3 Complex Variables

The **imaginary number** [1] i is a solution of the equation:

$$\boxed{x^2 + 1 = 0.}$$

• This idea of i was introduced to answer the above question. But then it results in many interesting results, beautiful theory and useful applications.

• In general a **complex** number $a + bi$, where a and b are both real numbers and $i^2 = -1$. We use $\mathbb{C}$ to denote the set of complex number:

$$\boxed{\mathbb{C} = \{a + bi : a, b \in \mathbb{R}, i^2 = -1\}.}$$

It is clear that the set of all real numbers $\mathbb{R}$ is a proper subset of $\mathbb{C}$.

Example 1. *We note that*

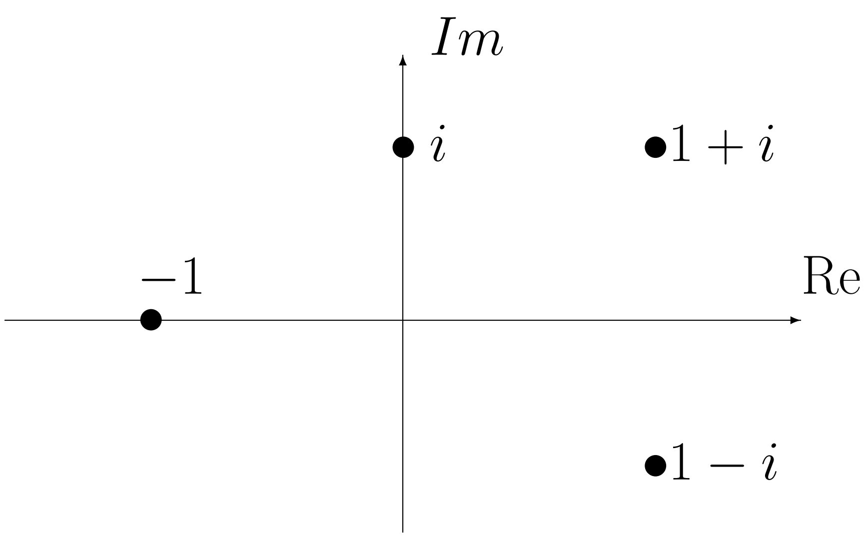
$$1 + i, 1 - i, -1 \quad \text{and} \quad i$$

are complex numbers.

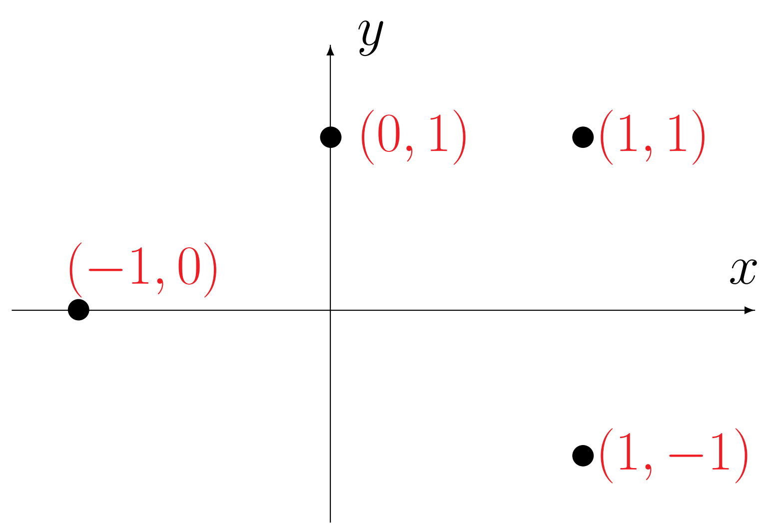
[1] The earliest reference to the square root of a negative number perhaps is the work of a Greek mathematician, Heron of Alexandria, in the 1st century AD. The impetus to study complex numbers proper first arose in the 16th century when algebraic solutions for the roots of cubic and quartic polynomials (Taken from Wikipedia).

3.1 The Complex Plane

• The four complex numbers can be represent in the following complex plane. It is similar to the case of $\mathbb{R}^2$ except the x-axis represents the **real part** of the complex number and the y-axis represents the **imaginary part** of the complex number.



The Complex Plane



The $\mathbb{R}^2$ Plane

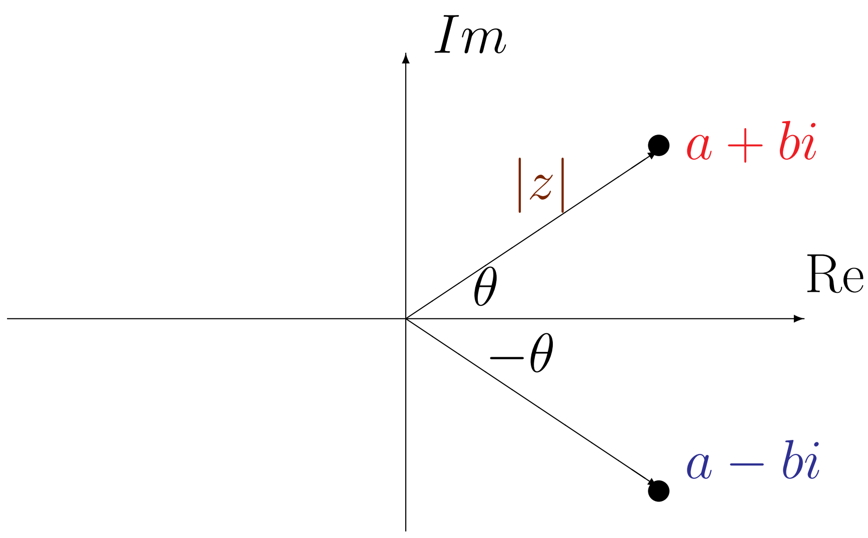
3.2 The Argand Diagram

3.2.1 Conjugate

Given a complex number $z = a + bi$, the real part $Re(z) = a$, the imaginary part $Im(z) = b$ and the **conjugate** of z, denoted by $\overline{z}$ is given by $a - bi$.

3.2.2 Modulus

Given a complex number $z = a + bi$, the modulus of z, denoted by $|z|$ is given by $\sqrt{a^2 + b^2}$.



The Argand Diagram

3.3 Addition of Two Complex Numbers

• Let $z_1 = a + bi$ and $z_2 = c + di$ where a, b, c, d are real. We first note that if

$$z_1 = a + bi = z_2 = c + di$$

then we have

$$(a - c) + (b - d)i = 0$$

and therefore we must have $a = c$ and $b = d$.

• For addition, we define

$$z_1 + z_2 = (a + bi) + (c + di) = (a + c) + (b + d)i.$$

It is very similar to the addition of **two vectors** in $\mathbb{R}^2$:

$$(a, b) + (c, d) = (a + c, b + d).$$

Example 2. *We have*

$$(2 + i) + (3 - 2i) = (2 + 3) + (1 - 2)i = 5 - i.$$

Similarly in $\mathbb{R}^2$, *we can write*

$$(2, 1) + (3, -2) = (5, -1).$$

Theorem 1. *For any two complex numbers z_1 and z_2, we have*

$$|z_1 + z_2| \le |z_1| + |z_2|.$$

Proof. Let $z_1 = a + bi$ and $z_2 = c + di$, where a, b, c, d are real numbers, then we are to show that

$$|z_1 + z_2| = \sqrt{(a+c)^2 + (b+d)^2} \le \sqrt{a^2 + b^2} + \sqrt{c^2 + d^2} = |z_1| + |z_2|.$$

The above is true if

$$ac + bd \le \sqrt{a^2 + b^2}\sqrt{c^2 + d^2}.$$

Now we have

$$(ac + bd)^2 \le (a^2 + b^2)(c^2 + d^2)$$

because

$$2abcd \le (ad)^2 + (bc)^2 \quad \text{as} \quad 0 \le (ad - bc)^2.$$

□

Remark 1. *The result can be generalized as follows:*

$$\boxed{|z_1 + \cdots + z_n| \le |z_1| + \cdots + |z_n|.}$$

Note that we can apply the following argument:

$$|z_1 + \cdots + z_n| = |(z_1 + \cdots + z_{n-1}) + z_n| \le |z_1 + \cdots + z_{n-1}| + |z_n|.$$

3.4 Multiplication of Two Complex Numbers

We can define the **product** of two complex numbers as follows.

Theorem 2. *Let* $z_1 = a + bi$ *and* $z_2 = c + di$*, then we have*

$$z_1 \cdot z_2 = (ac - bd) + (ad + bc)i.$$

Proof. We have

$$\begin{aligned} z_1 \cdot z_2 &= (a + bi) \cdot (c + di) \\ &= ac + adi + bci + bdi^2 \\ &= (ac - bd) + (ad + bc)i. \end{aligned}$$

□

Example 3. *We have*

$$(2 + i) \cdot (3 - 2i) = 6 + 3i - 4i - 2i^2 = 8 - i.$$

Remark 2. *For any two complex numbers* z_1 *and* z_2*,*

$$z_1 + z_2, \quad z_1 - z_2, \quad z_1 \cdot z_2 \quad \text{and} \quad z_1/z_2 \quad (z_2 \neq 0)$$

are also complex numbers.

Theorem 3. *We have* $|z|^2 = z \cdot \overline{z}$*.*

Proof. Let $z = a + bi$ where $a, b \in \mathbb{R}$, then we have

$$|z|^2 = a^2 + b^2$$

by definition. We also have

$$z \cdot \overline{z} = (a + bi)(a - bi) = a^2 - b^2 i^2 = a^2 + b^2.$$

Hence the result follows. □

- The above theorem can be applied to the following problem. Suppose that

$$\frac{1 + 2i}{1 - i} = a + bi$$

for some **real numbers** a and b. What are a and b? We know $\overline{1 - i} = 1 + i$ and $(1 - i) \cdot (1 + i) = 2$, it follows that

Example 4. *We have*

$$\begin{aligned} \frac{1 + 2i}{1 - i} &= \frac{1 + 2i}{1 - i} \times \underbrace{\frac{1 + i}{1 + i}}_{=1} \\ &= \frac{1 + 2i + i - 2}{1 + 1} = -\frac{1}{2} + \frac{3}{2} i \equiv a + bi. \end{aligned}$$

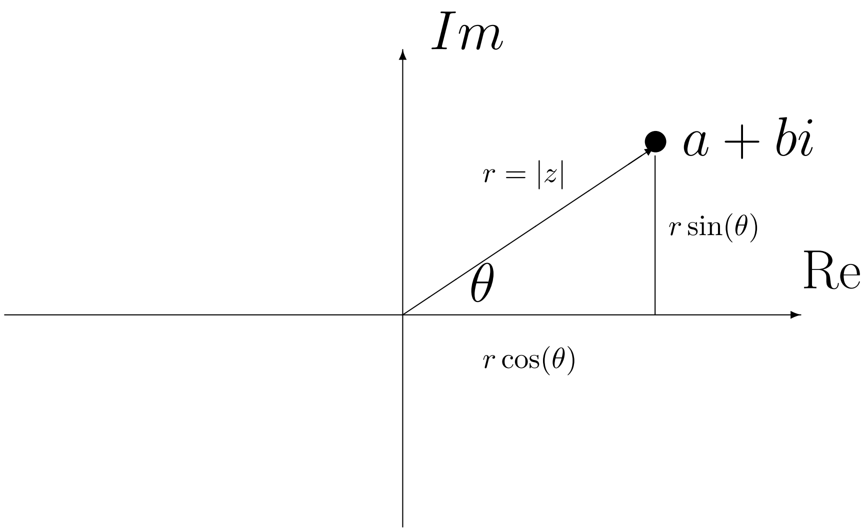
3.5 Polar Form

A complex number can be written in its **polar form** as follows:

$$z = a + bi = r\cos(\theta) + ri\sin(\theta).$$

Here $Arg(z) = \theta$ where $\theta \in [-\pi, \pi]$,

$$r = \sqrt{a^2 + b^2} = |z|, \quad \sin(\theta) = \frac{b}{\sqrt{a^2 + b^2}} \quad \text{and} \quad \cos(\theta) = \frac{a}{\sqrt{a^2 + b^2}}.$$



The Argand Diagram

Example 5. *Given the complex number*

$$z = 1 + i,$$

we can compute

$$|z| = \sqrt{z\overline{z}} = \sqrt{(1+i)(1-i)} = \sqrt{2}.$$

Then we can also compute the angle θ*, where*

$$\tan(\theta) = \frac{1}{1} = 1$$

thus θ *is* $\pi/4$.

Hence the polar form of

$$z = \sqrt{2}(\cos(\pi/4) + i\sin(\pi/4)).$$

Example 6. *Given the complex number*

$$z = 1 - \sqrt{3}i,$$

we can compute

$$|z| = \sqrt{z\overline{z}} = \sqrt{(1 - \sqrt{3}i)(1 + \sqrt{3}i)} = 2.$$

Then we can also compute the angle θ*, where*

$$\tan(\theta) = \frac{-\sqrt{3}}{1} = -\sqrt{3}$$

thus θ *is* $-\pi/3$*.*

Hence the polar form of

$$z = 2(\cos(\pi/3) + i\sin(-\pi/3)) = 2(\cos(\pi/3) - i\sin(\pi/3)).$$

Theorem 4. *We have*

$$\boxed{Arg(z_1 \cdot z_2) = Arg(z_1) + Arg(z_2)}$$

and

$$\boxed{Arg\left(\frac{z_1}{z_2}\right) = Arg(z_1) - Arg(z_2).}$$

Proof. Let

$$z_1 = r_1(\cos(\theta_1) + i\sin(\theta_1))$$

and

$$z_2 = r_2(\cos(\theta_2) + i\sin(\theta_2)).$$

Then we have

$$\begin{aligned} z_1 z_2 &= r_1 r_2 \left((\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2))\right. \\ &\quad \left.+(\cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2))i\right) \\ &= r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)). \end{aligned}$$

Because we have the following two identities:

$$\sin(A + B) = \sin(A)\cos(B) + \sin(B)\cos(A)$$

$$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B).$$

And

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{r_1}{r_2}\frac{(\cos(\theta_1)+i\sin(\theta_1))(\cos(\theta_2)-i\sin(\theta_2))}{(\cos(\theta_2)+i\sin(\theta_2))(\cos(\theta_2)-i\sin(\theta_2))}\\ &= \frac{r_1}{r_2}\frac{((\cos(\theta_1)\cos(\theta_2)+\sin(\theta_1)\sin(\theta_2))+(-\cos(\theta_1)\sin(\theta_2)+\sin(\theta_1)\cos(\theta_2))i)}{1}\\ &= \frac{r_1}{r_2}(\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)).\end{aligned}$$

Because we have the following two identities:

$$\sin(A-B) = \sin(A)\cos(B) - \sin(B)\cos(A)$$

$$\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B).$$

□

Example 7. *Let $z_1 = i$ and $z_2 = -i$. We have $Arg(i) = \frac{\pi}{2}$ and $Arg(z_2) = -\frac{\pi}{2}$. Therefore we have*

$$0 = Arg(1) = Arg(z_1 z_2) = Arg(z_1) + Arg(z_2) = \frac{\pi}{2} - \frac{\pi}{2} = 0$$

and

$$\pi = Arg(-1) = Arg\left(\frac{z_1}{z_2}\right) = Arg(z_1) - Arg(z_2) = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

3.6 Some Geometry of Complex Numbers

If $|z|^2 = 1$ and z is real then we know that $z = 1$ or $z = -1$. But if z is a complex number, then the answer is much more interesting.

- We write
$$z = x + yi$$
so if $|z| = 1$ then we have
$$|z| = \sqrt{x^2 + y^2} = 1$$
or
$$\boxed{x^2 + y^2 = 1.}$$
Thus if $|z| = 1$ then z can be any complex number on **the unit circle** centered at $0 + 0i$ of the complex plane.

- In other words, the following set
$$\{z : |z| = 1, z \in \mathbb{C}\}$$
contains all the complex numbers on the unit circle centered at $0+0i$ of the complex plane.

Example 8. *What are the complex numbers* z *satisfying* $|z-1| = 4$*?*

We let $z = x + yi$ *where* x *and* y *are real numbers. Then we have*

$$\boxed{(x-1)^2 + y^2 = 4^2 = 16}$$

which is a circle of radius 4 and centered at $1 + 0i$ *of the complex plane.*

Example 9. *Suppose we have*

$$Re(z^2) = 0$$

then the answer will be different and we have

$$Re(x^2 - y^2 + 2xyi) = 0$$

i.e.,

$$x^2 - y^2 = (x-y)(x+y) = 0.$$

Then z *can be any point on* **the lines***:*

$$\boxed{y = x \quad \text{or} \quad y = -x.}$$

3.7 Parametric Representation

• Given the straight line: $y = 2x + 1$ in $\mathbb{R}^2$. Then for each point (x, y) on the line, there is a $t \in \mathbb{R}$ (t is a parameter), such that $x = t$ and $y = 2t + 1$. Therefore the straight line can be represented in a **parametric form** as follows:

$$\{(t, 2t + 1) : t \in \mathbb{R}\}.$$

If the straight line is in the complex plane, then the corresponding set of complex numbers can be written as

$$\{t + (2t + 1)i : t \in \mathbb{R}\}.$$

• Similarly, the points on a unit circle centered at $(0, 0)$ in $\mathbb{R}^2$ can be written as

$$\{(\cos(t), \sin(t)) : t \in [0, 2\pi)\}.$$

If the unit circle is in the complex plane, then the corresponding set of complex numbers can be written as

$$\{\cos(t) + i \sin(t) : t \in [0, 2\pi)\}.$$

• Hence in the previous example, we can use the **parametric form** to represent the two lines:

$$\{t + ti : t \in \mathbb{R}\} \cup \{t - ti : t \in \mathbb{R}\}.$$

Theorem 5. *Let n be a non-negative integer and $r \neq 0$, we have*

$$\boxed{(r\cos(\theta) + ri\sin(\theta))^{n} = r^{n}\left(\cos(n\theta) + i\sin(n\theta)\right).}$$

Proof. We can show this by **mathematical induction**. When $n = 0$, we have

$$(r\cos(\theta) + ri\sin(\theta))^{0} = 1 = r^{0}\left(\cos(0) + i\sin(0)\right).$$

Thus the equality is true when $n = 0$. Assume that

$$(r\cos(\theta) + ri\sin(\theta))^{n} = r^{n}\left(\cos(n\theta) + i\sin(n\theta)\right). \tag{3.1}$$

Then we shall show

$$(r\cos(\theta) + ri\sin(\theta))^{n+1} = r^{n+1}\left(\cos((n+1)\theta) + i\sin((n+1)\theta)\right).$$

Now

$$\begin{aligned}
(r\cos(\theta) + ri\sin(\theta))^{n+1} &= (r\cos(\theta) + ri\sin(\theta))\, r^{n}\left(\cos(\theta) + i\sin(\theta)\right)^{n} \\
&= (r\cos(\theta) + ri\sin(\theta))\, r^{n}\left(\cos(n\theta) + i\sin(n\theta)\right) \quad \text{(by Eq. (3.1))} \\
&= r^{n+1}(\cos(\theta)\cos(n\theta) - \sin(\theta)\sin(n\theta)) \\
&\quad + r^{n+1}(\cos(\theta)\sin(n\theta) + \cos(n\theta)\sin(\theta))i \\
&= r^{n+1}(\cos((n+1)\theta) + i\sin((n+1)\theta)).
\end{aligned}$$

□

3.8.1 Principle of Mathematical Induction

The Principle of **Mathematical Induction** (M.I.) can be informally stated as follows:

We can establish the truth of a proposition if

(a) we can show that it follows from smaller instances of the same proposition, as well as

(b) we can establish the truth of the smallest instance (or instances) explicitly.

Theorem 6. *Let $S_1, S_2, S_3, \dots$ be statements such that*
(1) S_1 is true;
(2) Whenever Statement S_k is true, where $k \in \mathbf{N}$, the Statement S_{k+1} is true

Then all of the statements $S_1, S_2, \dots,$ are true.

Corollary 1. *Let n be a positive integer and $r \neq 0$, we have*

$$(r\cos(\theta) + ri\sin(\theta))^{-n} = r^{-n}\left(\cos(n\theta) - i\sin(n\theta)\right).$$

Proof. We have to show that

$$r^{-n}\left(\cos(n\theta) - i\sin(n\theta)\right) \cdot (r\cos(\theta) + ri\sin(\theta))^{n} = 1.$$

But we know that

$$(r\cos(\theta) + ri\sin(\theta))^{n} = r^{n}\left(\cos(n\theta) + i\sin(n\theta)\right)$$

and therefore we have

$$r^{-n}\left(\cos(n\theta) - i\sin(n\theta)\right) \cdot r^{n}\left(\cos(n\theta) + i\sin(n\theta)\right) = \cos^2(n\theta) + \sin^2(n\theta) = 1.$$

This is because we have the identity

$$\cos^2(A) + \sin^2(A) = 1.$$

The proof is then completed. □

Corollary 2. *Let n be a positive integer and $r \neq 0$, we have*

$$(r\cos(\theta) + ri\sin(\theta))^{\frac{1}{n}} = r^{\frac{1}{n}}\left(\cos(\frac{\theta}{n}) + i\sin(\frac{\theta}{n})\right).$$

Proof.

$$\begin{aligned}\left(r^{\frac{1}{n}}\left(\cos(\tfrac{\theta}{n}) + i\sin(\tfrac{\theta}{n})\right)\right)^n &= r\left(\cos(n\cdot\tfrac{\theta}{n}) + i\sin(n\cdot\tfrac{\theta}{n})\right) \\ &= r\left(\cos(\theta) + i\sin(\theta)\right).\end{aligned}$$

Hence the result follows. □

Corollary 3. *Let*

$$n = \frac{p}{q}$$

be a rational number (p, q are integers and $q \neq 0$) and $r \neq 0$, we have

$$(r\cos(\theta) + ri\sin(\theta))^{\frac{p}{q}} = r^{\frac{p}{q}}\left(\cos(\frac{p\theta}{q}) + i\sin(\frac{p\theta}{q})\right).$$

Proof. We note that

$$\left(r^{\frac{p}{q}}\left(\cos(\frac{p\theta}{q}) + i\sin(\frac{p\theta}{q})\right)\right)^{\frac{q}{p}} = r\left(\cos(\frac{p\theta}{q}) + i\sin(\frac{p\theta}{q})\right)^{\frac{q}{p}}$$

and

$$\left(\cos(\frac{p\theta}{q}) + i\sin(\frac{p\theta}{q})\right)^{\frac{q}{p}} = \left(\left(\cos(\frac{p\theta}{q}) + i\sin(\frac{p\theta}{q})\right)^{\frac{1}{p}}\right)^{q}.$$

Since

$$\left(\cos(\frac{p\theta}{q}) + i\sin(\frac{p\theta}{q})\right)^{\frac{1}{p}} = \left(\cos(\frac{\theta}{q}) + i\sin(\frac{\theta}{q})\right)$$

and

$$\left(\cos(\frac{\theta}{q}) + i\sin(\frac{\theta}{q})\right)^{q} = \left(\cos(\theta) + i\sin(\theta)\right).$$

The result follows. □

Remark: It seems that for $f(\theta) = \cos\theta + i\sin\theta$, we have for any "real number" r,

$$f(\theta)^r = f(r\theta).$$

What can be $f(\theta)$?

In fact, we have

$$\boxed{e^{\theta i} = \cos(\theta) + i\sin(\theta).}$$

We can recognize that the Taylor's series for any $x \in \mathbb{R}$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots +$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots +$$

$$\sin(x) = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + .$$

Thus by "letting $x = \theta i$"

$$\begin{aligned} e^{\theta i} &= 1 + \tfrac{\theta i}{1!} - \tfrac{\theta^2}{2!} - \tfrac{(\theta i)^3}{3!} + \tfrac{\theta^4}{4!} + \cdots + \\ &= 1 - \tfrac{\theta^2}{2!} + \tfrac{\theta^4}{4!} - \tfrac{\theta^6}{6!} + \cdots + \\ &\quad + i(\tfrac{\theta}{1!} - \tfrac{\theta^3}{3!} + \tfrac{\theta^5}{5!} + \cdots +) \\ &= \cos(\theta) + i\sin(\theta). \end{aligned}$$

Corollary 4.

$$\cos(x) = \frac{e^{xi} + e^{-xi}}{2} \quad \text{and} \quad \sin(x) = \frac{e^{xi} - e^{-xi}}{2i}$$

Theorem 7. *Show that*

$$\cos(2x) = \cos^2(x) - \sin^2(x) \quad \text{and} \quad \sin(2x) = 2\sin(x)\cos(x).$$

Proof.

$$(\cos(x) + i\sin(x))^2 = \cos^2(x) - \sin^2(x) + 2\cos(x)\sin(x)i.$$

But by De Moivre's theorem we have

$$(\cos(x) + i\sin(x))^2 = \cos(2x) + i\sin(2x).$$

By comparing coefficients, we have

$$\cos(2x) = \cos^2(x) - \sin^2(x) \quad \text{and} \quad \sin(2x) = 2\sin(x)\cos(x).$$

□

Corollary 5. *Let* $x = \pi/12$ *then*

$$\frac{\sqrt{3}}{2} = \cos(\pi/6) = \cos^2(\pi/12) - \sin^2(\pi/12) = 2\cos^2(\pi/12) - 1.$$

Thus we have $\cos(\pi/12) = \sqrt{\frac{1}{2}(1 + \frac{\sqrt{3}}{2})}$.

Theorem 8. *Show that*

$$\cos(3x) = 4\cos^3(x) - 3\cos(x) \quad \text{and} \quad \sin(3x) = 3\sin(x) - 4\sin^3(x).$$

Proof.

$$(\cos(x) + i\sin(x))^3 = \cos^3(x) - 3\sin^2(x)\cos(x) + (3\cos^2(x)\sin(x) - \sin^3(x))i.$$

But by De Moivre's theorem we have

$$(\cos(x) + i\sin(x))^3 = \cos(3x) + i\sin(3x).$$

By comparing coefficients, we have

$$\cos(3x) = \cos^3(x) - 3\sin^2(x)\cos(x)$$

and

$$\sin(3x) = 3\sin(x)\cos^2(x) - \sin^3(x).$$

Using the identity

$$\sin^2(x) + \cos^2(x) = 1$$

the result follows. □

3.10 The nth Root of Unity

For a positive integer n, the following equation has n roots:

$$z^n = 1$$

they are called **the nth roots of unity**. A complex number can be represented as

$$r\cos(x) + ir\sin(x) = re^{xi}$$

where in this case $r = 1$. Thus we can rewrite the equation as follows:

$$z^n = 1 = e^{2\pi ki}.$$

Hence we have the roots given by

$$z = e^{\frac{2k\pi i}{n}} = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right) \quad \text{for} \quad k = 0, \pm 1, \pm 2, \ldots, .$$

If we let $w = e^{\frac{2\pi i}{n}}$, then it is straightforward to check that

$$1, w, w^2, \ldots, w^{n-1}$$

are the n district roots. We call w an **nth root of unity** ($|w| = 1$ and $w^n = 1$). In fact, $1, w, w^2, \ldots, w^{n-1}$ are n distinct nth roots of unity.

Example 10. *Consider the equation:*

$$z^4 = 1$$

then we have $1, w, w^2, w^3$ *being the four distinct roots of unity where*

$$w = e^{\frac{2\pi i}{4}}.$$

However, if we take $w_1 = w^2$ *then we have*

$$w_1^2 = w^4 = 1,\ w_1^3 = w^6 = w^4 \cdot w^2 = w^2,\ w_1^4 = 1,\ w_1^5 = w^{10} = w^2,\ w_1^6 = w^{12} = 1, \cdots ,$$

Thus the powers of w_1 *cannot generate all the four distinct roots.*

But if we take $w_2 = w^3$ *then we have*

$$w_2^2 = w^6 = w^2, \quad , w_2^3 = w^9 = w, \quad w_2^4 = w^{12} = 1, \quad w_2^5 = w^{15} = w^3.$$

In this case, we can generate all the four distinct roots of the equation

$$z^4 = 1$$

by using the powers of w_2.

Example 11. *Let w be a complex number such that $w^3 = 1$ and $w \neq 1$.*

How to simplify

$$w^{2013} + w^{101} + w^{64}?$$

Since $w^3 = 1$, we have

$$w^{2013} + w^{101} + w^{64} = w^{671\times 3} + w^{3\times 33+2} + w^{3\times 21+1} = 1 + w^2 + w.$$

Furthermore, we note that

$$0 = w^3 - 1 = (w-1)(w^2 + w + 1).$$

Since $w \neq 1$, we must have $w^2 + w + 1 = 0$. Hence we have

$$w^{2013} + w^{101} + w^{64} = 0.$$

3.11 Some Complex Functions

In this section, we consider the **image** of a complex function.

3.11.1 Linear Function

Consider the following **linear function** $f : D \to \mathbb{C}$:

$$f(z) = cz + d \quad \text{where} \quad c, d \in \mathbb{R},\ c \neq 0 \text{ and } D \subset \mathbb{C}.$$

The **image** of D under the function f is the set of complex numbers

$$f(D) = \{z' : z' = f(z) \text{ for } z \in D\}.$$

(i) Suppose that

$$D = \{x + yi : ax + by = e\}$$

is a straight line. For $z' = x' + y'i \in f(D)$, we write

$$z' = x' + y'i = f(z) = c(x + yi) + d = cx + d + cyi$$

for some $z = x + yi \in D$.

To obtain the **relationship of x' and y'**, we proceed as follows. We first have

$$x = \frac{1}{c}(x' - d) \quad \text{and} \quad y = \frac{1}{c}y'.$$

We then have

$$ax + by = \frac{a}{c}(x' - d) + \frac{b}{c}y' = e.$$

Finally the relation between x' and y' is given by

$$ax' + by' = ad + ce.$$

which is another straight line.

That is to say, the image $f(D)$ is again a straight line.

Example 12. *Suppose*

$$f(z) = z + 1$$

and

$$D = \{x + yi : x - y = 1\}.$$

To find

$$f(D) = \{z' = x' + y'i : z' = x' + y'i = f(z) \text{ for some } z = x + yi \in D\}.$$

Now we have for $z' \in f(D)$*,*

$$z' = x' + y'i = f(z) = z + 1 = x + yi + 1 = (x + 1) + yi$$

for some $z \in D$*.*

To obtain the relation between x' *and* y'*, we have*

$$x' = x + 1 \quad \text{and} \quad y' = y.$$

Since $z = x + yi \in D$*, we have* $x - y = 1$*. Then we obtain the relationship between* x' *and* y' *as follows:*

$$1 = x - y = x' - 1 - y' \quad \text{or} \quad x' - y' = 2.$$

(ii) If

$$D = \{x + yi : x^2 + y^2 = 1\}$$

is the unit circle with center at zero then for $z' = x' + y'i \in f(D)$, we have

$$z' = x' + y'i = f(z) = cz + d = c(x + yi) + d.$$

for some $z = x + yi \in D$. Now we have

$$x = \frac{1}{c}(x' - d) \quad \text{and} \quad y = \frac{1}{c}y'.$$

Since $z = x + yi \in D$,

$$1 = x^2 + y^2 = \frac{1}{c^2}(x' - d)^2 + \frac{1}{c^2}y'^2.$$

Hence the relationship between x' and y' is given by

$$(x' - d)^2 + y'^2 = c^2$$

is another circle with center d and radius of $|c|$.[2]

Thus we see that the **above linear function** maps a straight line to a straight line and a circle to a circle.

[2]You may also use the parametric form of an unit circle: $x = \cos\theta$, $y = \sin\theta$ for $\theta \in [-\pi, \pi]$. Then we have $f(z) = x' + y'i = c(\cos\theta + i\sin\theta) + d = c\cos\theta + d + ci\sin\theta$. Hence we have $x' = c\cos\theta + d$ and $y' = c\sin\theta$. Finally we have $(x' - d)^2 + y'^2 = c^2(\cos^2\theta + \sin^2\theta) = c^2$ which is circle.

3.11.2 The Inverse Function

Consider the **inverse function** $f : D \to \mathbb{C}$:

$$f(z) = \frac{1}{z}.$$

(i) If $D = \{x + yi : cx + dy = e \text{ and } e \neq 0\}$ is a straight line.

Then we have for $z' = x' + y'i \in f(D)$,

$$z' = x' + y'i = f(z) = \frac{1}{z} = \frac{1}{x + yi} = \frac{x - yi}{x^2 + y^2} \quad \text{for some } z = x + yi \in D.$$

To obtain the relationship of x' and y' one can proceed as follows. We first have

$$x' = \frac{x}{x^2 + y^2} \quad \text{and} \quad y' = \frac{-y}{x^2 + y^2}.$$

We then have

$$\frac{x'}{y'} = \frac{-x}{y} \quad \text{and} \quad x'^2 + y'^2 = \frac{1}{x^2 + y^2} = \frac{-y'}{y}.$$

Now we have

$$cx + dy = e \quad \text{or} \quad \frac{cx}{y} + d = \frac{e}{y}.$$

Then we have

$$-c\left(\frac{x'}{y'}\right)+d=\frac{-e(x'^2+y'^2)}{y'}$$

or

$$-cx'+dy'=-e(x'^2+y'^2)$$

or

$$e(x'^2+y'^2)-cx'+dy'=0.$$

Now we have

$$\left(x'-\frac{c}{2e}\right)^2+\left(y'+\frac{d}{2e}\right)^2=\frac{c^2+d^2}{4e^2}.$$

which is a circle with center $\frac{c}{2e}-\frac{d}{2e}i$ and radius $\sqrt{\frac{c^2+d^2}{4e^2}}$.

(ii) If

$$D = \{x + yi : x^2 + y^2 = r^2\}$$

is the circle with center at zero and radius r then for $z' = x' + y'i \in f(D)$, we have

$$x' + y'i = f(z) = \frac{1}{z} = \frac{1}{x + yi} = \frac{x - yi}{x^2 + y^2} = \frac{x - yi}{r^2}$$

for some $z = x + yi \in D$. Now

$$x = r^2 x' \quad \text{and} \quad y = -r^2 y'$$

so we have

$$x'^2 + y'^2 = \frac{1}{r^2}$$

is a circle with radius $\frac{1}{r}$ and center 0 [3]. Thus if

$$D = \{x + yi : x^2 + y^2 = r^2 \leq 1\}$$

then

$$f(D) = \{x + yi : x^2 + y^2 = \frac{1}{r^2} \geq 1\}.$$

[3] You may also use the parametric form of a circle: $x = r\cos\theta$, $y = r\sin\theta$ for $\theta \in [-\pi, \pi]$. Then we have $f(z) = x' + y'i = \frac{1}{r\cos\theta + ir\sin\theta} = \frac{1}{r}(\cos\theta - i\sin\theta)$. Hence we have $x' = \frac{1}{r}\cos\theta$ and $y' = \frac{-1}{r}\sin\theta$. Finally we have $x'^2 + y'^2 = \frac{1}{r^2}$.

3.12 Hyperbolic Functions

We define a class of function called **Hyperbolic functions**. They are similar to the trigonometric functions.

(i) Hyperbolic sine of x:

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

(ii) Hyperbolic cosine of x:

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

Using the above two definitions, one can define the rest of the hyperbolic functions. For example the hyperbolic tangent and recall that

$$\tan(x) = \frac{\sin(x)}{\cos(x)}.$$

(iii) Hyperbolic tangent of x:

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

The remaining hyperbolic functions are just the inverse of the above (i)-(iii). Recall that

$$\cot(x) = \frac{1}{\tan(x)}.$$

(iv) Hyperbolic cotangent of x:

$$\coth(x) = \frac{1}{\tanh(x)} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

(v) Hyperbolic secant of x:

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)} = \frac{2}{e^x + e^{-x}}$$

(vi) Hyperbolic cosecant of x:

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)} = \frac{2}{e^x - e^{-x}}.$$

Theorem 9.

$$\sinh(-x) = -\sinh(x) \quad \text{and} \quad \cosh(-x) = \cosh(x).$$

Proof.

$$\sinh(-x) = \frac{e^{-x} - e^{x}}{2} = -\sinh(x)$$

and

$$\cosh(-x) = \frac{e^{-x} + e^{x}}{2} = \cosh(x).$$

□

Corollary 6.

$$\tanh(-x) = -\tanh(x)$$
$$\coth(-x) = -\coth(x)$$
$$\text{sech}(-x) = \text{sech}(x)$$
$$\text{csch}(-x) = -\text{csch}(x)$$

Theorem 10. *(a)*

$$\frac{d\sinh(x)}{dx} = \cosh(x).$$

(b)

$$\frac{d\cosh(x)}{dx} = \sinh(x).$$

(c)

$$\frac{d\tanh(x)}{dx} = \frac{d}{dx}\left(\frac{\sinh(x)}{\cosh(x)}\right) = \frac{\cosh^2(x) - \sinh^2(x)}{\cosh^2(x)} = \frac{1}{\cosh^2(x)}.$$

because we have

$$\cosh^2(x) - \sinh^2(x) = 1.$$

Theorem 11. *(a)*

$$\int \sinh(x)dx = \int \frac{e^x - e^{-x}}{2}dx = \frac{e^x + e^{-x}}{2} + C = \cosh(x) + C.$$

(b)

$$\int \cosh(x)dx = \int \frac{e^x + e^{-x}}{2}dx = \frac{e^x - e^{-x}}{2} + C = \sinh(x) + C.$$

(c)

$$\int \tanh(x)dx = \int \frac{\sinh(x)}{\cosh(x)}dx = \int \frac{d\cosh(x)}{\cosh(x)} = \log_e(\cosh(x)) + C.$$

3.13 Relation Between Hyperbolic Functions and Trigonometric Functions

Theorem 12.

$$i\sinh(x) = \sin(ix) \quad \text{and} \quad \cosh(x) = \cos(ix).$$

Proof. Recall that

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

and

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

and

$$\sin(x) = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + .$$

Thus

$$\sinh(x) = \frac{e^x - e^{-x}}{2} = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} +$$

and

$$\sin(ix) = i\left(\frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \right).$$

Then we have $i\sinh(x) = \sin(ix)$. Similarly one can show the second equality. $\square$

3.14 A Summary

1. Imaginary numbers.
2. The Argand diagram.
3. Arg and modulus and conjugate of a complex number.
4. Polar form.
5. De Moivre's theorem and its applications.
6. $e^{\theta i} = \cos(\theta) + i \sin(\theta)$ and trigonometric identities.
7. The nth roots of unity.
8. The image of a complex function.
9. Hyperbolic functions and trigonometric functions.