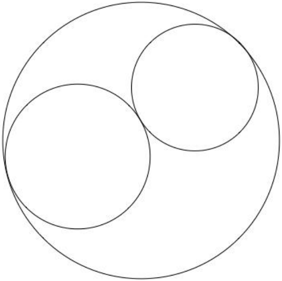
Apollonian Circle Packings and Beyond: Number Theory, Graph Theory and Geometric Statistics

Xin Zhang

3/29/2017

Construction of an Apollonian Circle Packing



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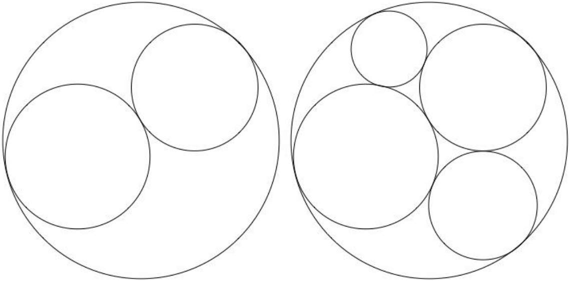

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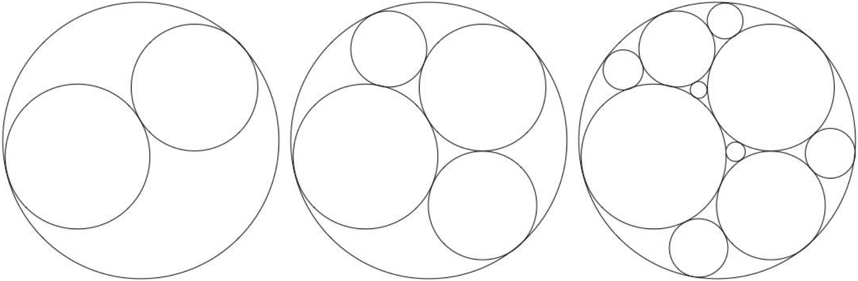


Figure: Construction of an Apollonian circle packing

An integral Apollonian circle packing

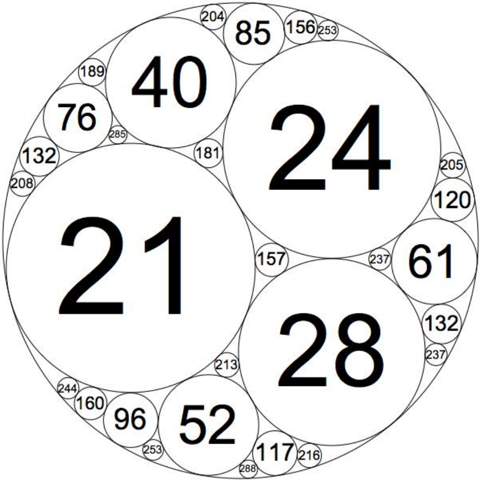


Figure: An integral Apollonian circle packing

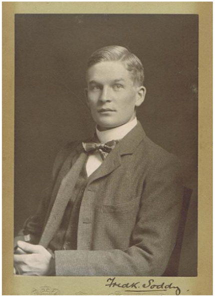
Descartes Quadratic Form

Theorem (Descartes)

The curvatures $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ *from any four mutually tangent circles in an Apollonian packing satisfy the following relation:*

$$2(\kappa_1^2 + \kappa_2^2 + \kappa_3^2 + \kappa_4^2) - (\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4)^2 = 0$$

Frederick Soddy and his poem



The Kiss Precise

FOR pairs of lips to kiss maybe
Involves no trigonometry.
'Tis not so when four circles kiss
Each one the other three.
To bring this off the four must be
As three in one or one in three.
If one in three, beyond a doubt
Each gets three kisses from without.
If three in one, then is that one
Thrice kissed internally.

Four circles to the kissing come.
The smaller are the benter.
The bend is just the inverse of
The distance from the centre.
Though their intrigue left Euclid dumb
There's now no need for rule of thumb.
Since zero bend's a dead straight line
And concave bends have minus sign,
The sum of the squares of all four bends
Is half the square of their sum.

To spy out spherical affairs
An oscular surveyor
Might find the task laborious,
The sphere is much the gayer,
And now besides the pair of pairs
A fifth sphere in the kissing shares.
Yet, signs and zero as before,
For each to kiss the other four
The square of the sum of all five bends
Is thrice the sum of their squares.

F. Soddy.

Figure: Soddy and his poem *The Kiss Precise* published in the journal *Nature*

Counting with multiplicity vs counting without multiplicity

Question (Counting with Multiplicity)

How many circles are there with curvatures bounded by T?

Counting with multiplicity vs counting without multiplicity

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Question (Counting without Multiplicity)

How many integers $< T$ appear in an integer Apollonian circle packing? What are they?

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Theorem (Kontorovich-Oh, JAMS, 2013)

Fix an Apollonian circle packing $\mathcal{P}$, and let $N(T)$ be the number of circles with curvatures $< T$. Then as $T \to \infty$,

$$N(T) \sim c_{\mathcal{P}} T^{\delta},$$

where $c_{\mathcal{P}} > 0$ depends on $\mathcal{P}$, and $\delta \approx 1.305688$ is the Hausdorff dimension of $\mathcal{P}$.

Counting without multiplicity

Definition

For a set of integers $\mathcal{Z}$, we define $\mathcal{A}_{\mathcal{Z}}$ the admissilbe set of $\mathcal{Z}$ by

$$\mathcal{A}_{\mathcal{Z}} = \{n \in \mathbb{Z} \mid n (mod\, q) \in \mathcal{Z} (mod\, q), \forall q \in \mathbb{N}\}$$

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$$n \in \mathcal{A}_{\mathcal{Z}} \text{ if and only if } n \in \mathcal{Z}(mod\ B)$$

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N.B.

If B exists, then $\mathcal{A}_{\mathcal{Z}}$ is a union of congruence classes mod B.

Example

If $\mathcal{Z} = \{x^2 + y^2 | x, y \in \mathbb{Z}\}$, then

$$\mathcal{A}_{\mathcal{Z}} = \{n \in \mathbb{Z} | n \equiv 0, 1, 2 (\text{mod } 4)\},$$

and 4 is the obstruction number.

Theorems

Theorem (Fuchs, Thesis, 2010)

For any primitive integral packing $\mathcal{P}$ (i.e. gcd{curvatures}=1), 24 is the local obstruction number.

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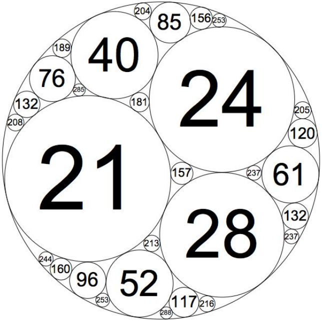


$$\mathcal{A} = \{n \in \mathbb{Z} | n \equiv 0, 4, 12, 13, 16, 21 (\text{mod } 24)\}$$

Theorem (Bourgain-Kontorovich, Inventiones, 2014)

Almost every integer in the admissible congruence classes mod 24 is a curvature.

Other integeral circle packings

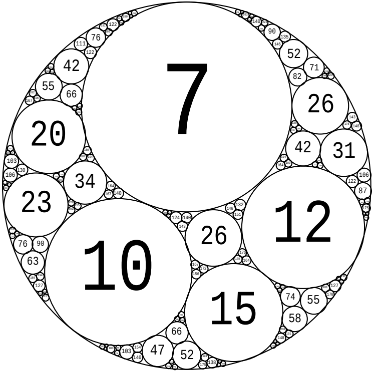
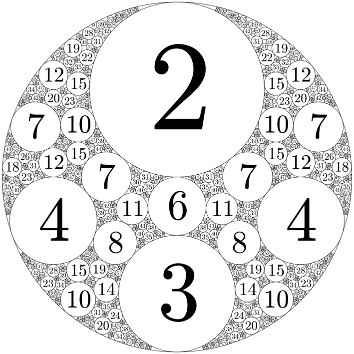


Figure: An Apollonian 3-circle packing, due to Butler-Mallows

Stange's circle packings (built from Bianchi groups)



Butler's circle packings

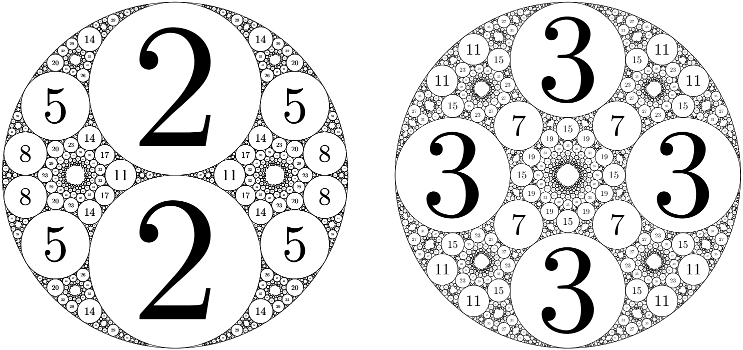


Figure: Butler's circle packings

Schmidt arrangements (built from Bianchi groups by Stange)

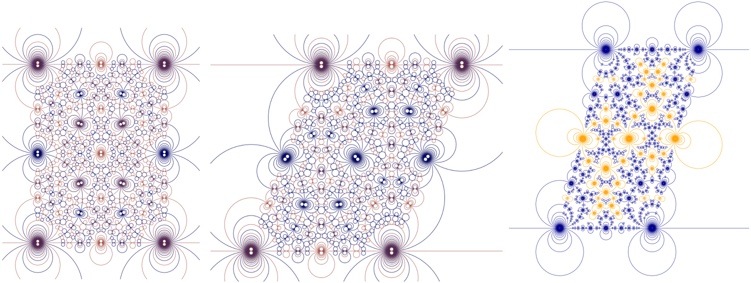


Figure: Schmidt arrangments from $\mathbb{Q}[\sqrt{-2}], \mathbb{Q}[\sqrt{-7}], \mathbb{Q}[\sqrt{-15}]$

Theorems

Theorem (Z, Crelle 2015)

For any primitive integral Apollonian-3 circle packing, the local obstruction is at 8, and almost every admissible integer is a curvature.

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Theorem (Fuchs-Stange-Z)

A density one theorem holds for all known integral circle packings.

Key: the symmetry group

Observation

There exists a finitely generated symmetry group $\Lambda_{\mathcal{P}} = \langle \mathcal{S} \rangle \subset PSL(2, \mathbb{C})$ acting on the circles from $\mathcal{P}$ by Möbius transformation.

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Example

If $\mathcal{P}$ is an Apollonian packing, then

$$\Lambda_{\mathcal{P}} = \left\{ \begin{pmatrix} 1 & 4i \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -2 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 2+2i & 4+3i \\ -i & -2i \end{pmatrix} \right\}$$

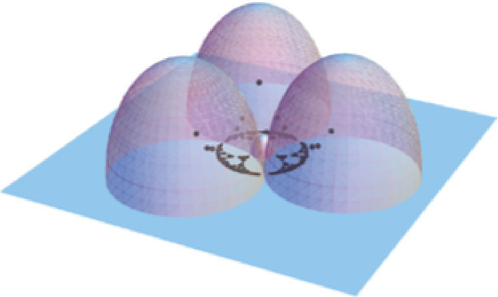

Figure: A fundamental domain and a point orbit for $\Lambda_{\mathcal{P}}$

Spectral property for $\Lambda_{\mathcal{P}}$

Let $\Delta_{\mathbb{H}} = -y^2 \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial y^2} \right)$ be the hyperbolic laplacian.
Then $\Delta_{\mathbb{H}}$ is a nonnegative symmetric operator on $L^2(\Lambda_{\mathcal{P}} \backslash \mathbb{H}^3)$.

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Theorem (Lax-Phillips, Journal of Functional Analysis, 1982)

The spectrum of $\Delta_{\mathbb{H}}$ on $L^2(\Lambda_{\mathcal{P}} \backslash \mathbb{H}^3)$ consists of a discrete part and a continuous part. The discrete part consists of finitely many eigenvalues

$$\lambda_0 = \delta(2 - \delta) < \lambda_1 \leq \cdots \lambda_n < 1.$$

Spectral gap property

Let $\Lambda(d)$ be the principal congruence subgroup of $\Lambda_{\mathcal{P}}$:

$$\Lambda(d) = \left\{ \gamma \in \Lambda | \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\text{mod } d) \right\}.$$

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Definition (Spectral gap property)

We say $\Lambda_{\mathcal{P}}$ has spectral gap property if there exists $\epsilon > 0$ such that for any $q \in \mathbb{Z}^+$, $\lambda_1(q) - \lambda_0(q) > \epsilon$.

Theorem (Bourgain-Gamburd-Sarnak, Acta Math. 2011)

The group Λ has spectral gap property if and only if $\{\mathcal{C}ay(\Lambda/\Lambda(q), \mathcal{S})\}_{q\in\mathbb{N}}$ is a family of expanders.

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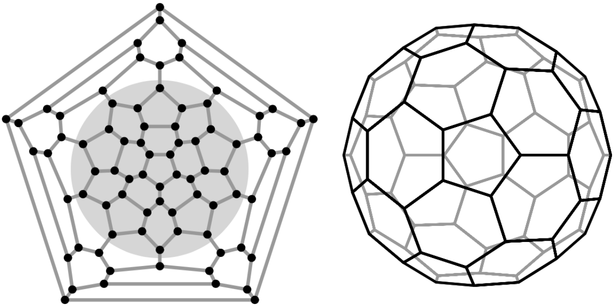

Combinatorial Spectral Gap

Let X be a connected graph, $V(X)$ be the set of vertices.

Definition

The discrete Laplacian Δ_X on $L^2(V(X))$ is defined by

$$\Delta_X f(v) = f(v) - \frac{1}{\#\{w : d(w,v) = 1\}} \sum_{d(w,v)=1} f(w)$$

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N.B.

The operator Δ_X is symmetric on $L^2(V(X))$, and

$$Spec(\Delta_X) = \{\lambda_0(X) = 0 < \lambda_1(X) \leq \cdots \leq \lambda_{|X|-1}(X)\}$$

Definition

For a family of graphs $\{X_i\}_{i\in\mathbb{Z}^+}$ *with* $|V_i| \to \infty$, *if there exists* $\epsilon > 0$ *such that* $\lambda_1(X_i) - \lambda_0(X_i) > \epsilon$, *then we say* $\{X_i\}_{i\in\mathbb{Z}^+}$ *is a family of expanders.*

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- The first explicit construction is given by Margulis: for a finite index subgroup $\Gamma = \langle \mathcal{S} \rangle$ of $SL(2, Z)$ $\{\mathcal{C}ay(\Gamma/\Gamma(q), S)\}_{q\in\mathbb{Z}^+}$ is a family of expanders. (Selberg's 3/16 Theorem)

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Techniques: Kazhdan's property T, propert τ, base change, trace formula

Question (Lubotzky's 1-2-3 Problem)

Is the family $\left\{\mathcal{C}ay\left(\left\langle\begin{pmatrix}1 & \pm i\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ \pm i & 1\end{pmatrix}\right\rangle\right)\right\}_{q\in\mathbb{Z}^+}$ *a family of expanders, where* $i = 1, 2, 3$ *and q restricted to primes?*

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- Generalization 2: Golsefidy-Varjü [GAFA, 2012]: any $\Gamma \subset GL(n, \mathbb{Z})$ has spectral gap property with q square free if and only if $Zcl(\Gamma)_0 = [Zcl(\Gamma)_0, Zcl(\Gamma)_0]$, based on Breuillard-Green-Guralnick-Tao [JEMS, 2015].

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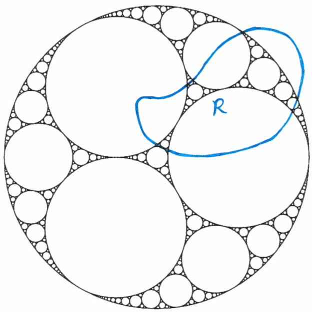
Corollary

Suppose $\Gamma \subset GL(n, \mathbb{Z})$ *and* $Zcl(\Gamma)$ *is simple. If* Γ *contains an arithmetic lattice, then* Γ *has spectral gap property with* $q \in \mathbb{Z}^+$.

Distribution of Circles

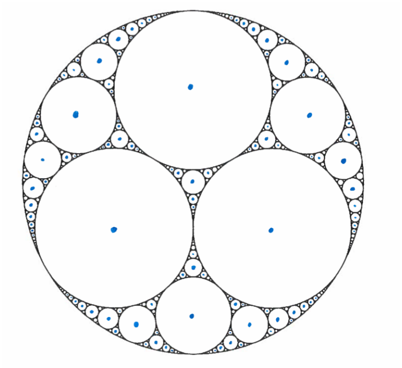
Theorem (Oh-Shah)

There exists a finite Borel measure μ on $\mathbb{C}$. Let $\mathcal{R}$ be any region with smooth boundary in $\mathbb{C}$ and $N_{\mathcal{R}}(T)$ be the number of circles in $\mathcal{R}$ whose curvatures are bounded by T, then as $T \to \infty$, $N_{\mathcal{R}}(T) \sim \mu(\mathcal{R}) T^{\delta}$.



Question

What about the fine scale structures of the Apollonian circle packing? Can we study some spatial statistic, e.g. pair correlation of an Apollonian circle packing?



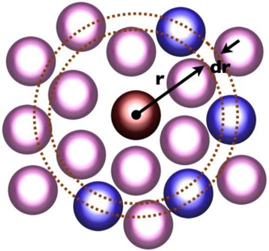
Definition

Let $\{X_i\}_{i \in \mathbb{N}^+}$ be a point process on a Riemannian maniforld $\mathcal{M}^n$, the pair correlation function F_N is defined by

$$F_N(s) = \frac{1}{N} \sum_{1 \le i < j \le N} \mathbb{1}\{d_N(X_i, X_j) < s\}$$

and the pair correlation density is defined by $f_N(s) = F'_N(s)$.

Pair correlation arises from statistical mechanics. It describes how density varies as a function of distance from a reference particle.



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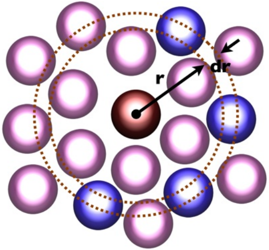

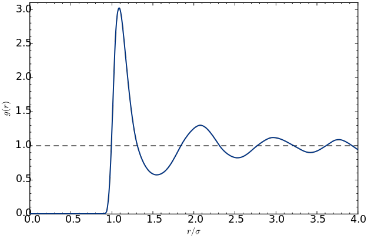

Figure: Pair correlation density of Lennard-Jones fluid

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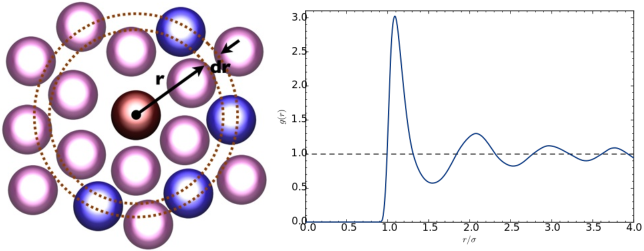

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Kirkwood-Buff solution theory: link the microscopic details to macroscopic properties.

Pair correlations in deterministic sequences

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  Farey sequences (Boca-Zaharescu, Journal of LMS 2004), Euclidean and hyperbolic lattice points (Boca-Popa-Zaharescu, Kelmer-Kontorovich, Marklof-Vinogradov)

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  [Dyson-Montgomery]: (Conjectural) The pair correlation of non trivial zeros of the Riemann zeta function agrees with the pair correlation of the a random Hermitian matrix.
- Limiting pair correlation determined in group orbits:
  Farey sequences (Boca-Zaharescu, Journal of LMS 2004), Euclidean and hyperbolic lattice points (Boca-Popa-Zaharescu, Kelmer-Kontorovich, Marklof-Vinogradov)

Feature in Apollonian circle packings

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Let $\{X_i\}_{i\in\mathbb{Z}^+}$ be the collection of centers, ordered by their radii from big to small, and F_N is the pair correlation function. Is there a limiting pair correlation as $N \to \infty$.

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- Normalization is different.

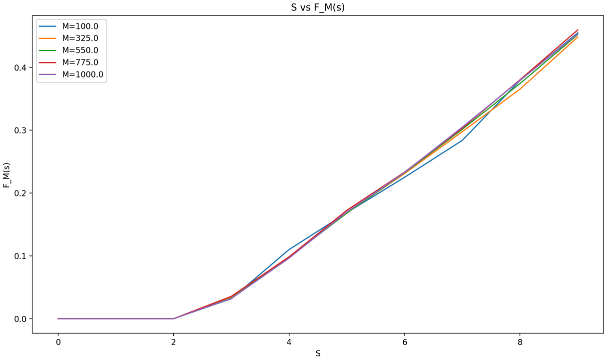

Figure: s vs $F_M(s)$

The ergodic theory behind it:

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- Equidistribution of the horocycle flow. (Kontorovich-Oh, JAMS, 2011; Mohammadi-Oh, JEMS, 2015)

A related problem

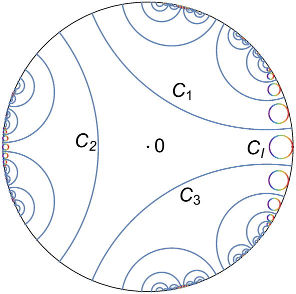

Figure: A Schottky group generated by three reflections. Each C_i cuts $\partial\mathbb{D}$ with the shorter arc of length $\frac{7\pi}{12}$; the radius of C_l is 0.08.

Theorem

There exists a limiting nearest neighbor spacing function h, where h is a continuous function supported at $[c, \infty)$ *for some* $c > 0$ *(this phenomenon is also called repulsion), and* $\int_0^\infty h(x)dx = 1$.

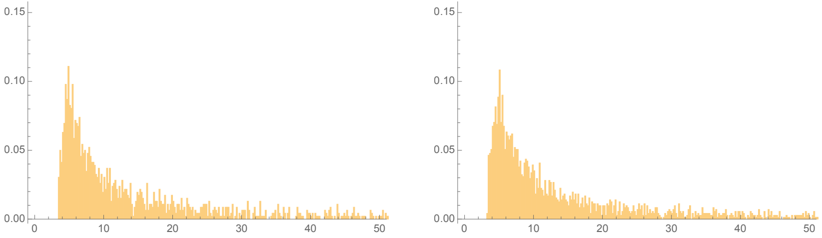

Figure: The histograms of h_N of different N

Fractal cosmology

Fractal sets are natural and universal.

Question

What can one say about the fine scale structure of a spiral galaxy?

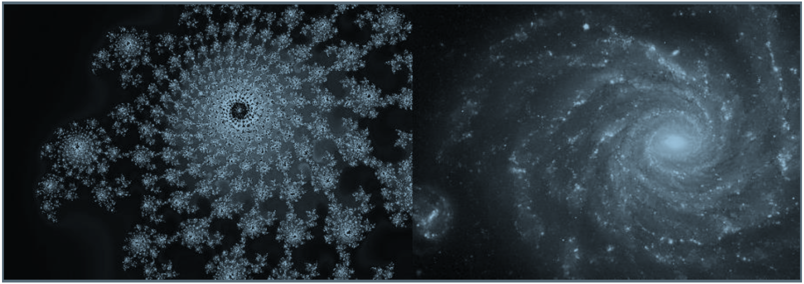

Figure: Image on the left depicts a sub-region of the Mandelbrot; image on the right is the famous Grand Spiral Galaxy (NGC 1232)

Thank you!