UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS
MATH1853
Tutorial 4

1. Andrew, Beatrix and Charles are playing with a crown. If Andrew has the crown, he throws it to Charles. If Beatrix has the crown, she throws it to Andrew or to Charles, with equal probabilities. If Charles has the crown, he throws it to Andrew or to Beatrix with equal probabilities. At the beginning of the game the crown is given to one of Andrew, Beatrix and Charles, with equal probabilities. What is the probability that, after the crown is thrown once, Andrew has it? that Beatrix has it? that Charles has it?

2. From a shuffled deck of 52 playing cards one card is drawn. Let A be the event that an ace is drawn, B the event that a diamond is drawn, and C the event that a red card (heart or diamond) is drawn. Which two of A, B, and C are independent? What if a joker is added to the deck?

3. Let S be a sample space, and let $A, B \subseteq S$ be two independent events. Let $A' = S - A$ and $B' = S - B$.
(a) Show that A' and B' are independent events. [Hint: $A' \cap B' = (A \cup B)'$.]
(b) How about A and A'?

4. The color of a person's eyes is determined by a single pair of genes. If they are both blue-eyed genes, then the person will have blue eyes; if they are both brown-eyed genes, then the person will have brown eyes; and if one of them is a blue-eyed gene and the other a brown-eyed gene, then the person will have brown eyes. (Because of the latter fact, we say that the brown-eyed gene is dominant over the blue-eyed one.) A newborn child independently receives one eye gene from each of its parents, and the gene it receives from a parent is equally likely to be either of the two eye genes of that parent. Suppose that Smith and both of his parents have brown eyes, but Smith's sister has blue eyes.
(a) What is the probability that Smith possesses a blue-eyed gene?
(b) Suppose that Smith's wife has blue eyes. What is the probability that their first child will have blue eyes?
(c) If their first child has brown eyes, what is the probability that their next child will also have brown eyes?

5. A ball is in any one of n boxes and is in the i-th box with probability P_i. If the ball is in box i, a search of that box will uncover it with probability α_i. Find the conditional probability that the ball is in box j, given that a search of box i did not uncover it.

6. In a lottery 10,000 tickets are sold for \$ 1 each. There are five prizes: \$ 5,000 (once), \$ 700 (once), \$ 100 (three times). What is the expected value of a ticket?

7. Suppose it is given that $E(X) = 1$ and $Var(X) = 3$. Find
(a) $E((1 + X)^2)$ and
(b) $Var(4 + 2X)$.

8. Suppose the demand of certain new product follows the uniform distribution on $[a, b]$ (where $a < b$). The probability density function takes the form:

$$f(x) = \begin{cases} K & a \le x \le b \\ 0 & \text{otherwise.} \end{cases}$$

Here K is a positive constant to be determined.
(a) Show that if $K = \frac{1}{b-a}$ then $f(x)$ is a probability density function.
(b) Find the probability that the demand of a new product lies in $[a, (b + 2a)/3]$.

9. Let $X_1, X_2, \ldots, X_n$ be n independent random variables sharing the same probability distribution with mean μ and variance σ^2. Let

$$\bar{X} = \frac{X_1 + X_2 + \ldots + X_n}{n}.$$

What are the values of $E(\bar{X})$ and $Var(\bar{X})$? What will happen when $n \to \infty$?

10. Determine the smallest integer n for which the probability of no 2 persons having the same birthday (365 days) in a group of n people is less than $1/2$.

11. A point is chosen at random on a line segment of length L. Interpret this statement, and find the probability that the ratio of the shorter to the longer segment is less than $1/4$.

12. A and B alternate rolling a pair of dice, stopping either when A rolls the sum 9 or when B rolls the sum 6. Assuming that A rolls first, find the probability that the final roll is made by A.