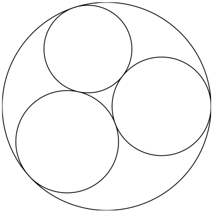
Pair correlation in Apollonian gaskets

Xin Zhang

12/04/2017

Construction of an Apollonian Circle Packing



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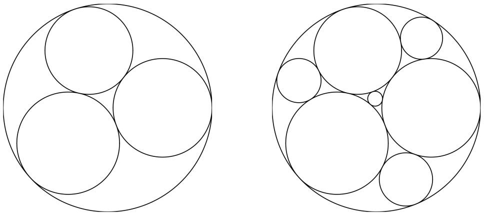

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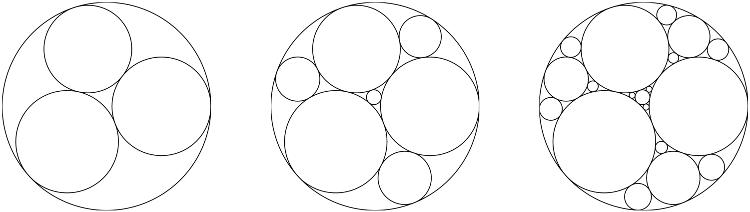


Figure: Construction of an Apollonian circle packing

An Apollonian circle packing

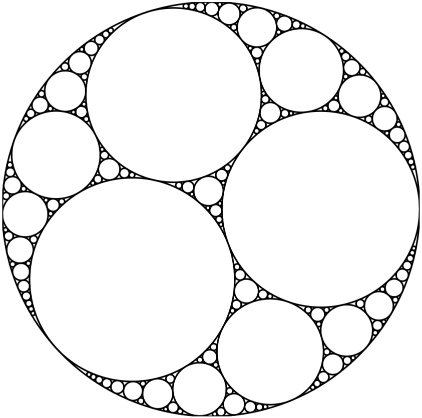


Figure: An Apollonian circle packing

Integral Apollonian circle packings

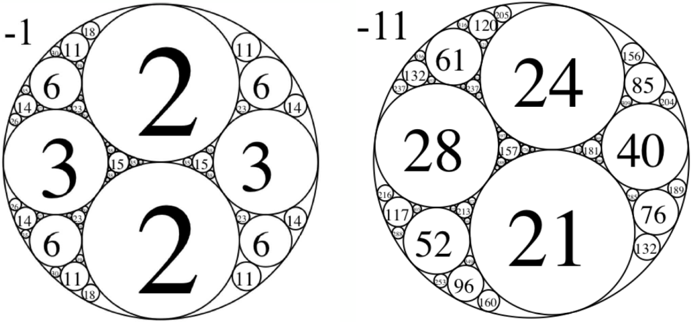
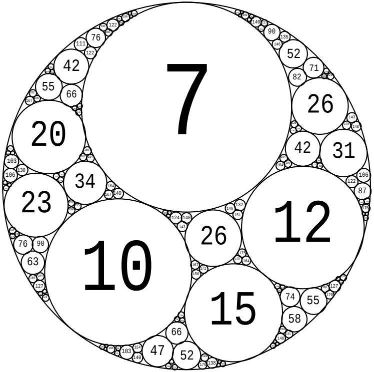


Figure: Other integral Apollonian circle packings

Other integral circle packings



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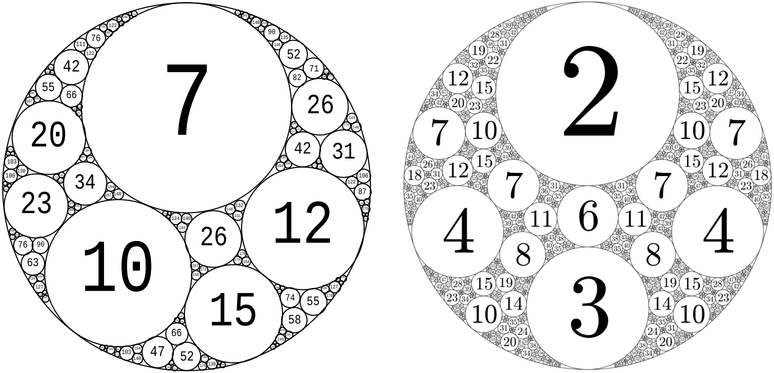

Figure: Guettler and Mallows' Apollonian 3-circle packing and Stange's $\mathbb{Q}[\sqrt{-2}]$-Apollonian packing

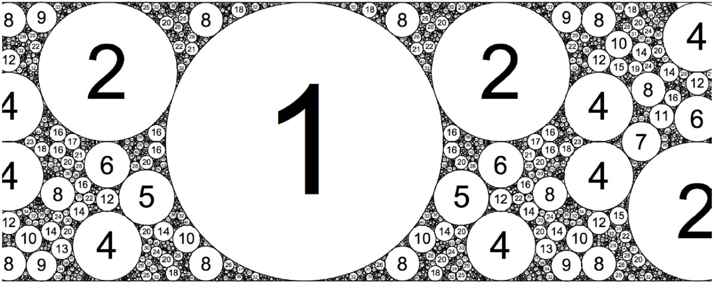

Figure: Kontorovich-Nakamura's integral crystallographic packing

Two basic questions

Question

Is there a law that governs the distribution of circles within a circle packing?

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What integers arise as curvatures from an integral circle packing?

Counting circles in an Apollonian packing

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Theorem (Kontorovich-Oh, 2011)

Fix an Apollonian circle packing $\mathcal{P}$, and let $\mathcal{P}_T$ be set of circles with curvatures $< T$. Then as $T \to \infty$,

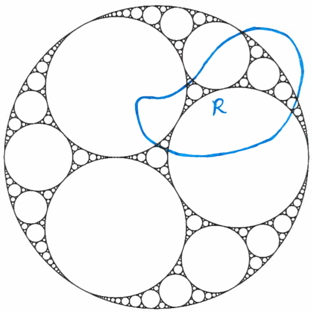
$$\#\mathcal{P}_T \sim c_{\mathcal{P}} T^{\delta},$$

where $c_{\mathcal{P}} > 0$ depends on $\mathcal{P}$, and $\delta \approx 1.305688$ is the Hausdorff dimension of $\mathcal{P}$.

Equidistribution of circles

Theorem (Oh-Shah, 2012)

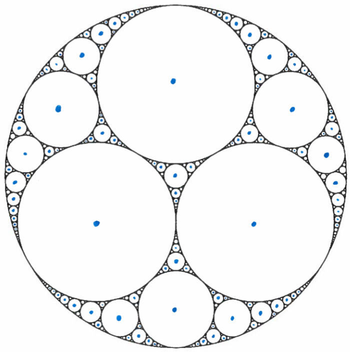
Let $\mathcal{R}$ be any region with smooth boundary in $\mathbb{C}$ and $\mathcal{P}_T(\mathcal{R})$ be the set of circles in $\mathcal{R}$ whose curvatures are bounded by T, then as $T \to \infty$, $\#\mathcal{P}_T(\mathcal{R}) \sim \mu(\mathcal{R})T^\delta$, where μ is a constant multiple of δ-dimensional Hausdorff measure supported on $\overline{\mathcal{P}}$.



The fine scale structure

Question

Let $\mathcal{P}_T$ be the set of circles with radius $> 1/T$ (curvature $< T$). How many circles are within the distance $10/T$ of a random circle? Is there a limit as $T \to \infty$?



Pair correlation

Definition

The pair correlation function $P_T(s)$ is defined as

$$P_T(s) = \frac{1}{\#\mathcal{P}_T} \sum_{C_1 \in \mathcal{P}_T} \sum_{\substack{C_2 \in \mathcal{P}_T \\ C_2 \neq C_1}} \mathbf{1}\{d(C_1, C_2) < s/T\},$$

where $d(C_1, C_2)$ is the Euclidean distance of the two circles C_1, C_2.

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Question

Is there a limit for P_T *as* $T \to \infty$*? If so what are some properties of the limiting pair correlation?*

Groups members: Weiru Chen, Mo Jiao, Calvin Kessler, Amita Malik and Xin Zhang. Work to appear at *Experimental Math*.

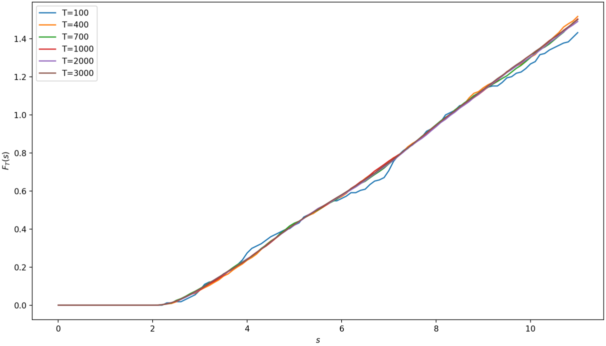

Figure: The plot for $P_T(s)$ with different T's

Experimental results from IGL

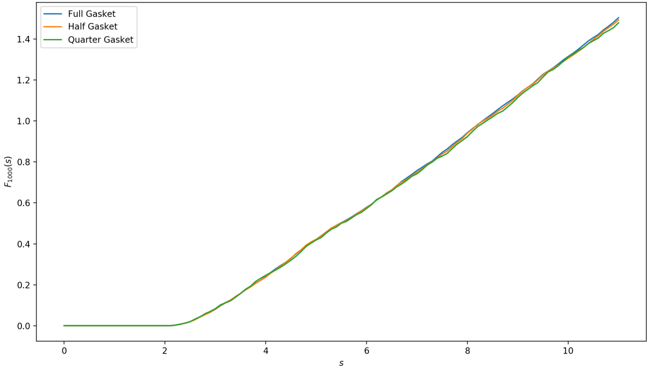

Figure: Pair correlation for the whole plane, half plane and the first quadrant

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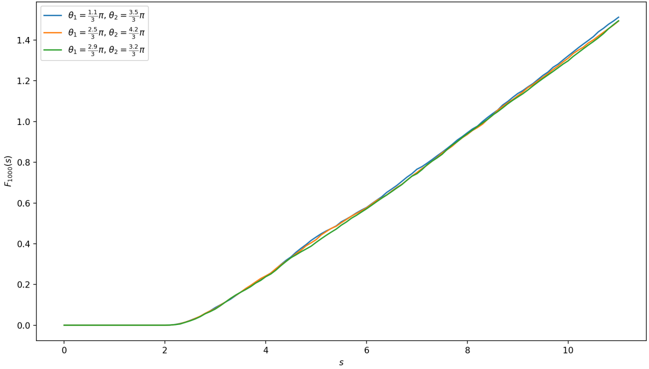

Figure: Pair correlation for different Apollonian gaskets

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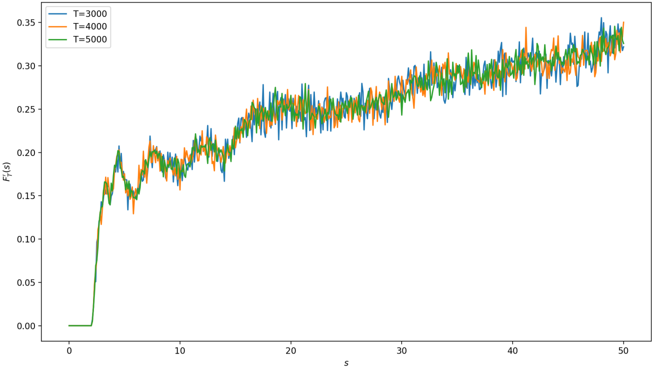


Figure: The empirical derivative $P'_T(s)$, with different T taken

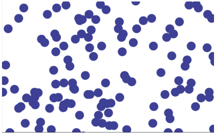
Limiting pair correlation

Theorem (limiting pair correlation, Z, 2017)

There exists a continuously differentiable function P, supported on $[c, \infty)$ *for some* $c > 0$*, such that*

$$\lim_{t \to \infty} P_T(s) = P(s).$$

Applications of pair correlation



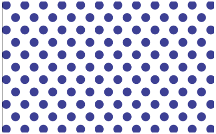


Figure: Arrangements of glass atoms vs metal atoms

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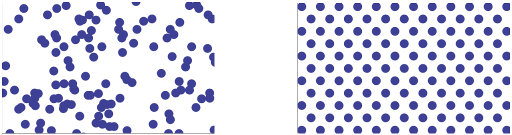

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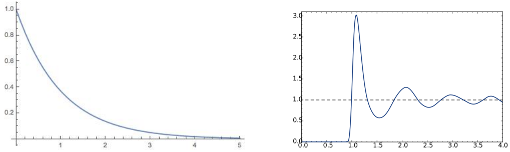

Figure: Pair correlations of glass atoms and metal atoms

Applications of pair correlation

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- Astronomers use pair correlation to predict the likelihood of finding one galaxy near another galaxy.

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  [Dyson-Montgomery]: (Empirically) The pair correlation of non trivial zeros of the Riemann zeta function agrees with the pair correlation of the eigenvalues from a random Hermitian matrix.

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The symmetry group

There exists a discrete group $\Gamma < PSL(2, \mathbb{C})$ whose limit set $\Lambda(\Gamma) = \overline{\mathcal{P}}$.

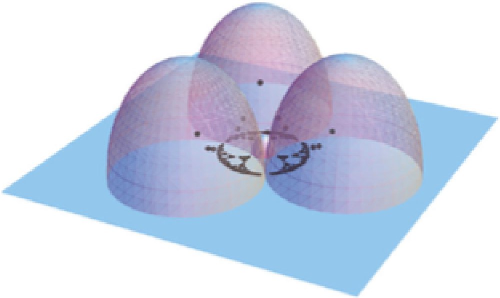

Figure: A fundamental domain of Γ and a point orbit of Γ

Subgroups of $PSL(2, \mathbb{C})$

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- $H := SU(1,1) \cup SU(1,1) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, where

  $SU(1,1) = \left\{ \begin{pmatrix} \xi & \eta \\ \bar{\eta} & \bar{\xi} \end{pmatrix} : \xi, \eta \in \mathbb{C}, |\xi|^2 - |\eta|^2 = 1 \right\}$.

The Patterson-Sullivan measure

Definition

The Patterson-Sullivan measure ν is the weak limit as $s \to \delta^+$ of the family of measures

$$\nu_{\boldsymbol{j},s} := \frac{1}{\sum_{\gamma\in\Gamma} e^{-sd(\boldsymbol{j},\gamma\boldsymbol{j})}} \sum_{\gamma\in\Gamma} e^{-sd(\boldsymbol{j},\gamma\boldsymbol{j})} \delta_{\gamma\boldsymbol{j}},$$

where $\delta_{\gamma\boldsymbol{j}}$ is the Dirac delta measure supported at the point $\gamma\boldsymbol{j}$.

Conformal measures on $T^1(\mathbb{H}^3)$

- $T^1(\mathbb{H}^3) \longmapsto \partial\mathbb{H}^3 \times \partial\mathbb{H}^3/\{diagonal\} \times \mathbb{R}$.
  $u \longmapsto (u^+, u^-, \beta_{u^-}(\boldsymbol{j}, \pi(u)),$
  where u^-, u^+ are the starting and ending points of u, $\pi(u)$ is the base point of u in $\mathbb{H}^3$, and β_{u^-} is the Buzeman function.

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- Identify $\mathrm{T}^1(\mathbb{H}^3) = PSL(2, \mathbb{C})/M$. These measures are can be lifted to right M-invariant measure on $PSL(2, \mathbb{C})$.
- These measures are Γ invariant, so descend to measures on $\Gamma \backslash \mathrm{T}^1(\mathbb{H}^3)$.

The map $\boldsymbol{q}$

Let S be the hemisphere based at the bounding circle, and let $g \in PSL(2, \mathbb{C})$. Define

$$\boldsymbol{q}(g) := \begin{cases} \text{the apex of } g(S), & \text{if } \infty \notin g(\partial S) \\ \infty, & \text{otherwise} \end{cases}$$

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- The set of centers $\mathcal{C}_T$ from $\mathcal{P}_T$ are the projection of apices with height $> 1/T$.

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- $\mathcal{C} = \{\Re(\boldsymbol{q}(\gamma)) : \gamma \in \Gamma/\Gamma_S\}$
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A detailed version of the limiting pair correlation

Theorem (limiting pair correlation, Z, 2017)

The pair correlation density P' is explicitly given by

$$P'(s) = \frac{\delta}{2\mu_H^{\mathrm{PS}}(\Gamma_H \backslash H)} \int_{h \in \Gamma_H \backslash H} \sum_{\substack{\gamma \in \gamma_H \backslash (\Gamma - \Gamma_H) \\ \boldsymbol{q}(h^{-1}\gamma^{-1}) \in V_s}} \frac{|(\Re(\boldsymbol{q}(h^{-1}\gamma^{-1}))|^{\delta}}{s^{\delta+1}} d\mu_H^{\mathrm{PS}}(h),$$

where $V_s = \{z + h\boldsymbol{j} : |z| < s, h > 1\} \cup \{z + h_i : \frac{|z|}{h} < s, h \leq 1\}$, *and* μ_H^{PS} *is the pullback of the visual map* $H \to \widehat{\mathbb{C}} : h \to h(X_0)$.

Convergence of correlations

Let $E \subset \mathbb{C}$ be an open set with ∂E piecewise smooth, and $\mathbf{\Omega} = \prod_{1 \leq i \leq k} \Omega_i \subset \mathbb{C}^k$, where $\Omega_i, 1 \leq i \leq k$ are bounded open subset of $\mathcal{C}$ with piecewise smooth boundaries.

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Let $\boldsymbol{r} = \langle r_1, \cdots, r_k \rangle$ be a multi-index, where $r_i \in \mathbb{Z}^{\ge 0}$, and at least one $r_i > 0$.
For $z \in \mathbb{C}$, let

$$\mathcal{B}_T(\Omega_i, z) := (\frac{1}{T}\Omega_i + z) \cap \mathcal{C}_T$$

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$$\mathcal{B}_T(\Omega_i, z) := (\frac{1}{T}\Omega_i + z) \cap \mathcal{C}_T$$

Want to understand

$$\int_{\mathbb{C}} \prod_{1 \leq i \leq k} \mathbf{1}\{\#\mathcal{B}_T(\Omega_i, z) = r_i\} \chi_E(z) dz$$

as $T \to \infty$.

Convergence of correlations

Define a function on $PSL(2, \mathbb{C})$:

$$F_{\mathbf{\Omega},\boldsymbol{r}}(g) := \prod_{1 \le i \le k} \mathbf{1}\{\#(\boldsymbol{q}(g^{-1}\Gamma/\Gamma_S) \cap \Omega_i^* = r_i\},$$

where $\Omega_i^*, 1 \le i \le k$ are the "infinite chimneys" based at Ω_i:

$$\Omega_i^* := \{z + r\boldsymbol{j} : z \in \Omega_i, r \in (1, \infty)\}.$$

Then

$$\int_{\mathbb{C}} \prod_{1 \le i \le k} \mathbf{1}\{\#\mathcal{B}_T(\Omega_i, z) = r_i\}\chi_E(z)dz = \int_{\mathbb{C}} F_{\mathbf{\Omega},\boldsymbol{r}}(n_z a_T)\chi_E(z)dz$$

Equidistribution of expanding horospheres

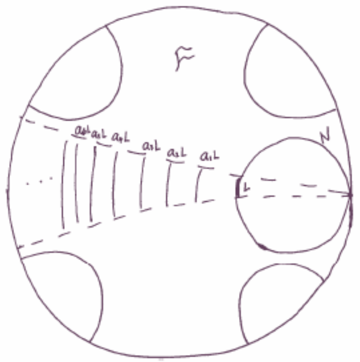
Theorem (Mohammadi-Oh, JEMS, 2015)

Let $f \in C^\infty(N)$ and $\Psi \in C_c^\infty(\Gamma\backslash G)$. We have

$$\lim_{T\to\infty} T^{2-\delta} \int_N f(n)\Psi(na_T)dn = \frac{m^{\mathrm{BR}}(\Psi)\mu_N^{\mathrm{PS}}(f)}{m^{\mathrm{BMS}}(\Gamma\backslash G)}.$$

A remark

Mohammadi-Oh does not hold for certain non-compactly supported test functions.
E.G., Choose (f, Ψ) such that $Supp(f) \subset L$ and $Supp(\Psi) \subset$ a small neighborhood of $\cup_{i=1}^{\infty} a_i L$.



A hierarchy lemma

Our contention: $F_{\Omega,r}$ is neither continuous nor compactly supported.

A hierarchy lemma

Our contention: $F_{\mathbf{\Omega},\boldsymbol{r}}$ is neither continuous nor compactly supported.
For two bounded functions Ψ_1, Ψ_2 on $\Gamma\backslash G$, let $\mathcal{D}(\Psi_1), \mathcal{D}(\Psi_2)$ be the closure of the discontinuities of Ψ_1, Ψ_2.

Lemma

If (f, Ψ_1) satisfies Mohammadi-Oh, $\Psi_2 \ll \Psi_1$, and $m^{BR}(\mathcal{D}(\Psi_1)), m^{BR}(\mathcal{D}(\Psi_2)) = 0$, then (f, Ψ_2) satisfies Mohammadi-Oh.

Convergence of correlations

- Certain pair (χ_E, Ψ_0) relates to the counting of circles. By Oh-Shah's theorem on counting circles, (χ_E, Ψ_0) satisfies Mohammadi-Oh.
- $F_{\mathbf{\Omega},\boldsymbol{r}} \ll \Psi_0$.
- $D(\Psi_0), D(F_{\mathbf{\Omega},\boldsymbol{r}})$ are contained in some algebraic subvarieties of $\Gamma\backslash G$ of codimension > 1, so $m^{\mathrm{BR}}(D(\Psi_0)), m^{\mathrm{BR}}(D(F_{\mathbf{\Omega},\boldsymbol{r}})) = 0$

Proposition

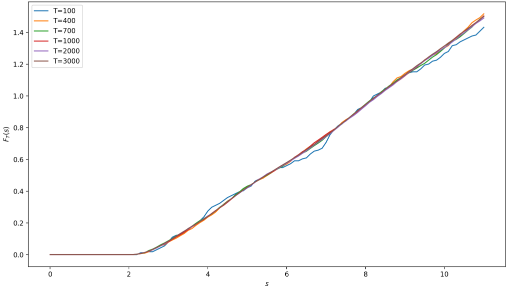
We have

$$\lim_{T\to\infty} T^{2-\delta} \int_{\mathbb{C}} F_{\mathbf{\Omega},\boldsymbol{r}}(n_z a_T)\chi_E(z)dz = \frac{m^{BR}(F_{\mathbf{\Omega},\boldsymbol{r}})w(E)}{m^{BMS}(\Gamma\backslash G)},$$

where w is the pullback measure of μ_N^{PS} under the map $z \to n_z$.

Question (Curt McMullen)

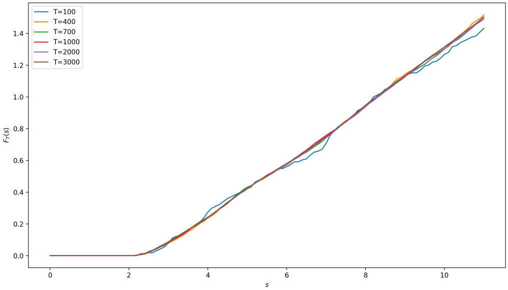
Can one say something about the growth of the limiting pair correlation P?



Growth of the limiting pair correlation

Question (Curt McMullen)

Can one say something about the growth of the limiting pair correlation P?

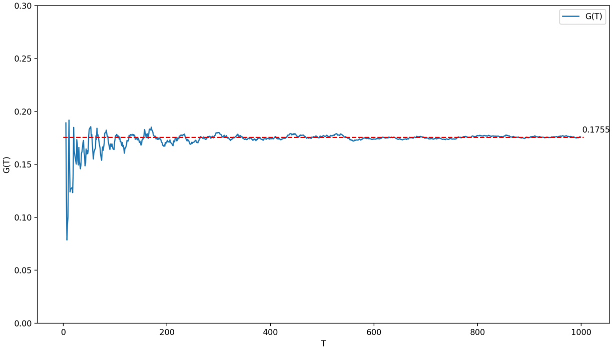


Empirically, $P(s) \sim 0.072 s^{\delta}$.

Electrostatic energy

The electrostatic energy function $G(T)$ is defined by

$$G(T) := \frac{1}{T^{2\delta}} \sum_{\substack{C_1, C_2 \in \mathcal{C}_T \\ p \neq q}} \frac{1}{d(C_1, C_2)}.$$



Question

What can one say about the fine scale structure of a spiral galaxy?

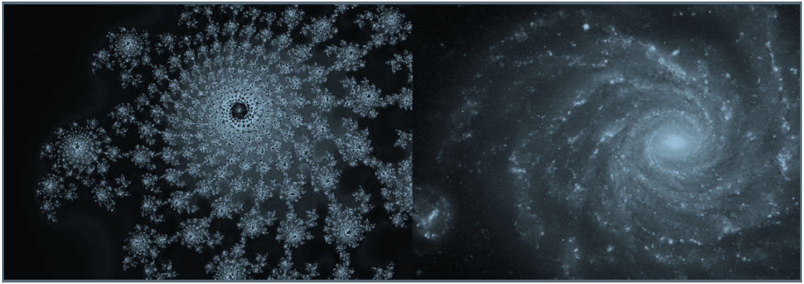


Figure: Image on the left depicts a sub-region of a Julia set; image on the right is the famous Grand Spiral Galaxy (NGC 1232)

Thank you!