IMPRIMITIVE PARAMETRIZATION OF ANALYTIC CURVES AND FACTORIZATIONS OF ENTIRE FUNCTIONS

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Abstract

Let $f(z)$ and $g(z)$ be transcendental entire functions. W.H.J. Fuchs and G.D. Song proved that if $(f(z), g(z))$ parametrizes some complex algebraic curve, then f and g must have a transcendental common right factor. In this paper, we shall prove this result by a different method which also allows us to prove a similar result for some transcendental curves. We then use this result to solve some factorization problems of entire functions.

1. Introduction and Main Results.

Let $f(z), g(z)$ be entire functions of one complex variable and $\Phi(x, y)$ be a complex polynomial in both x and y. The pair $(f(z), g(z))$ is called a parametrization of a complex algebraic curve defined by $\Phi(x, y) = 0$ if $\Phi(f(z), g(z)) \equiv 0$ on the complex plane. The parametrization $(f(z), g(z))$ is called imprimitive if there exists a non-linear entire function $h(z)$ such that $f(z) = f_1(h(z))$ and $g(z) = g_1(h(z))$, where f_1, g_1 are analytic on the image of h which will be denoted by Im(h). By the Little Picard Theorem, Im(h) can either be $\mathbb{C}$ or $\mathbb{C} - \{a\}$ for some complex number a. When both f_1, g_1 are entire, we call h a common right factor of f and g. In [5], W.H.J. Fuchs and G.D. Song proved that if both f, g are transcendental and $(f(z), g(z))$ parametrizes some complex algebraic curve, then the parametrization must be imprimitive. In fact, they proved something more.

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Theorem A. Let $f(z), g(z)$ be two transcendental entire functions and $\Phi(x, y)$ be a non-zero complex polymonial such that $\Phi(f(z), g(z)) \equiv 0$ on the complex plane, then there exists a transcendental entire function $h(z)$ such that $f(z) = f_1(h(z))$ and $g(z) = g_1(h(z))$, where f_1, g_1 are both rational functions with at most one pole.

Example 1. Let $f(z) = \sin z$, $g(z) = \cos z$ and $\Phi(x, y) = x^2 + y^2 - 1$, then $\Phi(f(z), g(z)) \equiv 0$ on $\mathbb{C}$. In this case, we can take $h(z) = e^{iz}, f_1(z) = \frac{i}{2}(z - z^{-1})$ and $g_1(z) = \frac{1}{2}(z + z^{-1})$ such that $f(z) = f_1(h(z))$ and $g(z) = g_1(h(z))$.

Theorem A has a lot of applications to the factorization and sharing value problems of entire functions. Notice that W.H.J. Fuchs and G.D. Song only considered the case $\Phi(x, y) = p(x) - q(y)$, where p, q are polynomials. However, their proof works for general Φ. Their proof is based on the following result of Picard ([11]), which can also be proved by using the Nevanlinna Theory (see [7], p.232).

Theorem B. Let $\Phi(x, y) \in \mathbb{C}[x, y]$ be an irreducible polynomial and $f(z), g(z)$ be non-constant meromorphic functions. Suppose that

$$\Phi(f(z), g(z)) \equiv 0$$

on the complex plane, then the complex algebraic curve defined by $\Phi(x, y) = 0$ has genus less than or equal to one.

In this paper, we shall give a different proof of Theorem A. Our method depends on a result of Grauert [6] about complex analytic equivalence relations. Using this method, we can prove result similar to Theorem A for some transcendental curve. In fact, we have

Theorem 1 . *Let $n \geq 1$ and $\Phi(x, y) = \sum_{i=0}^{n} a_i(x) y^i$ be a polynomial in y with entire functions $a_i(x)$ as coefficients such that $a_n \not\equiv 0$. Suppose that $f(z)$ and $g(z)$ are transcendental entire functions such that*

$$\Phi(f(z), g(z)) = \sum_{i=0}^{n} a_i(f(z)) g(z)^i \equiv 0$$

on the complex plane. Then there exists a transcendental entire fuction h such that $f(z) = f_1(h(z))$ and $g(z) = g_1(h(z))$, where f_1, g_1 are analytic on $\mathrm{Im(h)}$.

Example 2. Let $f(z) = \cos z$, $g(z) = \sin z e^{\cos z}$ and $\Phi(x, y) = (1 - x^2)e^{2x} - y^2$, then $\Phi(f(z), g(z)) \equiv 0$ because $g(z)^2 = \sin^2 z e^{2\cos z} = ((1-w^2)e^{2w}) \circ \cos z = ((1-w^2)e^{2w}) \circ f(z)$. In this case, we can take $h(z) = e^{iz}$, $f_1(z) = \frac{1}{2}(z + z^{-1})$, $f_2(z) = \frac{i}{2}(z - z^{-1})e^{\frac{1}{2}(z+z^{-1})}$ such that $f(z) = f_1(h(z))$, $g(z) = g_1(h(z))$ and f_1, f_2 are analytic on $\mathbb{C} - \{0\}$. Note that f_2 has an essential singularity at 0 .

To prove Theorem 1, we first establish in Section 2 a main lemma which gives a sufficient condition for the existence of a non-linear generalized common right factor of two entire functions. We then further develop some criteria on the existence of a transcendental entire common right factor in Section 3. These results in turn allow us to prove Theorem 1 and A. To illustrate the usefulness of the main lemma, let us state the follwing result which will be deduced from the main lemma in Section 2.

Theorem 2 . *Let $p(z)$ and $q(z)$ be two complex polynomials. If $p(x) - p(y)$ and $q(x) - q(y)$ have a common factor of degree $d \geq 2$ in $\mathbb{C}[x, y]$, then $p(z)$ and $q(z)$ have a common right factor of degree greater than or equal to d. Furthermore, $p(x) - p(y)$ and $q(x) - q(y)$ have a common factor of maximal degree d in $\mathbb{C}[x, y]$ if and only if $p(z)$ and $q(z)$ have a greatest common right factor of degree d.*

In Section 4, we shall mainly concern with the applications of those criteria to factorization problems of entire functions. We shall generalize some known results which were proved by several different argruments before. In fact, the main purpose of this paper is to provide a more systematic way of solving factorization problems of entire functions. Our method depends very much on the ideas and results in A. Eremenko and L. Rubel's paper [3]. Especially, most of the contents of the following section is taken from [3].

2. The Main Lemma.

In this section, we shall state and prove the main lemma (Lemma 2) which gives a sufficient condition on the existence of a non-linear entire common right factor for any two non-constant entire functions. Its proof is based on the following result of Grauert ([6]) on complex analytic equivalence relations. Throughout this paper, X will denote either $\mathbb{C}$ or $\mathbb{C} - \{a\}$, where a is a complex number.

Theorem C. Let R be any equivalence relation on X whose graph $G = \{(x, y) \in X \times X | xRy\}$ is a complex analytic subset of $X \times X$ containing no vertical or horizontal lines (i.e subsets of the form $\{x\} \times X$ or $X \times \{y\}$). Suppose that G is of pure dimension one (i.e. G is everywhere of the same dimension one). Then, there exists a holomorphic map h from $\mathbb{C}$ onto a Riemann surface S such that xRy if and only if $h(x) = h(y)$.

In the Appendix A of [3], A. Eremenko and L. Rubel gave a more elementary and direct proof when $X = \mathbb{C}$. The same proof also works for the case $X = \mathbb{C} - \{a\}$. The basic terminology and properties of complex analytic sets can be found in [1].

Definition 1 . *Let f be an analytic function on X. We say that h is a generalized right factor (denoted by $h \leq f$) of f if h is a holomorphic map from X to a Riemann surface S and there exists a holomorphic map f_1 from S to $\mathbb{C}$ such that $f = f_1 \circ h$.*

Note that the word "map" here always means a mapping between two Riemann surfaces while "function" means a mapping with its range in the complex plane.

Definition 2 . *Let f and g be entire functions. An entire function h is a greatest common right factor of f and g if*

(i) h is a right factor of both f and g.

(ii) every right factor of f and g must be a right factor of h.

Note that if h is a greatest common right factor, so is $L \circ h$ for any linear function L. It is proved in [3] that a greatest common right factor of f and g always exists and is unique up to a composition of a linear function. A non-constant holomorphic map k from X onto a Riemann surface S can induce an equivalence relation R in X defined by xRy if and only if $k(x) = k(y)$. Let $G_k = \{(x, y) \in X \times X| \quad k(x) = k(y)\}$, then G_k is a complex analytic set of pure dimension one which does not contain any vertical or horizontal line because k is non-constant. Such G_k is called the graph of equivalence relation induced by k. Let G_h and G_k be the graphs of the equivalence relation induced by surjective holomorphic maps $h : X \to S_1$ and $k : X \to S_2$ respectively, where S_1, S_2 are two Riemann surfaces. The following lemma states the relation between h and g when G_h is a subset of G_k (see [3], p.338).

Lemma 1 . *G_h is a subset of G_k if and only if $h \leq k$.*

Proof of Lemma 1. It is clear that if $h \leq k$, then $G_h \subseteq G_k$. Now suppose that $G_h \subseteq G_k$, this actually means that for any $x, y \in X$, $h(x) = h(y)$ implies $k(x) = k(y)$. Hence, the function $f : S_1 \to S_2$ defined by $f(s) = k(h^{-1}(s))$ is single valued. Cleary, $k(x) = f \circ h(x)$ on X. It remains to show that f is holomorphic on S_1. Take any $s \in \mathrm{Im(h)} = S_1$, let w be any point in $h^{-1}(s)$. We can always choose suitable local charts (α_i, U_i) of w and (β_j, V_j) of s such that $\beta_j \circ h \circ \alpha_i^{-1}(z) = z^n$ in a neighborhood of zero for some positive integer n. Let (γ_k, W_k) be a local chart of $k(w)$. Then $\gamma_k \circ k \circ \alpha_i^{-1}(z)$ is analytic near zero. Now, $\gamma_k \circ k \circ h^{-1} \circ \beta_j^{-1}(z) = \gamma_k \circ k \circ \alpha_i^{-1} \circ \alpha_i \circ h^{-1} \circ \beta_j^{-1}(z) = \gamma_k \circ k \circ \alpha_i^{-1} \circ z^{1/n}$ which is single valued, analytic in a deleted neighborbood of zero and continuous at zero. Hence, it is analytic at zero and f is holomorphic at s.

The following lemma is very crucial.

Lemma 2 . *Let f, g be two analytic functions on X. For $i = 1, ..., k$, $k \geq 2$, let $S_i = \{z_{in}\}_{n \in \mathbb{N}}$ be a sequence of distinct complex numbers with limit point z_i. Suppose that all the limit points z_i are distinct and for all $n \in \mathbb{N}$,*

$$(*) \begin{cases} f(z_{1n}) = f(z_{2n}) = \cdots = f(z_{kn}) \\ g(z_{1n}) = g(z_{2n}) = \cdots = g(z_{kn}). \end{cases}$$

Then there exists a holomorphic function $h(z) : X \to \mathbb{C}$ (depends only on f and g) satisfying $h \leq f$, $h \leq g$ and $h(z_1) = h(z_i)$ for all $2 \leq i \leq k$.

The proof of Lemma 2 is very similar to that of Theorem 1.1 in A. Eremenko and L. Rubel's paper [3]. For completeness, we sketch the proof below.

Proof of Lemma 2. Let G_f and G_g be the graphs of the equivalence relation induced by f and g respectively. Then $G_f \cap G_g$ is a complex analytic set (see [1], p.62), but may not have pure dimension one, so we consider its derived set $H = (G_f \cap G_g)'$ (i.e. the set of limit points). Then H is a pure dimension one complex analytic set and does not contain any vertical or horizontal line. The non-trival fact that H is still a graph of some equivalence relation is proved in ([3], p.338). By Theorem C, we conclude that H is a graph of the equivalence relation induced by some holomorphic map h from $\mathbb{C}$ to some Riemann surface S. Clearly, h depends only on f and g. Now H is a subset of both G_f and G_g, so from Lemma 1, we have $h \leq f$ and $h \leq g$. From the assumption $(*)$ of the lemma, we have $(z_{1n}, z_{jn}) \in G_f \cap G_g$ for all $2 \leq j \leq k$ and $n \in \mathbb{N}$. Therefore, for all $2 \leq j \leq k$, $(z_1, z_j) \in H = (G_f \cap G_g)'$ and hence $h(z_1) = h(z_j)$.

From the Uniformization Theorem, we know that S is conformally equivalent to S_0/G, where G is a fix-point free discrete group of isometries of S_0 and S_0 is any one of $\mathbb{C}_\infty, \mathbb{C}$ or the unit disk Δ. If $X = \mathbb{C}$, we claim that S_0 cannot be Δ. For otherwise, S will have a holomorphic universal covering from $S_0 = \Delta$. Since $X = \mathbb{C}$ is simply connected, h can be lifted to a holomorphic map from $\mathbb{C}$ to Δ, which must be constant by Liouville's Theorem. Hence, h must also be a constant which is a contradiction. For $X = \mathbb{C} - \{a\}$, we can get the same contradiction by considering $h(e^z + a)$ instead of h. Therefore, S_0 is either $\mathbb{C}_\infty$ or $\mathbb{C}$. From this fact, it is not difficult to show that S can only be one of the following: Riemann sphere, complex plane, punctured plane or torus (see [4], p.193). Since $h \leq f$ and $h \leq g$, there exist holomorphic maps h_1 and h_2 from S to $\mathbb{C}$ such that $f = h_1 \circ h$ and $g = h_2 \circ h$. If S is a sphere or a torus, then S is compact. As h_1 is holomorphic on S, $h_1$and hence h must be a constant which is a contradiction. Therefore, S is the whole plane or punctured plane and h is an entire function on $\mathbb{C}$. This completes the proof of Lemma 2.

Proof of Theorem 2. Let $R(x, y) \in \mathbb{C}[x, y]$ (with $\deg R(x, y) = n \geq 2$) be a common factor of $p(x) - p(y)$ and $q(x) - q(y)$. Assume without loss of generality that $\deg R(x, y) = \deg_y R(x, y) = n \geq 2$. We claim that there are only finitely many $(a, b) \in \mathbb{C}^2$ such that

$$R(a, b) = 0,\ R_y(a, b) = 0. \tag{1}$$

Since $p(x) - p(y) = R(x, y)S(x, y)$ for some $S(x, y) \in \mathbb{C}[x, y]$, $-p'(y) = R_y(x, y)S(x, y) + R(x, y)S_y(x, y)$. So if (a, b) satisfies (1), then $p(a) = p(b)$ and $p'(b) = 0$, which in turn can only be satisfied by finitely many (a, b).

Now, we can choose some a such that $R(a, y) = 0$ has n distinct solutions b_i with $R_y(a, b_i) \neq 0$. By the Implicit Function Theorem, for each $1 \leq i \leq n$, there exists a unique analytic function $w_i(x)$ on an open set A_i of a such that $R(x, w_i(x)) \equiv 0$ on A_i. Hence, we have on an open neighborhood $A = \cap_{i=1}^{n} A_i \neq \phi$ of $x = a$,

$$\begin{cases} p(x) = p(w_1(x)) = \cdots = p(w_n(x)) \\ q(x) = q(w_1(x)) = \cdots = q(w_n(x)). \end{cases}$$

Take a sequence of distinct terms, $\{a_k\}_{k \in \mathbb{N}}$ such that $\lim_{k \to \infty} a_k = a$. Define $z_{ik} = w_i(a_k)$ and $S_i = \{z_{ik}\}_{k \in \mathbb{N}}$. According to the Implicit Function Theorem, all z_{ik} are distinct and $\lim_{k \to \infty} z_{ik} = \lim_{k \to \infty} w_i(a_k) = b_i$, hence the sequences S_i satisfy the requirements in Lemma 2. Therefore, there exists an entire function $h(z)$ which is a generalized right factor of both p and q. Note that h must be a polynomial, for otherwise p and q will be transcendental. Since, all b_i are dinstinct and

$$h(b_1) = \cdots = h(b_n),$$

h is a polynomial of degree at least $n \geq 2$ and we have proved the first part of Theorem 2.

We first remark that for any non-constant polynomial $k(z)$, we always have $k(x) - k(y) = (x - y)K(x, y)$ for some $K(x, y) \in \mathbb{C}[x, y]$. So if $p(z)$ and $q(z)$ have a common right factor $h(z)$ which is of degree n, then $p(x) - p(y) = \{h(x) - h(y)\}P(x, y)$ and $q(x) - q(y) = \{h(x) - h(y)\}Q(x, y)$ has a common factor $h(x) - h(y)$ which is of degree n. Now, let $R(x, y) \in \mathbb{C}[x, y]$ be a common factor of $p(x) - p(y)$ and $q(x) - q(y)$ with maximal degree. Then for any common right factor h of p and q, $h(x) - h(y)$ divides $R(x, y)$.

If $n = \deg R(x, y) = 1$, then it follows from the above argument that any common right factor of $p(z)$ and $q(z)$ must be of degree one. Hence, the greatest common right factor is of degree one. If $n = \deg R(x, y) \geq 2$, from the first part of this theorem, there exists a common right factor h which is of degree at least n. On the other hand, h must have degree less than or equal to n because $h(x) - h(y)$ divides $R(x, y)$. Hence, $\deg h = n$ and $h(x) - h(y) = cR(x, y)$ where c is a non-zero complex number. It remains to show that h is a greatest common right factor of p and q. Let k be any common right factor of p and

q. Then $k(x)-k(y)$ divides $R(x,y)=\frac{1}{c}\{h(x)-h(y)\}$. This is equivalent to $G_k \subseteq G_h$. By Lemma 1, we have $k \leq h$. Since k is a polynomial, k is actually a right factor of h.

3. Criteria for the existence of a non-linear common right factor

In many situations, Lemma 2 is not so easy to use because of the difficulties in finding the sequences required in the lemma. In this secton, we shall deduce the following two useful criteria on the the existence of a non-linear common right factor of two entire functions.

Theorem 3 . *Let f and g be two holomorphic functions defined on X. Suppose that there exists a non-constant entire function of two complex variables $\Phi(x,y)$ such that $\Phi(f(z),g(z)) \equiv 0$ on X. Suppose further that there exist $n \geq 2$ distinct points $z_1, \ldots\ldots, z_n$ such that $\Phi_y(f(z_i),g(z_i)) \neq 0$, $f'(z_i) \neq 0$ for all i and*

$$\begin{cases} f(z_1)=f(z_2)=\cdots=f(z_n) \\ g(z_1)=g(z_2)=\cdots=g(z_n). \end{cases}$$

Then, there exists a holomorphic function $h: X \to \mathbb{C}$ (independent of k and z_i) with $h \leq f$, $h \leq g$ and $h(z_1)=h(z_i)$ for all $2 \leq i \leq n$.

Proof of Theorem 3. For each $2 \leq i \leq n$, define $a_i(s,t)=f(z_i+t)-f(z_1+s)$. Then $a_i(0,0)=0$ and $\frac{\partial a_i}{\partial t}(0,0)=f'(z_i)\neq 0$. According to the Implict Function Theorem, there exits a unique analytic function ϕ_i defined in a neighborhood A_i of $s=0$ such that $\phi_i(0)=0$ and $a_i(s,\phi_i(s))=0$ on A_i , i.e.

$$f(z_1+s)=f(z_i+\phi_i(s)). \tag{2}$$

Since $\Phi(f(z_1),g(z_1))=0$ and $\Phi_y(f(z_1),g(z_1))\neq 0$, by the Implict Function Theorem again, there exists a unique analytic function $k: W \to k(W)$ such that $\Phi(w,k(w))\equiv 0$ on an open neighborhood W of $f(z_1)$. Note that $f(z_1)\in W, g(z_1)=g(z_i)\in k(W)$. Since $0\in A_i, \phi_i(0)=0$, f and ϕ_i are continuous, we can choose A_i small enough that

$$f(z_1+s)=f(z_1+\phi_i(s))\in W, \qquad g(z_1+s), \quad g(z_i+\phi_i(s))\in k(W).$$

On the other hand, we always have

$$\Phi(f(z_1+s),g(z_1+s))=0, \qquad \Phi(f(z_i+\phi_i(s)),g(z_i+\phi_i(s)))=0.$$

Since for each $2\le i\le n$, $f(z_1+s)=f(z_i+\phi_i(s))$ on $E=\cap_{i=2}^{n}A_i\ne\phi$ of $s=0$, it follows from the uniqueness part of the Implict Function Theorem that $g(z_1+s)=g(z_i+\phi_i(s))$ on E for all $2\le i\le n$. Take a sequence $\{s_i\}_{i\in\mathbb{N}}$ of distint complex numbers such that $\lim_{i\to\infty}s_i=0$. It is clear that we get the required sequences in Lemma 2 and the result follows.

We shall prove the following result which is slightly more general than Theorem 1 and will be used to prove Theorem A in Section 4.

Theorem 4 . *Let $n\ge 1$ and $\Phi(x,y)=\sum_{i=0}^{n}a_i(x)y^i$ be a polynomial in y with entire functions $a_i(z)$ as coefficients such that $a_n\not\equiv 0$. Suppose that $f,g:X\to\mathbb{C}$ are holomorphic functions defined on X such that*

$$\Phi(f(z),g(z))=\sum_{i=0}^{n}a_i(f(z))g(z)^i\equiv 0$$

on X. If both f and g are transcendental (i.e. they have an essential singularity at infinity), then there exists a holomorphic function h defined on X such that $h\le f,h\le g$ and $h^{-1}(s)$ is infinite for some complex number s.

Remark 1. Let $f(z)=z^2,g(z)=ze^{z^2}$ and $\Phi(x,y)=xe^{2x}-y^2$. Then $\Phi(f(z),g(z))\equiv 0$ on $\mathbb{C}$ because $ze^{2z}\circ f=ze^{2z}\circ z^2=z^2e^{2z^2}=z^2\circ(ze^{z^2})=z^2\circ g$. Note that there doesn't exist any transcendental entire h with $h\le f$ and $h\le g$. Therefore, the condition that f and g are both transcendental is needed.

Remark 2. Let $f(z)=e^z+z,g(z)=e^z$ and $\Phi(x,y)=e^x-ye^y$. Then $\Phi(f(z),g(z))\equiv 0$ because $e^z\circ f=e^z\circ(e^z+z)=e^ze^{e^z}=ze^z\circ(e^z)=ze^z\circ g$. Note that there doesn't exist any transcendental entire function h with $h\le f$ and $h\le g$, because f is prime (for the definition, see Section 4). Therefore, the condition that $\Phi(x,y)$ is a polynomial in y is also needed.

Proof of Theorem 4. Define $E = \{f(z) | \quad \Phi_y(f(z), g(z)) = 0 \text{ and } \Phi(f(z), g(z)) = 0\} \cup \{f(z) | \quad f'(z) = 0\}$. Then E is a countable set. Since f is transcendental, it follows from Little Picard theorem that we can choose $A \in \mathbb{C} - E$ so that the equation $f(z) = A$ has infinitely many distinct roots $\{z_n\}_{n\in\mathbb{N}}$. Hence $\Phi(A, g(z_n)) = \Phi(f(z_n), g(z_n)) = 0$ for all n. So $g(z_n)$ are roots of the equation $\Phi(A, y) = 0$ which has only finitely many roots. Hence there exists an infinite subsequence of $\{z_n\}_{n\in\mathbb{N}}$ (which we denote by the same $\{z_n\}_{n\in\mathbb{N}}$) such that $g(z_1) = g(z_n)$ for all z_n. Note that $f(z_1) = f(z_n) = A$ for all n and $\Phi_y(f(z_n), g(z_n)) \neq 0, \quad f'(z_n) \neq 0$. By Theorem 3, there exists a holomorphic function h with $h \leq f$, $h \leq g$ and $h(z_1) = h(z_n)$ for all $n \in \mathbb{N}$. As all z_n are distinct, $h^{-1}(h(z_1))$ is infinite.

4. Some applications

In this section, we shall show how Theorem 3, 4 can be used to solve some factorization problems of entire functions. Let us recall some basic definitions. As an analogue to prime numbers, we define a non-linear entire function F to be prime if F cannot be expressed as a composition of two non-linear entire functions. Examples of prime entire functions are polynomials of prime degrees, $e^z + z$, ze^{z^n}, $\sin z e^{\cos z}$, etc (see [2] for more examples). In fact there are plenty of prime functions as Y. Noda proved in [10] that for any transcedental entire function f, $f(z) + az$ is prime for all $a \in \mathbb{C} - E_f$ where E_f is some countable set. In order to prove F to be prime, we need to prove that it is pseudo-prime first, i.e. F cannot be expressed as a composition of two transcendental entire functions. In [13], N. Steinmetz proved a very useful criterion for an entire function to be pseudo-prime. It says that any entire solution of a linear complex differential equation with polynomial coefficients is pseudo-prime. It follows that e^z is pseudo-prime. Given two prime entire functions f and g, it is natural to ask whether $f \circ g$ is uniquely factorizable, which means, if we express $f \circ g = f_1 \circ g_1$ for non-linear entire functions f_1 and g_1, then $f = f_1 \circ L$ and $g = L^{-1} \circ g_1$ for some linear L. Note that $f \circ g$ is not always uniquely factorizable. For example, $ze^{2z} \circ z^2 = z^2 \circ (ze^{z^2})$ where all the factors involved are prime. In general, even

$f \circ g$ is uniquely factorizable, it is usually difficult to prove it. In [8], T. Kobayashi showed that for the prime functions $e^z + z$ and ze^z, their composition $F(z) = ze^z \circ (e^z + z)$ is uniquely factorizable. He proved this result by using Nevanlinna theory. Using Theorem 3, we can give a much simpler proof.

Suppose that $ze^z \circ (e^z + z) = f_1 \circ g_1$ for some non-linear entire functions f_1, g_1. Since $e^z + z$ has infinity many zeros which are all simple, so does $F(z) = (e^z + z)e^{e^z+z}$. Therefore, f_1 must have at least one zero. If f_1 has only finitely many zeros, $a_1, ..., a_n$, then there exists some a_j such that $g_1(z) = a_j$ has infinitely many roots, for otherwise $f_1 \circ g_1$ will only have finite number of zeros. If f_1 has infinite number of zeros $\{a_i\}_{i \in \mathbb{N}}$, then there exists some a_j such that $g_1(z) = a_j$ has at least two distinct roots because g_1 is non-linear. In any case, we can get some a_j and two distinct z_1, z_2 such that $f_1(a_j) = 0$ and $g_1(z_1) = g_1(z_2) = a_j$. Note that z_1, z_2 are zeros of $F(z) = (e^z + z)e^{e^z+z}$. Hence, $e^{z_1} + z_1 = e^{z_2} + z_2 = 0$. Let $\Phi(x, y) = xe^x - f_1(y)$, then $\Phi(e^z + z, g_1(z)) \equiv 0$. Since $e^z + z$ and $F(z) = (e^z + z)e^{e^z+z}$ has simple zeros only, $e^{z_i} + 1 \neq 0$ and $\Phi_y(e^{z_i} + z_i, g_1(z_i)) = f_1'(g_1(z_i)) \neq 0$ for $i = 1, 2$. By Theorem 3, there exists a non-linear entire function h with $h \leq e^z + z$ and $h \leq g_1$. Hence, $e^z + z = h_1 \circ h$ and $g_1 = h_2 \circ h$, where h_1, h_2 are analytic on Im(h).

If the image of h is $\mathbb{C} - \{a\}$, then $h = a + e^q$ for some entire function q. We may assume $a = 0$ so that $e^z + z = h_1(e^w) \circ q(z)$. The primeness of $e^z + z$ will force $q(z)$ to be linear. Hence $e^z + z$ is periodic which is impossible. Therefore, the image of h must be the whole plane. This implies that both h_1, h_2 are entire. $e^z + z$ is prime and h is non-linear, so h_1 must be linear. It follows that $g_1(z) = h_2 \circ h_1^{-1} \circ (e^z + z)$. From $ze^z \circ (e^z + z) = f_1 \circ g_1$, we get $ze^z = f_1 \circ h_2 \circ h_1^{-1}(z)$. The fact that ze^z is prime and f_1 is non-linear will force $L = h_2 \circ h_1^{-1}$ be to linear and we are done.

Clearly, by using very similar arguments, we can actually prove the following more general result.

Theorem 5 . *Let $f(z), g(z)$ be two prime entire functions. Suppose that f is of the form $ze^{\alpha(z)}$, where $\alpha(z)$ is a nonconstant entire function. If $g(z)$ is non-periodic and has infinitely many zeros, all but finitely many of them are simple, then $f(g(z))$ is uniquely factorizable.*

Remark 3. When α is a polynomial or periodic entire function, it is known that $ze^{\alpha(z)}$ is prime.

The following result was first proved in [9].

Theorem 6 *Let f be a transcendental entire function not of the form $h \circ q$, where h is a peroidic entire function and q is a polynomial. If f is pesudo-prime (i.e. f cannot be expressed as a composition of two transcendental entire functions), then so is $p \circ f$, where p is a non-constant polynomial.*

Remark 4. In [12], the periodic function $f(z) = \sin z e^{\cos z}$ was showed to be pesudo-prime while $w^2 \circ f(z) = ((1-w^2)e^{2w}) \circ \cos z$ is not. Therefore, the condition that f is not of the form $h \circ q$ is needed. Whether similar result holds for $f \circ p$ remains open.

Proof of Theorem 6. Assume that $p \circ f$ is not pseudo-prime, i.e. $p \circ f = k \circ g$ for some transcendental entire functions k and g. Applying Theorem 4 to $\Phi(x, y) = k(x) - p(y)$, we get a transcendental entire function h with $h \leq f$ and $h \leq g$. Hence $f = h_1 \circ h$ and $g = h_2 \circ h$, where h_1, h_2 are analytic on the image of h. If the image of h is $\mathbb{C} - \{a\}$, then $h = a + e^q$ for some entire function q. We may assume $a = 0$ so that $f(z) = h_1(e^w) \circ q(z)$. The pseudo-primeness of f will force $q(z)$ to be a polynomial. This contradicts the assumption on the form of f. Therefore, Im(h) $= \mathbb{C}$ and both h_1 and h_2 are entire. Hence, $p \circ h_1(z) \equiv k \circ h_2(z)$ on $\mathbb{C}$. Since h is transcendental, h_1 must be a polynomial as f is pseudo-prime. It follws that $p \circ h_1$ is a polynomial which is impossible as k is assumed to be transcendental. Therefore, $p \circ f$ must be pseudo-prime.

Finally, we proved Theorem A with an alternative method.

Proof of Theorem A. By Theorem 4, there exists a transcendental entire function h such that $f = f_1 \circ h$ and $g = g_1 \circ h$, where f_1, g_1 are analytic on $X =$ Im(h). By the Little Picard Theorem, $X = \mathbb{C}$ or $\mathbb{C} - \{a\}$ for some complex number a. Now, we have $\Phi(f_1(z), g_1(z)) \equiv 0$ on X. Using the Great Picard Theorem and the fact that Φ is a polynomial in both x and y, it is not difficult to show that f_1 and g_1 must be both transcendental (i.e with an essential singularity at infinity) or both not. Suppose, both f_1 and g_1 are transcendental. It follows from Theorem 4 that there exists a transcendental holomorphic function $h_0 : X \to \mathbb{C}$ such that $h_0 \leq f_1$ and $h_0 \leq g_1$. This implies that $h_0 \circ h \leq f, g$ and hence $G_{h_0 \circ h} = (G_{h_0 \circ h})' \subset (G_f \cap G_g)' = G_h$. By Lemma 1, there exists a holomorphic function h_1 such that $h_1 \circ h_0 \circ h = h$ and hence, $h_1 \circ h_0 \equiv id_X$ on X. This is impossible as h_0 is transcendental. Therefore, we must have both of f_1 and g_1 are rational and holomorphic on X. f_1 and g_1 can have at most one pole as $X = \mathbb{C}$ or $\mathbb{C} - \{a\}$.

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