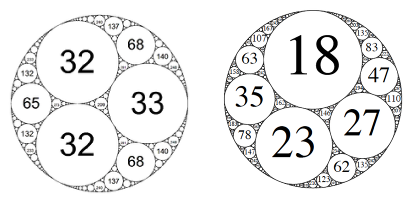
Finding integers from group orbits

Jake Shin, Yike Xu, Catherine Zhang, Xin Zhang
Team Leader: Junxian Li
Faculty mentor: Xin Zhang

Introduction

Motivation

1. **Curvatures of Apollonian Circle Packings**: Apollonian circle packings arise by repeatedly filling the interstices between mutually tangent circles with further tangent circles. It is possible for every circle in such a packing to have integer radius of curvature, and we call such a packing an integral Apollonian circle packing. Consider the curvatures, or reciprocals of the radii. It is already proved that if we start with integer radius in each circle, the curvatures will still be integers. We want to see what integers are there appeared to be the curvatures of Apollonian circle packings.



2. **Zaremba's Conjecture** : Conjecture Z (Zaremba 1972). Every natural number is the denominator of a reduced fraction whose partial quotients are absolutely bounded. That is, there exists some absolute C $>$ 1 so that for each d there is some (b, d) = 1, so that b/d = $[a_1, ..., a_k]$ with max $a_j \in C$.

Connection to Group Orbits

- Given a subgroup $\Gamma < GL_d(\mathbb{Z})$ and a vector $v_0 \in \mathbb{Z}^d$, we are interested in the group orbit:
$$\mathcal{O} := \Gamma \cdot v_0$$
and the set of represented integers:
$$\mathcal{S} := < w_0, \mathcal{O} > \subset \mathbb{Z},$$
- Many problems can be reduced to the study of the integers coming from the group orbits.
  - Curvatures of Apollonian Circle Packings:
  $$\Gamma \text{ is finitely generated and } \Gamma < SL_4(\mathbb{Z})$$
  - Zaremba's Conjecture, Γ is finitely generated and $\Gamma < PSL_2(\mathbb{Z})$, where $\Gamma = \Gamma_{\mathcal{C}} =< \begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix} : a \in \mathcal{C} >$

Local To Global Phenomenon

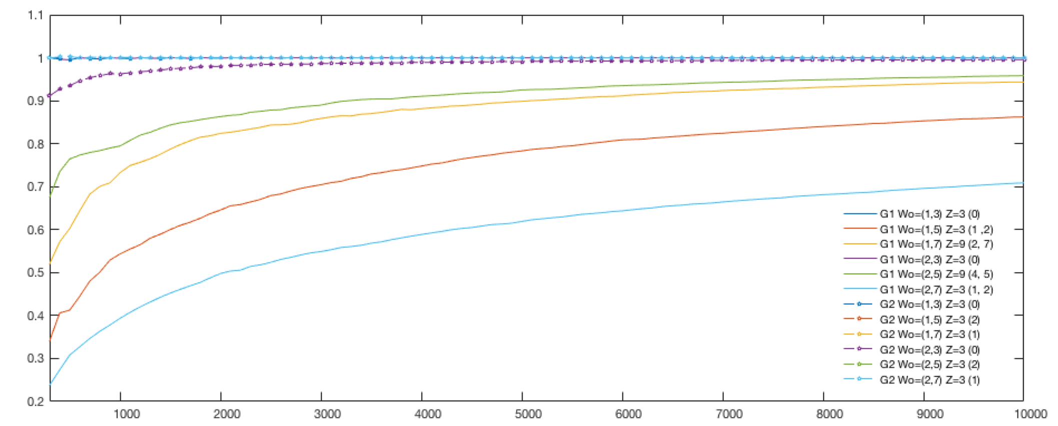
- The Admissible Set:
$$\mathcal{A} = \{n \in \mathbb{Z} : n \in \mathcal{S} \pmod q\} \text{ for any } q \in \mathbb{N}$$
Strong Approximation Property implies that $\exists\ \mathcal{Z} \in \mathbb{Z}$, such that
$$n \in \mathcal{A} \iff n \in \mathcal{A}(mod \mathcal{Z})$$
where $\mathcal{Z}$ is defined as the local obstruction.
- The local global conjecture:
$$n \in \mathcal{A} \iff n \in \mathcal{S}$$

Observations

A weaker version of the local global conjecture is:
$$r_m := \frac{|\mathcal{S} \cap [-m, m]|}{|\mathcal{A} \cap [-m, m]|} \to 1, \text{ as } m \to \infty.$$

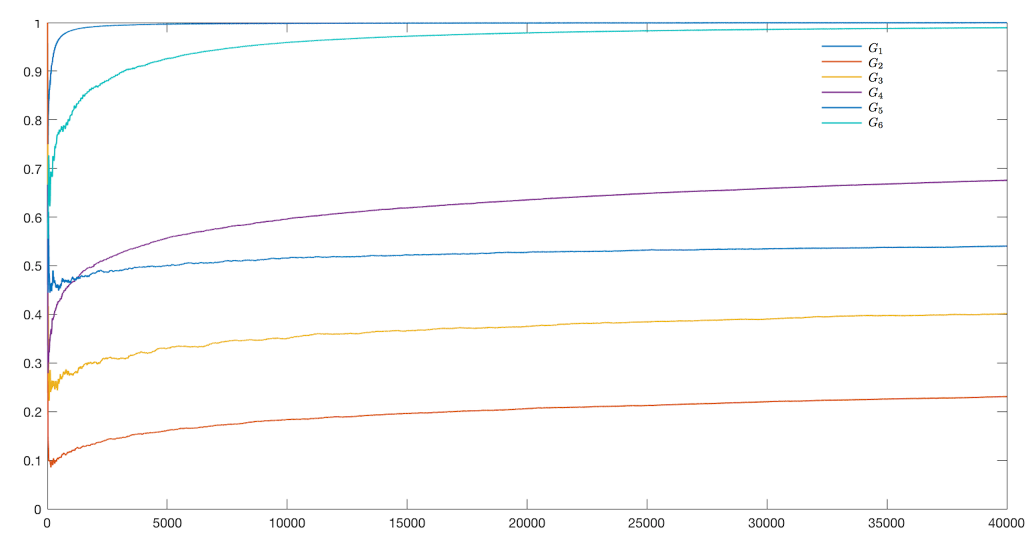
We investigated the behavior of r_m in the following ways:

- How does r_m differ from one subgroup of $SL_2(\mathbb{Z})$ to another?
- How does r_m change for various w_0?



Graphs of r_m for $m \le 10000$ of G_1 and G_2 with various w_0, where $G_1 =< \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} >$ and $G_2 =< \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} >$ and Z represents the local obstruction and the congruent classes of each case.

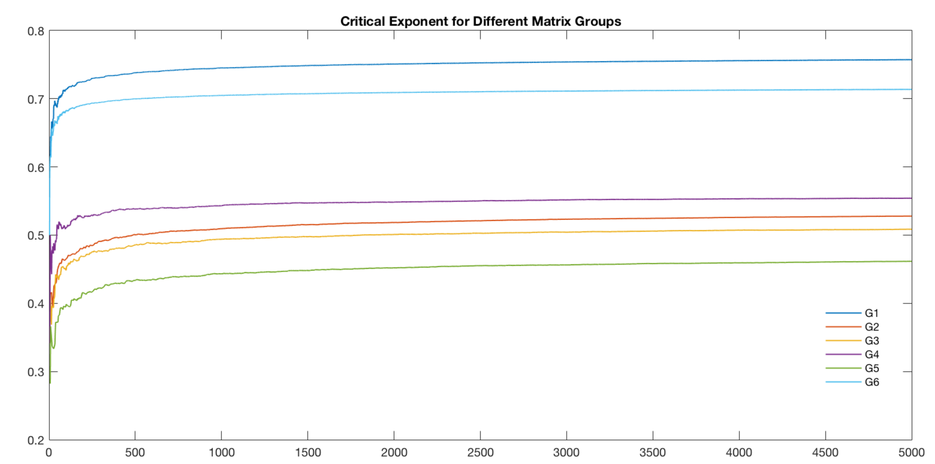
- We notice that for most of the cases, $r_m \to 1$ as $m \to \infty$.
- G_2 has a lower growth rate.



Graphs of r_m for $m \le 40000$ of Different Groups with w_0 $(2, 5)$.

- For $G_1 =< \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 9 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} >$, $r_m \to 0.9996$, as $m \to 40000$.
- For $G_2 =< \begin{pmatrix} 1 & 7 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix} >$, $r_m \to 0.2327$, as $m \to 40000$.
- For $G_3 =< \begin{pmatrix} 1 & 7 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} >$, $r_m \to 0.3995$, as $m \to 40000$.
- For $G_4 =< \begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} >$, $r_m \to 0.674$, as $m \to 40000$.
- For $G_5 =< \begin{pmatrix} -2 & 5 \\ -3 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 9 & 1 \end{pmatrix} >$, $r_m \to 0.5410$, as $m \to 40000$.
- For $G_6 =< \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} >$, $r_m \to 0.9903$, as $m \to 40000$.
- The differences of the limit of r_m is due to the different structure of these subgroups, especially the growth property.

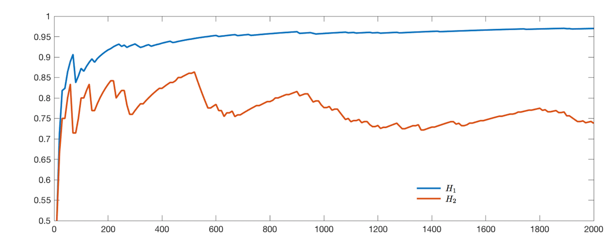
The **Critical Exponent** δ: Let $N(T) := \#\{\gamma \in \Gamma : \|\gamma\| \le T\}$, then $N(T) \sim C_\Gamma T^{2\delta}$. The critical exponent controls the rate of growth of $N(T)$. And it is believed that if $\delta \ge \frac{1}{2}$, then the local-global conjecture should hold. Based on the results of our experiment, the lower the δ is, the longer time it takes for r_m to approach 1, given $r_m \to 1$. This suggests that if the conjecture is true, it is difficult to prove the conjecture when δ is really small.



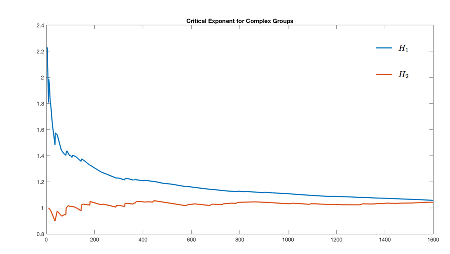
Critical Exponents Imposed by Different Matrix Groups

- G1: $\delta \approx 0.7644$.
- G2: $\delta \approx 0.5582$.
- G3: $\delta \approx 0.5391$.
- G4: $\delta \approx 0.5709$.
- G5: $\delta \approx 0.4823$.
- G6: $\delta \approx 0.5391$.

- In the case of $SL_2(\mathbb{Z}[i])$, we experimented with two subgroups H_1
$$< \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 3i \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 3i & 1 \end{pmatrix} >$$
and H_2
$$< \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix} > .$$
The integers are from the quadratic form $c_1^2 + c_2^2 + d_1^2 + d_2^2$ in complex matrices $\begin{pmatrix} a_1 i + a_2 & b_1 i + b_2 \\ c_1 i + c_2 & d_1 i + d_2 \end{pmatrix}$



The weak local-global conjecture for H_1 and H_2.



Critical Exponent for H_1 and H_2.

Future Directions

Theoretical:

- For a fixed subgroup Γ, what is the connection between w_0 and r_m? How is the local obstruction related to the r_m?
- For different subgroups, how does δ affect r_m?

Computational:

- Improvement our computational algorithm to make it more efficient.

References

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