MATH 1853 Linear Algebra, Probability and Statistics

Instructor: Prof. Wai-Ki CHING
Room 414, **Run Run Shaw Building**, Email: wching@hku.hk.
Consultation Hours: TUE 09:30 - 11:30 and FRI 09:30 - 10:30.
Time Table: (A) 08:30-09:20 MON,TUE, FRI (LE4) and (B) 12:30-13:20 MON,THU, FRI (LE5).

Demonstrators:
Dr. Fai Lung TSANG
Room 316, **Run Run Shaw Building**, E-mail: fail@hku.hk
Consultation hours: FRI 11:30 - 13:30.

Assessment Method: Assignments (10% + 10 %) and Final Exam (40% + 40%).

Website: http://hkumath.hku.hk/course/MATH1853/
Username: **MATH1853** Password: **RR414**.

Help Room: Opening hours: **Mondays, Tuesdays and Thursdays** from **3:00pm** to **6:00pm** in **Room 404**, **Run Run Shaw Building**.

What is Engineering?

• **Engineering combines the fields of science (Physics) and mathematics to solve real world problems that improve the world around us.**

• What really distinguishes an engineer is his ability to implement ideas in a cost effective and practical approach.

• The ability to take **a thought**, or **abstract idea**, and translate it into reality is what separates an engineer from other fields of science and mathematics.

[Taken from http://whatisengineering.com/]

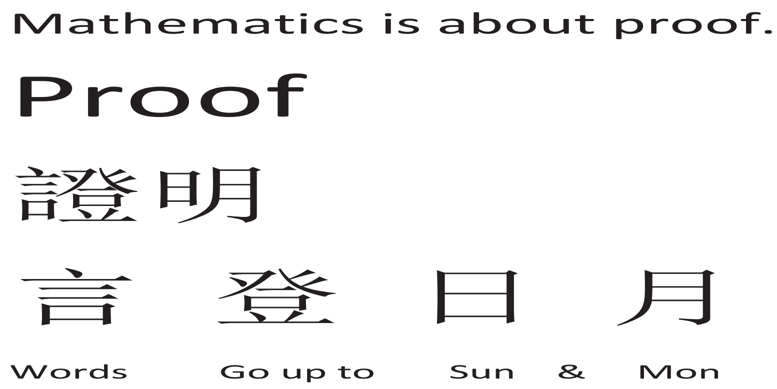

Figure 1: **What is Mathematics?**

What is Mathematics?

• Mathematics is a **language**. Language is the **dress of thought** (Samuel Johnson). Moreover, the limits of my language are the limits of my world (Ludwig Wittgenstein).

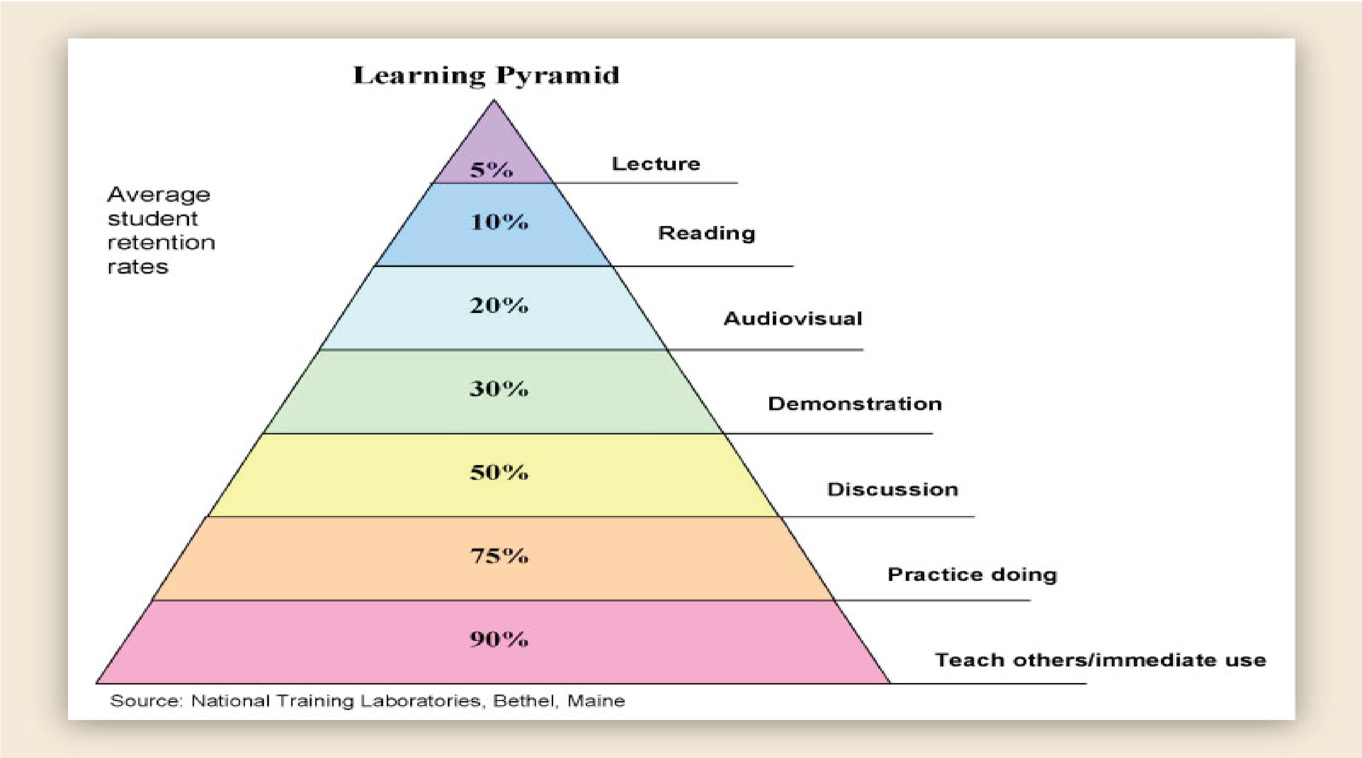

Figure 2: The Learning Pyramid.

The Learning Pyramid

• The percentages represent the average "retention rate" of information following teaching or activities by the method indicate. But the theory is controversial.

[Taken from http://drwilda.com/2013/03/06/what-is-the-learning-pyramid/]

(0) **A Review on Set Theory and Basic Calculus.**

(1) **Elementary Complex Variables:** Arithmetic of complex numbers; Modulus argument and conjugate; Basic properties and operations of complex numbers; Argand diagram; Applications in plane geometry; De Moivre's theorem; Transformations of the complex plane; Applications to trigonometric identities; nth roots of a complex number; Complex functions; Relationships between trigonometric functions and hyperbolic functions.

(2) **Basic Probability Laws; Random Variables, Probability Distribution, Expectation and Variance:** A review on history of probability; Permutations and combinations; Samples spaces and events; Basic probability rules; The concepts of conditional probability and independent events; Bayes' Theorem; The concepts and examples of discrete and continuous random variables; The concepts of discrete and continuous probability distribution and its mathematical formulae; The concepts of expectation and variance; Using the relevant formulae to solve simple problems.

(3) **Binomial, Geometric, and Poisson Distribution; Normal Distribution:** The concepts and properties of the binomial distribution, the geometric distribution, and the Poisson distribution etc; Applications: calculating probabilities involving those distributions; The concepts and properties of the normal distribution; Standardization of a normal variable and use of the standard normal distribution table; Applications of the normal distribution.

(4) **Sampling distribution, Point Estimates and Confidence Interval:** The concepts of sample statistics and population parameters; The sampling distribution of the sample mean from a random sample of size n; the concept of point estimates including sample mean, sample variance; The concept of confidence interval; The confidence interval for a population mean with known and unknown population variances.

1 Set Theory*

This section aims to introduce basic knowledge of sets as a language of mathematics.

Definition 1. *A* **set** *is a well-defined collection of objects.*

The keyword here is "**well-defined**".

- The set of **numbers** $1, 2, 3, 4$, and 5.
- The set of all **cities** in Europe.

Clearly there is no ambiguity of what or who belongs to the sets defined. How about the following sets?

- The set of all **beautiful women**?
- The set of all **young people**?

Since beauty or youth for that matter belongs to the eyes of the beholders, it is not clear who should belong to those sets. The objects of these "sets" are NOT **well-defined** here though they can be studied using **Fuzzy set theory**.

1.1 Elements

Definition 2. *Given a set, the objects in the set are called the* **elements** *of the captured set.*

Thus in the set of numbers $1, 2, 3, 4$ and 5, the elements are precisely $1, 2, 3, 4$ and 5.

For example, we may use the letter S to denote the set of numbers $1, 2, 3, 4$ and 5.

We say that x is an element of S if $x = 1, 2, 3, 4$, or 5.

We denote this membership by using the symbol "$\in$".

Thus $2 \in S$ and $4 \in S$.

The symbol "$\notin$" means "**is not an element of**", or equivalently, "**does not belong to**".

Thus $6 \notin S$.

1.2 Three methods to specify the elements of a given set

1. **Listing Method:** In this method, we simply write down all the elements explicitly and enclose them by braces. Take for example,

$$A = \{2, 4, 6, 8, 10\}, \quad \text{and} \quad B = \{\text{Alan}, \text{Anita}, \text{Leon}, \text{Gigi}, \text{Jacky}, \text{Simon}\}.$$

Question: What is this set $\{3, 5, 7, \ldots, \}$? Is it well defined?

2. **Statement Method:** In this method, the elements are specified by a descriptive sentence. For example A is the set of all even numbers between 0 and 10, or C is the set of all my students.

3. **Rule Method:** In this method, we specify the properties that the objects in the set must satisfy. For instance,

$$A = \{x \ : \ x \text{ is even and } 2 \leq x \leq 10\} = \{x \mid x \text{ is even and } 2 \leq x \leq 10\}$$

where the symbol ":" (**Or equivalently "|"**) means "**such that**". Here the letter x is irrelevant and one may use some other symbols. The last equation above is read as "A is a set containing all x such that x is even and is between 2 and 10".

1.3 Equality of sets

Definition 3. *Two sets are said to be* **equal** *if they have the same elements.*

We remark that we say "**the same elements**" and not "**the same number of** elements".

Thus if

$$A = \{2, 4, 6, 8, 10\},$$
$$B = \{x \ : \ 0 < x \leq 10 \quad \text{and} \quad x \text{ is even}\}$$

and

$$C = \{2, 4, 6, 8, 11\}$$

then we have

$$A = B$$

but

$$A \neq C.$$

We have

$$\{2, 2, 3, 3\} = \{2, 3\} = \{3, 2\}.$$

1.4 The Cardinality of a Set

Definition 4. *Given a set A, the number of elements in the set is called the* **cardinality** *of A and is denoted by $|A|$.*

If a set has finite number of elements, then it is called a **finite set**, otherwise it is called an **infinite set**.

Example 1.

$$|\{2,2,3,3\}| = |\{2,3\}| = |\{3,2\}| = 2.$$

On the other hand, the sets $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{I}$, $\mathbf{R}$ and $\mathbf{C}$ are all **infinite sets** of numbers.

1. $\mathbf{N} = \mathbb{N} = \{1, 2, 3, \ldots, \}$.
2. $\mathbf{Z} = \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
3. $\mathbf{Q} = \mathbb{Q} = \{\frac{p}{q} : p \in \mathbf{Z}, q \in \mathbf{Z}, q \neq 0\}$.
4. $\mathbf{R} = \mathbb{R} = \{x : x \text{ is a real number}\}$.
5. $\mathbf{I} = \mathbb{I} = \{x : x \in \mathbf{R} \text{ but } x \notin \mathbf{Q}\}$.
6. $\mathbf{C} = \mathbb{C} = \{x + yi : x \in \mathbf{R}, y \in \mathbf{R}, i^2 = -1\}$.

1.5 The Empty Set

Definition 5. *The* **empty set** *is the set containing no elements, i.e., the cardinality of the set is* 0*.*

Examples of the empty set are the followings

$$D = \{x \ : \ x \in \mathbf{R} \text{ and } x > 10 \text{ and } x < 0\}$$

and

$$E = \{x \ : \ x \text{ is a man with four legs and eight arms}\}.$$

The empty set is usually denoted by

$$\boxed{\phi \quad \text{or} \quad \{\}.}$$

Thus we may write $\phi = D = E$.

Remark 1. However, we note that the set $\{\phi\}$ is **NOT** an empty set, but a set with the only **ONE** element, namely the **empty set**.

Exercise 1. What is this set $\{\{\phi\}\}$?

1.6 The Universal Set

Definition 6. *The* **universal set** *denoted by U is the set that contains all the objects* **under consideration**.

The words "under consideration" are put here to avoid the famous **Russell's paradox** which states that there is no set that can contain every object.

There is a well-known paradox of the atheistic Russell:

"no gods can be omnipotent, as he cannot create a stone that he cannot bear".

1.7 The Concept of Subsets

Definition 7. *A set A is a* **subset** *of a set B if all elements of A are also elements of B. In this case we write $A \subseteq B$.*

If $A \subseteq B$, but $A \neq B$, then we say A a **proper subset** of the set B and we denote the relation by $A \subset B$. Thus we also have $\{1, 2\} \subset \{1, 2, 3\}$.

Remark 2. Sometimes people use $A \subsetneq B$ to represent A is a proper subset of B and $A \subset B$ to represent A is a subset of B. Here we adopt the former definitions.

• The set of all women in Hong Kong is a proper subset of the set of all people in Hong Kong and

$$\mathbf{N} \subset \mathbf{Z} \subset \mathbf{Q} \subset \mathbf{R} \subset \mathbf{C}.$$

For any set A, we obviously have $A \subseteq A$, $\phi \subseteq A$ and $A \subseteq U$.

However, unlike numbers, not all the sets can be compared in size.

Example 2. $\{1, 2\} \not\subseteq \{2, 3\}$ and $\{2, 3\} \not\subseteq \{1, 2\}$.

1.8 The Venn Diagram

In a **Venn diagram**, sets are represented by any enclosed regions such as circles, ovals or rectangles.

Elements in a set are represented by **dots** in the corresponding enclosed region.

Thus $a \in A$ is represented as follows in Figure 1.1.

Since all sets are subsets of a universal set, we usually enclose all circles by a **large rectangle** representing the **universal set** U.

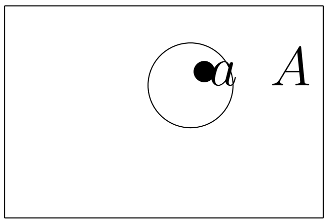

Figure 1.1.

Subsets of a given set are represented by **inner circles**.

Figure 1.2 shows that $a \in A \subseteq B$ and $b \in B$ but $b \notin A$.

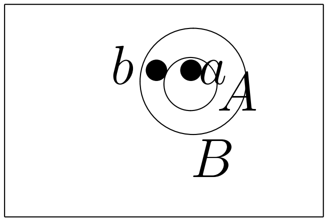

Figure 1.2.

If two sets have common elements, but are not equal, then we draw them as two overlapping circles with the common elements listed in the overlapping area.

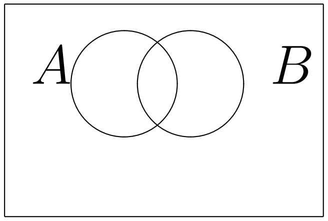

Figure 1.3.

For example, if

$$A = \{1, 2, 3\} \quad \text{and} \quad B = \{2, 4, 6\}.$$

Here A and B have a **common element** 2.

Then we have the representation in Figure 1.4.

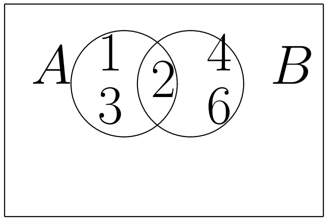

Figure 1.4.

Exercise 2. What's wrong with the figure?

1.9 Set Operations

The **union** of two given sets A and B, denoted by $A \cup B$ is the set of all elements that are either in set A or in set B, i.e.,

$$A \cup B = \{x \ : \ x \in A \text{ or } x \in B\} = B \cup A.$$

Example 3. If

$$A = \{1, 2, 3, 4\} \quad \text{and} \quad B = \{2, 4, 6, 8\},$$

then

$$A \cup B = \{1, 2, 3, 4, 6, 8\}.$$

In fact, **the set of all real numbers** is just the union of the sets of **rational** and **irrational** numbers, i.e.,

$$\mathbf{R} = \mathbf{Q} \cup \mathbf{I}.$$

In particular, a real number is either a rational number or an irrational number.

From the definition, one can see that given any set A, we have

$$A \cup U = U \quad \text{and} \quad A \cup \phi = A.$$

The union of two sets can be depicted by using the **Venn diagrams**.

We just shade the interior of the circles representing A and B respectively.

Then the resulting shaded region is the union of the sets A and B.

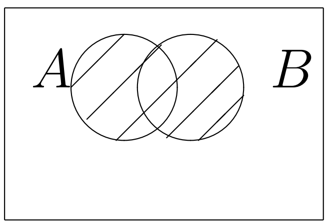

Figure 1.5.

1.9.1 Intersection

The **intersection** of two given sets A and B, denoted by $A \cap B$ is the set of all elements that are common to both sets, i.e.

$$A \cap B = \{x \ : \ x \in A \text{ and } x \in B\} = B \cap A.$$

Example 4. If

$$A = \{1, 2, 3, 4\} \quad \text{and} \quad B = \{2, 4, 6, 8\}$$

then $A \cap B = \{2, 4\}$. Furthermore, if we let $C = \{1, 3, 5, 7\}$, then $B \cap C = \phi$.

Two sets A and B are said to be **disjoint** if $A \cap B = \phi$, i.e., the two sets have no element in common.

The set of all rational numbers and set of all irrational numbers are disjoint as no number can be both rational and irrational. In set notations, we have $\mathbf{Q} \cap \mathbf{I} = \phi$.

Remark 3. For any set A, we clearly have $A \cap \phi = \phi$ and $A \cap U = A$.

The **intersection** of two sets can also be described by using Venn diagrams.

We shade the interior of the circles representing A and B by two different shading.

Then the region shaded with both kinds of shading will be the intersection of A and B.

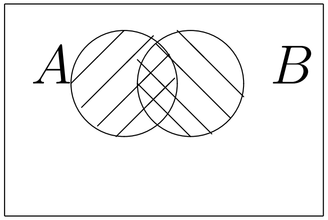

Figure 1.6.

Exercise 3. *Use Venn Diagram to demonstrate that for any three non-empty sets A, B, C, we have*

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

and

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

We remark that Venn Diagram can only serve as a demonstration but not a proof.

1.9.2 Complement

Given a set A, the **complement** of A, denoted by $\bar{A}$ or A', is the set of all elements that are in the universal set U **but not** in A, i.e.

$$\bar{A} = A' = \{x \ : \ x \in U \text{ and } x \notin A\}.$$

If $U = \mathbf{R}$, then $\bar{\mathbf{Q}} = \mathbf{I}$, as any real numbers that are not rational must be irrational.

The complement of a set A can also be described by using Venn diagrams as follows:

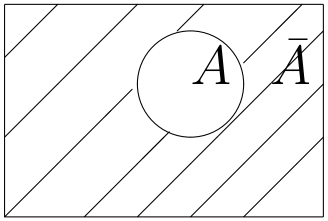

Figure 1.7.

1.9.3 Relative complement

Now we are ready to define subtraction of sets. The **relative complement** of A in B, denoted by $B - A$, (**Or equivalently** $B \backslash A$) is defined as

$$B - A = B \cap \bar{A} = \{x \ : \ x \in B \text{ and } x \notin A\}.$$

Example 5. If

$$A = \{1, 2, 3, 4\} \quad \text{and} \quad B = \{2, 4, 6, 8\},$$

then

$$A - B = \{1, 3\} \quad \text{and} \quad B - A = \{6, 8\}.$$

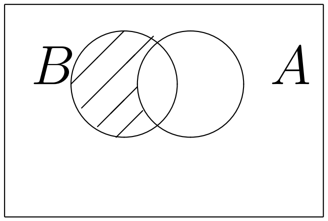

Figure 1.8.

Exercise 4. What is the relation between A and B if $B - A = \phi$?

2 A Quick Review on Differentiation and Integration*

This section provides a quick review on the results of differentiation and integration of a single variable function $f(x)$.

2.1 Differentiation of a Function

The geometric meaning of the **derivative** of a function $f(x)$, at a point x_0 denoted by

$$\boxed{f'(x_0) \quad \text{or} \quad \left.\frac{df(x)}{dx}\right|_{x=x_0}.}$$

at a point x_0 is the slope of the **tangent line** that passing through the point $(x_0, f(x_0))$.

Since the derivative of $f(x)$ at $x = x_0$ is the instantaneous rate of change of f at x_0, we see that if the derivative is positive (negative), that means $f(x)$ is increasing (decreasing) when x is increased by a small amount at the point x_0.

2.2 Some Rules for Differentiation

In the following, we present some useful rules without proof. These rules can simplify the calculations of derivatives.

(i) $\dfrac{dc}{dx} = 0$ for any constant c.

(ii) $\dfrac{dx^n}{dx} = nx^{n-1}$.

(iii) $\dfrac{de^x}{dx} = e^x$.

(iv) $\dfrac{d\log_e x}{dx} = \dfrac{1}{x}$.

(v) $\dfrac{d\sin x}{dx} = \cos x$.

(vi) $\dfrac{d\cos x}{dx} = -\sin x$.

(vi) $\dfrac{d(af(x) + bg(x))}{dx} = a \cdot (\dfrac{df(x)}{dx}) + b \cdot (\dfrac{dg(x)}{dx})$.

2.3 Some Advanced Rules for Differentiation

(i) **Product Rule:**

$$\frac{d[u(x) \cdot v(x)]}{dx} = \frac{du(x)}{dx} \cdot v(x) + u(x)\frac{dv(x)}{dx}.$$

In the "prime" notation, we have

$$\boxed{(u(x)v(x))' = u(x)'v(x) + u(x)v(x)'.}$$

(ii) **Quotient Rule:**

$$\boxed{\left[\frac{u(x)}{v(x)}\right]' = \frac{d\left[\frac{u(x)}{v(x)}\right]}{dx} = \frac{u(x)'v(x) - u(x)v(x)'}{v(x)^2}.}$$

(iii) **Chain Rule:** Suppose that $u = f(y)$ where $y = y(x)$, i.e. $u(x) = f(y(x))$. Then

$$\boxed{\frac{du(x)}{dx} = \frac{df(y(x))}{dx} = \frac{df}{dy} \cdot \frac{dy}{dx}.}$$

Example 6. (a) By product rule, we have

$$\frac{d\,\overbrace{e^x}^{u}\,\overbrace{x^{2013}}^{v}}{dx} = \frac{d\,\overbrace{e^x}^{u}}{dx} \cdot \overbrace{x^{2013}}^{v} + \overbrace{e^x}^{u}\,\frac{d\,\overbrace{x^{2013}}^{v}}{dx} = x^{2012}e^x(x+2013).$$

(b) By quotient rule, we have

$$\frac{d}{dx}\left(\frac{\overbrace{e^x}^{u}}{\underbrace{x^2}_{v}}\right) = \frac{\overbrace{(e^x)'}^{u'}\,\overbrace{x^2}^{v} - \overbrace{e^x}^{u}\,\overbrace{(x^2)'}^{v'}}{\underbrace{(x^2)^2}_{v^2}} = \frac{e^x x^2 - e^x 2x}{x^4} = e^x\left(\frac{x-2}{x^3}\right).$$

(c) Using the chain rule with $u = e^{x^{2013}} = e^{y(x)}$ where $y(x) = x^{2013}$, we have

$$\frac{de^{x^{2013}}}{dx} = \frac{de^{x^{2013}}}{dy} \cdot \frac{dy}{dx} = \frac{de^y}{dy} \cdot \frac{dy}{dx} = e^y \cdot 2013 \cdot x^{2012} = 2013x^{2012}e^{x^{2013}}.$$

Or simply,

$$\frac{de^{x^{2013}}}{dx} = \frac{de^{x^{2013}}}{dx^{2013}} \cdot \frac{dx^{2013}}{dx} = e^{x^{2013}} \cdot (2013x^{2012}) = 2013x^{2012}e^{x^{2013}}.$$

2.4 Definite Integrals

Let $y = f(x)$ and $a < b$ be two given real numbers such that

$$f(x) \geq 0 \quad \text{for} \quad a \leq x \leq b.$$

The **definite integral** of $y = f(x)$ between the points a and b is defined as the area of the region **bounded by the curve**

$$y = f(x)$$

and the lines

$$x = a, \quad x = b \quad \text{and} \quad y = 0$$

and is denoted by

$$\boxed{\int_a^b f(x)dx.}$$

Here if $f(x) < 0$ then the area is negative.

2.5 Some Rules of Definite Integral

(i) For any real constant c,

$$\int_a^b c\, f(x)dx = c \int_a^b f(x)dx.$$

(ii) $\displaystyle\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx.$

(iii) For $a \leq b \leq c$,

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx.$$

(iv) $\displaystyle\int_a^a f(x)dx = 0.$

(v) $\displaystyle\int_a^b f(x)dx = -\int_b^a f(x)dx.$

• How to find
$$\int_a^b f(x)dx?$$
This can be answered by the following proposition.

Theorem 1. *For any differentiable function $g(x)$, we have*
$$\int_a^b g'(x)dx = g(b) - g(a) \equiv g(x)\Big|_a^b.$$
Thus if you are given $f(x)$, then you need to find a function $g(x)$ such that
$$\boxed{g'(x) = f(x).}$$
This can be regarded as an "**anti-derivative problem**".

Remark: Here we define
$$\int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx$$
and similarly
$$\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty}\int_t^b f(x)dx.$$

Theorem 2. *For any continuous function f,*

$$\boxed{\frac{d}{dx}\int_a^x f(t)dt = f(x).}$$

From the previous theorem, one can see that if

$$g(x) = \int_a^x f(t)dt \quad \text{then} \quad \frac{dg(x)}{dx} = f(x).$$

Thus $f(x)$ is the derivative of $g(x)$. And we say that $g(x)$ is an **anti-derivative** of $f(x)$ if $g'(x) = f(x)$.

We note that if $g(x)$ is an anti-derivative of $f(x)$, then for any real number c, $g(x)+c$ is also an anti-derivative of f. Given $f(x)$, the anti-derivatives of $f(x)$ are denoted by

$$\int f(x)dx$$

and it is sometimes called the **indefinite integral** of $f(x)$. Here are some examples of indefinite integrals.

Example 7. (a)

$$\int x^2 dx = \frac{x^3}{3} + c$$

because

$$\frac{d}{dx}\left(\frac{x^3}{3} + c\right) = x^2 + 0 = x^2.$$

(b)

$$\int e^x dx = e^x + c$$

because

$$\frac{d}{dx}\left(e^x + c\right) = e^x + 0 = e^x.$$

(c)

$$\int \frac{1}{x} dx = \log_e(x) + c.$$

because

$$\frac{d}{dx}\left(\log_e(x) + c\right) = \frac{1}{x} + 0 = \frac{1}{x}.$$

We note that if

$$\int f(x)dx = g(x) + c$$

then

$$\int_a^b f(x)dx = g(x)\Big|_a^b = g(b) - g(a).$$

The following is an example.

Example 8. The Exponential Function:

$$F(t) = \int_0^t e^{-x}dx = -e^{-x}\Big|_0^t = -e^{-t} + e^0 = 1 - e^{-t}.$$

Example 9. The Uniform Function:

$$U(x) = \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{if otherwise.} \end{cases}$$

$$F(t) = \int_0^t U(x)dx = t \quad \text{for } t \in [0,1].$$

Example 10. *Compute*

$$\int_{-\infty}^{\infty} e^{-|x|} dx.$$

We note that

$$e^{-|x|} = \begin{cases} e^{-x} & \text{if } x \geq 0 \\ e^{x} & \text{if } x < 0. \end{cases}$$

Therefore we have

$$\int_{-\infty}^{\infty} e^{-|x|} dx = \int_{-\infty}^{0} e^{x} dx + \int_{0}^{\infty} e^{-x} dx.$$

We also note that

$$\int_{0}^{\infty} e^{-x} dx = \int_{-\infty}^{0} e^{x} dx$$

and therefore

$$\int_{-\infty}^{\infty} e^{-|x|} dx = 2 \int_{0}^{\infty} e^{-x} dx = -2e^{-x} \Big|_{0}^{\infty} = 2 \cdot 1 = 2.$$

Question: *Compute*

$$\int_{-\infty}^{\infty} x e^{-|x|} dx.$$

Example 11. We have

$$1 + x + x^2 + \cdots + = \frac{1}{1-x} \quad \text{for} \quad |x| < 1.$$

So for $|t| < 1$

$$t + \frac{t^2}{2} + \frac{t^3}{3} + \cdots + = \int_0^t 1 + x + x^2 + \cdots + dx = \int_0^t \frac{1}{1-x} dx = -\log_e(1-t).$$

We have

$$\boxed{\log_e(1-t) = -t - \frac{t^2}{2} - \frac{t^3}{3} - \cdots - \quad \text{for} \quad |t| < 1}$$

and therefore

$$\boxed{\log_e(1-t) \approx -t}$$

for very small t. This is a very useful approximation for $\log_e(1-t)$ when t is very small.

2.6 Integration by Substitution

In this section, we introduce the method of **integration by substitution**.

Basically the method is very simple and can be described as follows. Suppose that

$$x = g(u)$$

is a function of u, then

$$\begin{aligned} \int f(x)dx &= \int f(g(u))d(g(u)) \\ &= \int f(g(u))\frac{dg(u)}{du}du \\ &= \int f(g(u))g'(u)du. \end{aligned}$$

Example 12. The integral

$$\int (x+1)^3 dx$$

can be computed as

$$\int (x+1)^3 dx \;=\; \int (x^3+3x^2+3x+1)dx = \frac{x^4}{4} + x^3 + \frac{3}{2}x^2 + x + c_1.$$

Using the substitution

$$u = x+1 \quad \text{or} \quad x = u-1$$

then

$$\frac{du}{dx} = 1,$$

and we have

$$\int (x+1)^3 dx = \int u^3 du = \frac{u^4}{4} + c_2 = \frac{(x+1)^4}{4} + c_2.$$

We remark that the constant c_1 can be different from the constant c_2.

2.7 Integration by Parts

In this section, we are going to introduce another very useful integration technique, **integration by parts**. Recall that the product rule of differentiation gives

$$\frac{d(f(x)g(x))}{dx} = f'(x)g(x) + g'(x)f(x). \tag{2.1}$$

Hence if we integrate both sides of (2.1), we get

$$\boxed{f(x)g(x) = \int f'(x)g(x)dx + \int g'(x)f(x)dx,} \tag{2.2}$$

or equivalently,

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx. \tag{2.3}$$

This is called the **integration by parts** formula. Very often, the formula is written as

$$\int f(x)dg(x) = f(x)g(x) - \int g(x)df(x) \tag{2.4}$$

or simply

$$\boxed{\int fdg = fg - \int gdf.} \tag{2.5}$$

We remark for definite integral, from (2.3), we have

$$\boxed{\int_a^b f(x)g'(x)dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)dx.} \tag{2.6}$$

Let us see how this formula can help us in the calculation of integrals.

Example 13. Let us compute

$$\int_0^1 xe^x dx$$

by integration by parts. We let

$$f(x) = x \quad \text{and} \quad g(x) = e^x$$

then

$$f'(x) = 1 \quad \text{and} \quad g'(x) = e^x.$$

Thus the indefinite integral can be computed as follows:

$$\int \underbrace{x}_{f} \underbrace{e^x}_{g'} dx = \int \underbrace{x}_{f} \, d \underbrace{e^x}_{g} = \underbrace{x}_{f} \underbrace{e^x}_{g} - \int \underbrace{e^x}_{g} d \underbrace{x}_{f} = xe^x - e^x + c.$$

Hence we have

$$\int_0^1 xe^x dx = xe^x - e^x\Big|_0^1 = 1.$$

2.8 Taylor's Series

• Taylor's series: for any $x \in \mathbb{R}$, we have the followings: [1]

$$\boxed{e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots +}$$

$$\boxed{\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots +}$$

$$\boxed{\sin(x) = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + .}$$

• In the next part, we shall introduce a special number i such that $i^2 = -1$. The above series can be linked up by

$$\boxed{e^{ix} = \cos(x) + i \cdot \sin(x).}$$

Check the following:

$$\begin{aligned} e^{ix} &= 1 + \frac{xi}{1!} + \frac{x^2 i^2}{2!} + \frac{x^3 i^3}{3!} + \frac{x^4 i^4}{4!} + \frac{x^5 i^5}{5!} + \cdots + \\ &= 1 + \frac{xi}{1!} - \frac{x^2}{2!} - \frac{x^3 i}{3!} + \frac{x^4}{4!} + \frac{x^5 i}{5!} + \cdots + \end{aligned}$$

[1] Here $n! = 1 \times 2 \times 3 \times \cdots \times n$ for $n \in \mathbf{N}$ and $0!$ is defined to be equal to 1.