Gap Distribution on Circle Packings

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Our setting

Given a configuration of finitely many circles, we can generate a circle packing by circle inversions.
Examples:

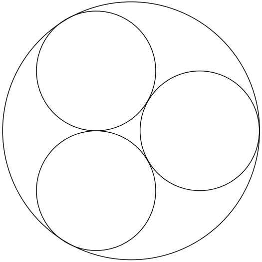

Figure : An Apollonian Configuration

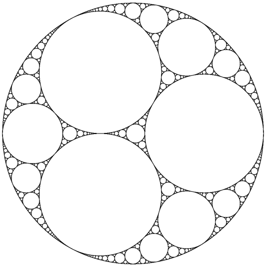

Figure : An Apollonian Circle Packing

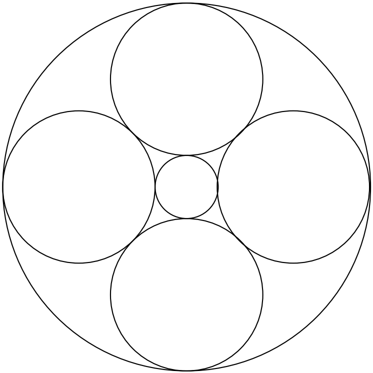

Figure : An Apollonian 3-Circle Configuration

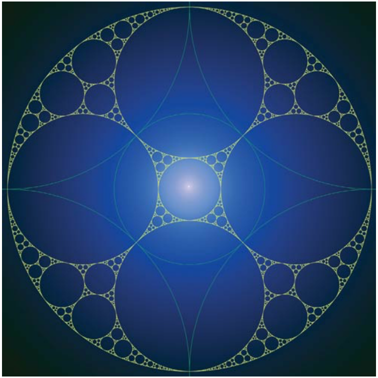

Figure : An Apollonian 3-Circle Packing

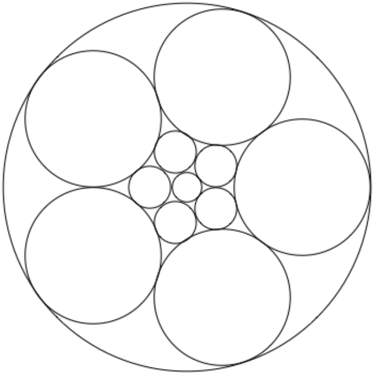

Figure : An Apollonian 9-Circle Configuration

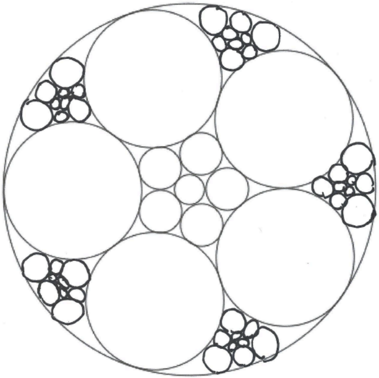

Figure : An Apollonian 9-Circle Packing

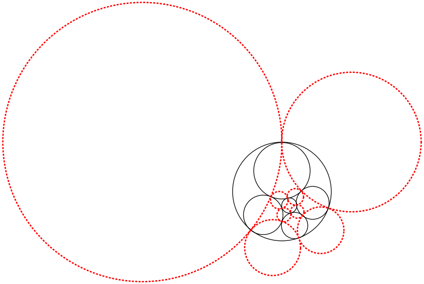

Figure : Dual Circles

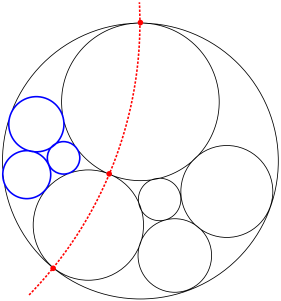

Figure : Circle Inversion

We fix a circle packing $\mathcal{P}$, pick a circle C_0 from $\mathcal{P}$, and let $\mathcal{P}_0$ be the circles tangent to C_0. We want to study how the tangencies on $\mathcal{P}_0$ are distributed.

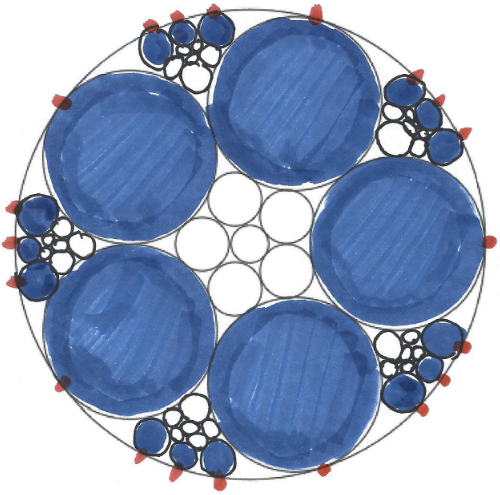

Figure : P_0: the bounding circle
C_0: the blue circles

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Let $\mathcal{I}$ be an arc on C_0, and $l(\mathcal{I})$ be the standard arclength measure. Let $\mathcal{A}_{T,\mathcal{I}}$ be the collection of tangencies on $\mathcal{I}$, whose corresponding tangent circles have curvatures $\leq T$.

Fact

There is a positive constant $c_{\mathcal{P},C_0}$, independent of $\mathcal{I}$, such that, as T goes to infinity,

$$\#\mathcal{A}_{T,\mathcal{I}} \sim l(\mathcal{I}) c_{\mathcal{P},C_0} T$$

Gap Distribution

Let $\{x^i_{T,\mathcal{I}}\}$ be the sequence of tangencies in $\mathcal{A}_{T,\mathcal{I}}$ ordered by counterclockwise direction. The nearest neighbor gaps, or spacings between the tangencies are the arclength distance between $x^i_{T,\mathcal{I}}$ and $x^{i+1}_{T,\mathcal{I}}$, denoted by

$$d(x^i_{T,\mathcal{I}}, x^{i+1}_{T,\mathcal{I}})$$

and the mean spacing is

$$\langle d_{T,\mathcal{I}} \rangle = \frac{l(\mathcal{I})}{\#\mathcal{A}_{T,\mathcal{I}}}$$

We define the gap distribution function to be

$$F_{T,\mathcal{I}}(s) = \frac{1}{\#\mathcal{A}_{T,\mathcal{I}}} \cdot \# \left\{ x^i_T : \frac{d(x^i_T, x^{i+1}_T)}{\langle d_{T,\mathcal{I}} \rangle} \leq s \right\}$$

If $\mathcal{A}_{T,\mathcal{I}}$ were distributed like a lattice, then

$$\lim_{T\to\infty} F_{T,\mathcal{I}}(s) = \begin{cases} 0 & \text{if } s < 1 \\ 1 & \text{if } s \geq 1 \end{cases}$$

If $\mathcal{A}_{T,\mathcal{I}}$ were distributed like a random sequence, then

$$\lim_{T\to\infty} F_{T,\mathcal{I}}(s) = \int_0^s e^{-t} dt = 1 - e^{-s}$$

We will show that the $\mathcal{A}_{T,\mathcal{I}}$ are distributed differently from the above two scenarios. More precisely,

Theorem (Rudnick-Z, Main Theorem)

There exists a continuous piecewise smooth function F, which is independent of $\mathcal{I}$ such that

$$\lim_{T\to\infty} F_{T,\mathcal{I}}(s) = F(s).$$

F is also conformal invariant: let $M \in SL(2,\mathbb{C}), \tilde{\mathcal{P}} = M(\mathcal{P}), \tilde{C}_0 = M(C_0)$ and $\tilde{F}$ be the gap distribution function of $\tilde{C}_0$ from $\tilde{\mathcal{P}}$, then $\tilde{F} = F$.

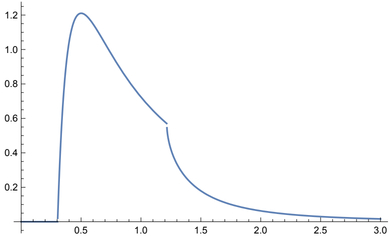

Figure : The density $F^{'}(s)$ of the gap distribution for classical Apollonian packings, which is the same as that discovered by Hall (1970) for Farey sequences.

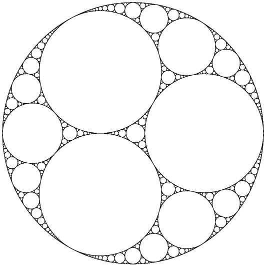

Figure : An Apollonian Circle Packing

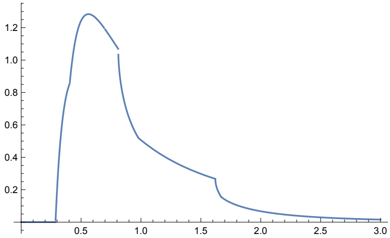

Figure : The density $F^{'}(s)$ of the gap distribution for Apollonian 3-circle packings.

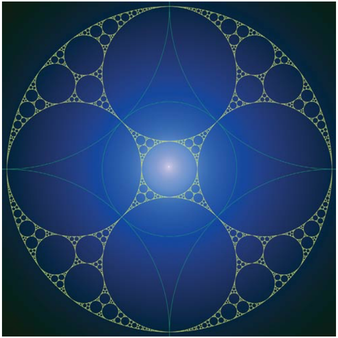

Figure : An Apollonian 3-Circle Packing

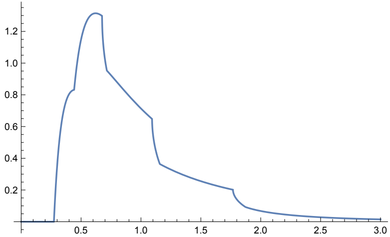

Figure : The density $F'(s)$ of the gap distribution for Apollonian 9-circle packings.

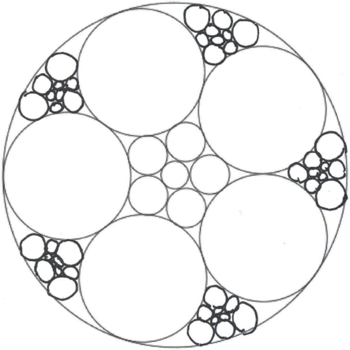

Figure : An Apollonian 9-Circle Packing

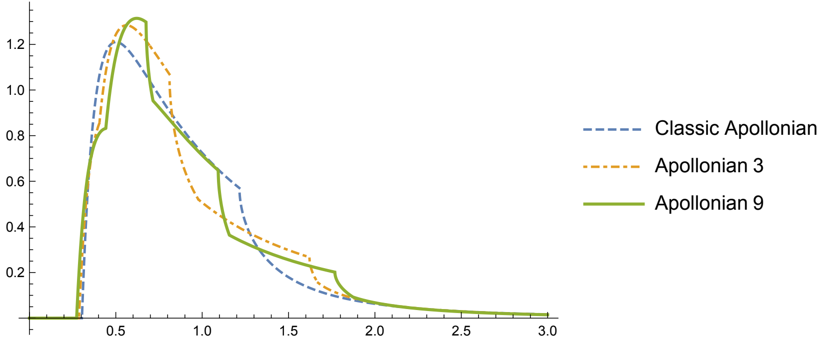

Figure : The density $F^{'}(s)$ of the gap distributions for classical Apollonian, Apollonian 3-circle, and Apollonian 9-circle packings.

Ingredients of the Proof

- Reduction to a hyperbolic lattice point counting problem in $PSL(2, \mathbb{R})$. A typical such problem is to count lattice points asymptotically in an expanding subset of $PSL(2, \mathbb{R})$

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- Tools from spectral theory of automorphic forms (Good's Theorem).

Anton Good's Theorem in Iwasawa Decomposition

Recall the Iwasawa Decomposition: $PSL(2,\mathbb{R}) = G = NAK$ (Iwasawa Decomposition), where

$$N = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{R} \right\}$$

$$A = \left\{ \begin{pmatrix} a^{-\frac{1}{2}} & 0 \\ 0 & a^{\frac{1}{2}} \end{pmatrix} \mid a \in \mathbb{R}^+ \right\}$$

$$K = PSO(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mid \theta \in [0,\pi] \right\}$$

For each $\gamma \in PSL(2,\mathbb{R})$, there exist unique $n(\gamma) \in \mathbb{R}$,$a(\gamma) \in \mathbb{R}^+$,$\theta(\gamma) \in [0,\pi)$ such that

$$\gamma = \begin{pmatrix} 1 & n(\gamma) \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a(\gamma)^{-\frac{1}{2}} & 0 \\ 0 & a(\gamma)^{\frac{1}{2}} \end{pmatrix} \cdot \begin{pmatrix} \cos\theta(\gamma) & -\sin\theta(\gamma) \\ \sin\theta(\gamma) & \cos\theta(\gamma) \end{pmatrix}$$

the Haar measure $d\mu$ of G can be written as $d\mu = dn da d\theta$.

Theorem (Good)

Assume Γ is a lattice in $PSL(2,\mathbb{R})$ and has a cusp at ∞. Let $\mathcal{I}$ be a bounded interval in $\mathbb{R}$, and $\mathcal{J}$ be an interval in $[0,\pi)$, then as $T \to \infty$,

$$\#\{\gamma \in \Gamma \mid {\scriptstyle a(\gamma)\leq T, n(\gamma)\in\mathcal{I}, \theta(\gamma)\in\mathcal{J}}\} \sim \frac{l(\mathcal{I})l(\mathcal{J})}{\pi V(\Gamma)} T.$$

Corollary

Suppose ∞ is a cusp for Γ. For any compact subset Ω in the right half plane $\{(c,d) \mid c \geq 0\}$with piecewise smooth boundary,

$$\#\left\{\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \in \Gamma : n(\gamma) \in \mathcal{I} \text{ , } (c_\gamma, d_\gamma) \in \sqrt{T}\Omega\right\}$$

$$\sim \frac{2l(\mathcal{I})m(\Omega)}{\pi V(\Gamma)} T$$

as $T \to \infty$, where m is the standard Lebesgue measure in $\mathbb{R}^2$.

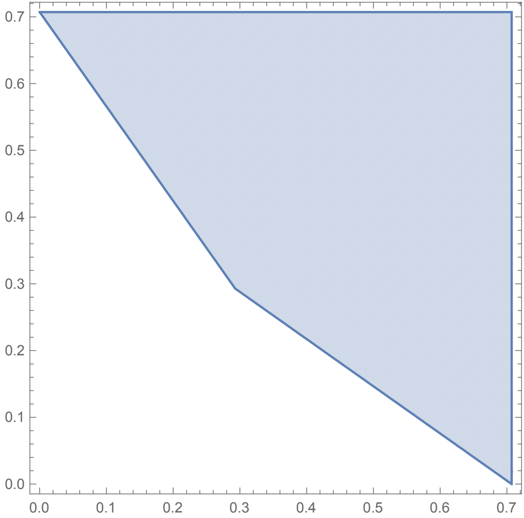

Figure : The region Ω for Apollonian 3-Circle Packings