Local-global principles in circle packings

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Integral Apollonian circle packings

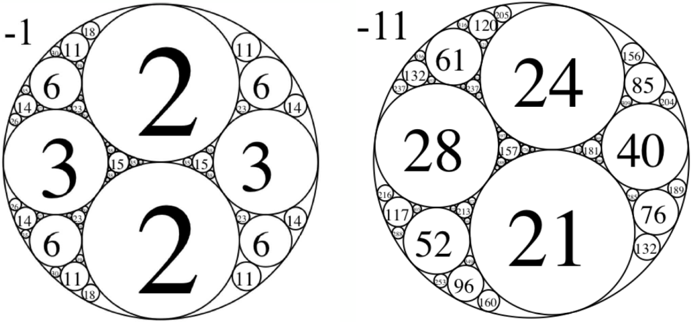

Figure: Integral Apollonian circle packings

Integral circle packings of other conformal types

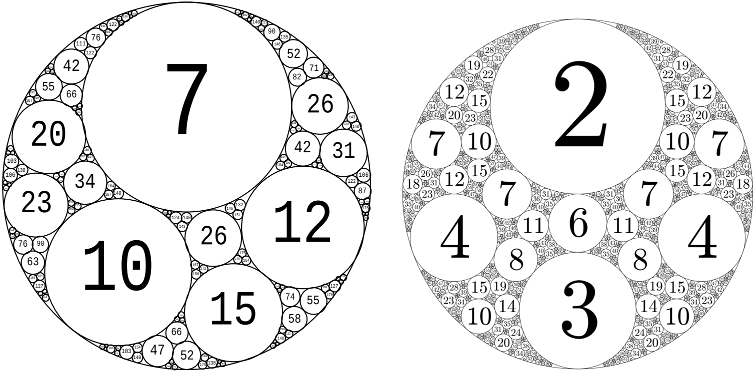

Figure: Guettler and Mallows' Apollonian 3-circle packing and Stange's $\mathbb{Q}[\sqrt{-2}]$-Apollonian packing

Other integral circle packings

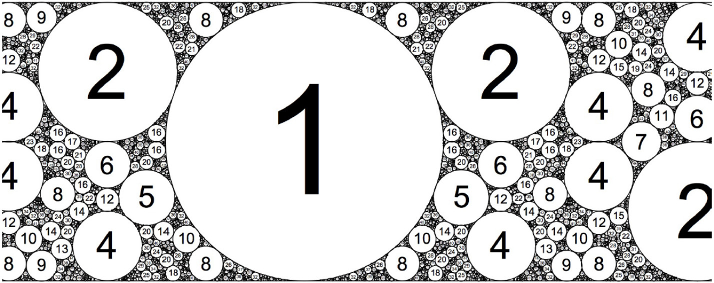

Figure: Kontorovich-Nakamura's integral crystallographic packing

A basic questions

Question

What integers arise as curvatures from an integral circle packing?

A brief overview of the work on Apollonian packings

Theorem (Descartes)

The curvatures $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ from any four mutually tangent circles in an Apollonian packing satisfy the following relation:

$$Q(\kappa_1, \kappa_2, \kappa_3, \kappa_4)$$
$$=2(\kappa_1^2 + \kappa_2^2 + \kappa_3^2 + \kappa_4^2) - (\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4)^2 = 0$$

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$$Q(\kappa_1, \kappa_2, \kappa_3, \kappa_4) = 2(\kappa_1^2 + \kappa_2^2 + \kappa_3^2 + \kappa_4^2) - (\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4)^2 = 0$$

Corollary

Fix $\kappa_2, \kappa_3, \kappa_4$. The two solutions $\kappa_1^{(1)}, \kappa_1^{(2)}$ for κ_1 in the quadratic equation

$$Q(\kappa_1, \kappa_2, \kappa_3, \kappa_4) = 0$$

satisfies

$$\kappa_1^{(1)} + \kappa_1^{(2)} = 2(\kappa_2 + \kappa_3 + \kappa_4)$$

Corollary

Fix a quadruple of curvatures of four mutually tangent circles $\mathbf{r} = (\kappa_1, \kappa_2, \kappa_3, \kappa_4)$. *The set of curvatures K is given by* $\cup_{i=1}^{k} \langle \mathcal{A} \cdot \mathbf{r}, e_i \rangle$, *where* $\mathcal{A} = \langle S_1, S_2, S_3, S_4 \rangle < O_Q$, *and*

$$S_1 = \begin{pmatrix} -1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, S_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 2 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$S_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 2 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, S_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 2 & -1 \end{pmatrix}$$

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- Choose a spin homomorphism $\rho : PSL(2,\mathbb{C}) \to SO_Q$, and work with $\Lambda = \rho^{-1}(A \cap SO_Q) \subset PSL(2,\mathbb{Z}[i])$, where

$$\Lambda = \left\langle \pm \begin{pmatrix} 1 & 4i \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} -2 & i \\ i & 0 \end{pmatrix}, \pm \begin{pmatrix} 2+2i & 4+3i \\ -i & -2i \end{pmatrix} \right\rangle$$

Definition

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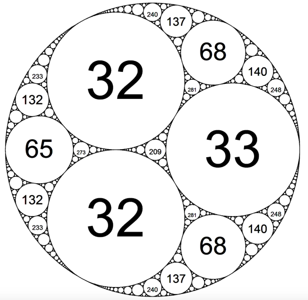

Figure: Admissible congruence classes mod 24 are $\{0, 8, 9, 12, 17, 20\} (\mathrm{mod}\ 24)$

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Why 24?

Let $\Lambda(m)$ be the principle congruence subgroup of Λ of level m: $\left\{\lambda \in \Lambda : \lambda \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod m\right\}$, then for any $q \in \mathbb{Z}^+$,

$$\Lambda(24)(\text{mod } q) = SL(2, \mathbb{Z}[i])(24)(\text{mod } q).$$

The Schmidt arrangement

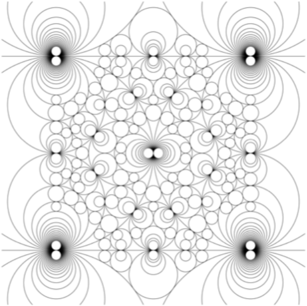

Figure: The orbit $SL(2, \mathbb{Z}[i]) \cdot \widehat{\mathbb{R}}$

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Then $\mathcal{P}$ is an Apollonian packing, and any other integral Apollonian packing is a Möbius transform of $\mathcal{P}$ by a matrix $M \in SL(2, \mathbb{Z}[i])$.

In general, one can replace $\mathbb{Z}[i]$ by any ring of integers $\mathcal{O}_K$ of an imaginary quadratic field K (e.g. $K = \mathbb{Q}[\sqrt{-5}]$, then $\mathcal{O}_K = \mathbb{Z}[\sqrt{-5}]$). Consider the maximal set $\mathcal{P}$ of circles from $SL(2, \mathcal{O}_K)$ satisfying:

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- $\Lambda_{\mathcal{P}}$ contains a congruence subgroup $\Gamma_{\mathcal{P}}$ of $SL(2, \mathbb{Z})$.

Statement of the problem

Let Δ be the discriminant of $\mathcal{O}_K$. For any $g = \begin{pmatrix} A_g & B_g \\ C_g & D_g \end{pmatrix} \in PSL(2, \mathcal{O}_K)$, g sends the horizontal line $\widehat{\mathbb{R}} + \frac{\sqrt{\Delta}}{2}$ to a circle of curvature

$$\kappa\left(g\left(\widehat{\mathbb{R}} + \frac{\sqrt{\Delta}}{2}\right)\right) = \sqrt{-\Delta}|C_g|^2 + 2\Im(\overline{C_g}D_g)$$

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The Problem

Let $\Lambda = \langle \mathcal{S} \rangle$ be a finitely generated subgroup of $PSL(2, \mathcal{O}_K)$, and Λ contains a congruence subgroup Γ of $PSL(2, \mathbb{Z})$. Let $M \in PSL(2, \mathcal{O}_K)$. Study the set of integers

$$\mathcal{K} = \frac{\kappa(M\Lambda(\widehat{\mathbb{R}} + \frac{\sqrt{\Delta}}{2}))}{\sqrt{-\Delta}} = \left\{ \frac{\kappa(M\lambda(\widehat{\mathbb{R}} + \frac{\sqrt{\Delta}}{2}))}{\sqrt{-\Delta}} : \lambda \in \Lambda \right\},$$

where Δ is the discriminant of $\mathcal{O}_K$.

The main theorem

Theorem (Fuchs-Stange-Z)

Let $\mathcal{K}(N) = \mathcal{K} \cap [0, N]$*. There exists a positive integer L, such that*

$$\#\mathcal{K}(N) = cN + O(N^{1-\eta})$$

for some $\eta > 0$*, where*

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Conjecture (Local-global conjecture)

$$\#\mathcal{K}(N) = cN + O(1)$$

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Comment: (1) and (2) allows us to use Lax-Phillips' Theory to count points of Λ. (3) allows us to count points of Λ and its congruence subgroups $\Lambda(q)$ with uniform control on the error terms.

Shifted quadratic forms

Fix $\lambda \in \Lambda$. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, the congruence subgroup of $SL(2,\mathbb{Z})$. A computation shows that

$$\frac{\kappa\left(M\lambda\begin{pmatrix} a & b \\ c & d \end{pmatrix}(\mathbb{R}+\frac{\sqrt{\Delta}}{2})\right)}{\sqrt{-\Delta}} = |C_{M\lambda}a + D_{M\lambda}c|^2 + \frac{2\Im(\overline{C_{M\lambda}}D_{M\lambda})}{\sqrt{-\Delta}}$$

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Define $f_\lambda(a, c) = |C_{M\lambda} a + D_{M\lambda} c|^2 + \frac{2\Im(\overline{C_{M\lambda}} D_{M\lambda})}{\sqrt{-\Delta}}$. Then $f_\lambda(a, c) \in \mathcal{K}$ if a, c satisfies some congruence condition and $\gcd(a, c) = 1$.

It is classical that

$$\#\left(\{f_\lambda(a,c) | a, c \in \mathbb{Z}, \gcd(a,c) = 1\} \cap [0, N]\right) \gg_\lambda \frac{N}{(\log N)^{1/2}}$$

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Idea to prove the theorem: Take many such quadratic forms, hoping to cover most of the admissible numbers from $[0, N]$.

Setup of the ensemble

Choose two parameters T, X, where $T = N^{\frac{1}{200}}, X = N^{\frac{99}{200}}$, so that $T^2X^2 = N$. Let $\mathcal{B}_T = \{\lambda \in \Lambda : \|\lambda\| < T\}$, where $\left\|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right\| = \sqrt{a^2+b^2+c^2+d^2}$. Define

$$R(n) = \sum_{\lambda \in \mathcal{B}_T} \sum_{\substack{x,y \le X \\ \gcd(x,y)=1}} \mathbf{1}\{f_\lambda(a,c) = n\}$$

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$R(n) > 0$ for all sufficiently large admissible numbers $\Longrightarrow$ The local-global conjecture.

We have a good understanding of the sum over $R(n)$:

$$\widehat{R}(0) = \sum_{n\in\mathbb{Z}} R(n) = \sum_{n\in\mathbb{Z}} \sum_{\lambda\in\mathcal{B}_T} \sum_{\substack{x,y\le X \\ \gcd(x,y)=1}} \mathbf{1}\{f_\lambda(a,c)=n\}$$

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We also have good estimate for

$$\widehat{R}(\frac{r}{q}) = \sum_{n\in\mathbb{Z}} R(n) e\left(\frac{rn}{q}\right)$$

for q small.

The circle method

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The main contribution of the above integral comes from $\mathfrak{M}$, the union of small neighborhoods of $\frac{r}{q}$, with $q < Q_0$, where Q_0 is a small power of T (recall T is a small power of N).
Write

$$\begin{aligned} R(n) =& \int_{\mathfrak{M}} \widehat{R}(\theta) e(-n\theta) d\theta + \int_{[0,1]-\mathfrak{M}} \widehat{R}(\theta) e(-n\theta) d\theta \\ =& M(n) + E(n) \end{aligned}$$

The stratagy of the proof

The total input

$$\widehat{R}(0) = \sum_{\gamma \in \mathcal{B}_T} \sum_{\substack{x,y \leq X \\ (x,y)=1}} \asymp T^{2\delta} X^2,$$

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Indeed, the major arc analysis shows that for each $n \in [N/2, N]$ admissible,

$$M(n) \gg \frac{T^{2\delta} X^2}{T^2 X^2} = T^{2\delta - 2}.$$

The stratagy of the proof

We can not show $E(n) \ll T^{2\delta-2-\eta}$ for each $n \in [N/2, N]$ admissible, but we can show $E(n)$ is small on average, by establishing an l^2 bound:

$$\sum_{n\in\mathbb{Z}} E(n)^2 = \int_{[1,0]-\mathfrak{M}} |\widehat{R}(\theta)|^2 d\theta \ll T^{4\delta-4-\eta} N.$$

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This implies that for most $n \in [N/2, N]$ admissible, $R(n) = M(n) + E(n) > 0$, with a power savings on the exceptional set.

Major arc analysis

To estimate $\int_{\mathfrak{M}} \widehat{R}(\theta)e(-n\theta)d\theta$, we evaluate $\widehat{R}(\frac{r}{q})$ when q is small. The main player is the λ-sum:

$$\begin{aligned}
\widehat{R}(\frac{r}{q}) &= \sum_{\substack{x,y\le X \\ \gcd(x,y)=1}} \sum_{\lambda\in\mathcal{B}_T} e(f_\lambda(x,y)\frac{r}{q}) \\
&= \sum_{\substack{x,y\le X \\ \gcd(x,y)=1}} \sum_{\lambda_0\in\Lambda/\Lambda(q)} e(f_{\lambda_0}(x,y)\frac{r}{q}) \sum_{\substack{\lambda\in\mathcal{B}_T \\ \lambda\equiv\lambda_0 (\mathrm{mod}\ q)}} 1
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Effective lattice point counting (Lee-Oh, Vinogradov, Mohammadi-Oh) $\Longrightarrow$

$$\sum_{\substack{\lambda\in\mathcal{B}_T \\ \lambda\equiv\lambda_0 (\mathrm{mod}\ q)}} 1 = \frac{c}{\#\Lambda/\Lambda(q)} T^{2\delta} + O(T^{2\delta-\epsilon})$$

It is important that there exists $\epsilon > 0$ independent of q.

Spectral gap property of Λ

Definition

Let $\{X_i\}_{i\in\mathbb{N}}$ be a family of k-regular, finite, connected graphs with $|X_i| \to \infty$. Let M_i be the adjacency matrix of X_i. It has eigenvalues

$$k = \lambda_0(M_i) > \lambda_1(M_i) \geq \lambda_2(M_i) \geq \cdots \geq \lambda_s(M_i) \geq -k.$$

$\{X_i\}_{i\in\mathbb{N}}$ is an *expander family* if $\exists \epsilon > 0$ such that $k - \lambda_1(M_i) \geq \epsilon$ for all i.

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Let $G = \langle \mathcal{S} \rangle$ is a finitely generated, infinite subgroup of $GL_n(\mathbb{Z})$, and let $A \subset \mathbb{Z}^+$. G has *spectral gap property* with respect to A if $\{Cay(G/G(q), \mathcal{S})\}_{q \in A}$ is a family of expanders.

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- *The connected component of $Zcl(G)$ is perfect,*
  $A = \{q : q \text{ square free}\}$ (Salehi Golsefidy-Varju), or
  $A = \{p^m : m \in \mathbb{Z}^+\}$ (Salehi Golsefidy)

Spectral gap property of Λ

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Lemma (Varju)

Let G be a finite group with a finite symmetric generating set $\mathcal{S}$. Suppose $G_1, \cdots, G_l \leq G$ with $S \cap G_i$ generates G_i, and that for each $g \in G$, there exist $g_i \in G_i$ such that $g = g_1 g_2 \cdots g_l$. Then

$$|\mathcal{S}| - \lambda_1(G, \mathcal{S}) \geq \min_{1 \leq i \leq l} \left\{ \frac{|\mathcal{S} \cap G_i|}{|\mathcal{S}|} \cdot \frac{|\mathcal{S}| - \lambda_1(G_i, \mathcal{S} \cap G_i)}{2l^2} \right\}$$

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In our application, $G = \Lambda/\Lambda(q)$, G_i are conjugates of $\Gamma/\Gamma(q)$.

A generalization

Theorem (Salehi Golsefidy-Z)

Let Λ_1 and Λ_2 be two finitely generated subgroups of $GL_n(\mathbb{Z})$. For $i = 1, 2$, let $Zcl(\Lambda_i)^\circ$ be the Zariski-connected component of the Zariski-closure of Λ_i in $GL_n(Q)$. Suppose $\Lambda_2 \leq \Lambda_1$ and the normal closure of $Zcl(\Lambda_2)^\circ$ in $Zcl(\Lambda_1)^\circ$ is $Zcl(\Lambda_1)^\circ$. Then if Λ_2 satisfies the spectral gap property with respect to some $A \subset \mathbb{Z}^+$, then Λ_1 satisfies the spectral gap property with respect to A.

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Corollary

Let $\Lambda \leq GL_n(\mathbb{Z})$. Assume $Zcl(\Lambda)^\circ$ is perfect:

$$[Zcl(\Lambda)^\circ, Zcl(\Lambda)^\circ] = Zcl(\Lambda)^\circ.$$

Assume further that Λ contains a Zariski-dense subgroup of $SL_d(\mathbb{Z})$, then Λ satisfies the spectral gap property with respect to $\mathbb{Z}^+$.

Minor arc analysis

We evaluate $\widehat{R}(\frac{r}{q})$ for q large. The x, y-sum plays the main role:

$$\widehat{R}(\frac{r}{q}) = \sum_{\lambda \in \mathcal{B}_T} \sum_{x,y \leq X} e(f_\lambda(x,y)\frac{r}{q})$$

$$= \sum_{\lambda \in \mathcal{B}_T} \sum_{x_0, y_0 \in \mathbb{Z}/q\mathbb{Z}} e(f_\lambda(x_0, y_0)\frac{r}{q}) \sum_{\substack{x,y \leq X \\ x \equiv x_0, y \equiv y_0}} 1$$

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We do not get enough cancellation from the exponential sum $\sum_{x_0,y_0\in\mathbb{Z}/q\mathbb{Z}} e(f_\lambda(x_0,y_0)\frac{r}{q})$.

Minor arc analysis

By taking norm square of $\widehat{R}(\frac{r}{q})$ and sum over $r \in \mathbb{Z}/q\mathbb{Z}^\times$, we encounter Kloosterman-Salié type sum

$$\sum_{x \in \mathbb{Z}/q\mathbb{Z}^\times} e\left(\frac{ax + bx^{-1}}{q}\right)\chi(x),$$

where χ is a character of $\mathbb{Z}/q\mathbb{Z}^\times$.
Kloosterman's elementary bound gives

$$\left|\sum_{x \in \mathbb{Z}/q\mathbb{Z}^\times} e\left(\frac{ax + bx^{-1}}{q}\right)\right| \ll q^{3/4}$$