). Thus in any case $\kappa = 1 + k/n$ or $\kappa = 1$, which proves b).

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