On the error term in an asymptotic formula for
the symmetric square $L$-function

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Abstract. Recently Wu proved that for all prime $q$,

$$
\sum_f L(1, \text{sym}^2 f) = \frac{\pi^4}{432} q + O(q^{27/28} \log^B q)
$$

where $f$ runs over all normalized newforms of weight 2 and level $q$.
Here we show that $27/28$ can be replaced by $9/10$.

1. Introduction. Let $q$ be a prime and

$$
\Gamma_0(q) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{Z}) : q|c \right\}.
$$

We denote by $S_2(q)$ the space of all holomorphic cusp forms for $\Gamma_0(q)$ of weight 2. With respect to the inner product

$$
\langle f, g \rangle = \int_{\Gamma_0(q) \backslash \mathbb{H}} f(z)\overline{g(z)}\, dx dy,
$$

$S_2(q)$ is a finite dimensional Hilbert space, and there is an orthogonal basis $B_2(q)$ (which is the set of all normalized newforms in $S_2(q)$) such that

(i) each $f \in B_2(q)$ is a common eigenvector of all Hecke operators $T_n$ with $(n, q) = 1$,
i.e. when $f \in B_2(q)$ and $(n, q) = 1$,

$$
T_nf = \lambda_f(n)f.
$$

(ii) the Fourier expansion of $f \in B_2(q)$ is

$$
f(z) = \sum_{n=1}^{\infty} \lambda_f(n)\sqrt{n}e(nz)
$$

where $e(\alpha) = e^{2\pi i \alpha}$, $\lambda_f(n)$ is the eigenvalue in (i) if $(n, q) = 1$ and $\lambda_f(n)^2 = l^{-1}\lambda_f(m)^2$ if $n = lm$ where $l$ is a power of $q$ and $(m, q) = 1$ (see [3, (2.19) and (2.24)]).

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For the properties of $\lambda_f(n)$, it is known that they are all real, and satisfy the Deligne’s bound $|\lambda_f(n)| \leq \tau(n)$. (Here and in the sequel $\tau(n) = \sum_{d|n} 1$ is the divisor function.) Moreover we have

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \epsilon_q(d) \lambda_f\left(\frac{mn}{d^2}\right)$$

where $\epsilon_q$ is the principal character mod $q$. In particular, we see that $\lambda_f(1) = 1$.

Associated to each $f \in B_2(q)$, we define the symmetric square $L$-function by

$$L(s, \text{sym}^2 f) = \zeta_q(2s) \sum_{n=1}^{\infty} \lambda_f(n^2)n^{-s} \quad \text{for } \Re s > 1,$$

where $\zeta_q(s) = \prod_{p \nmid q} (1 - p^{-s})^{-1}$. This $L$-function extends to an entire function over $\mathbb{C}$ and it satisfies a functional equation; more precisely, let us write

$$\Lambda(s, \text{sym}^2 f) = \left(\frac{q}{\pi^{3/2}}\right)^s \Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s+2}{2}\right)L(s, \text{sym}^2 f),$$

then we have $\Lambda(s, \text{sym}^2 f) = \Lambda(1-s, \text{sym}^2 f)$. Analogous to the Riemann zeta function, the values attained by $L(s, \text{sym}^2 f)$ in the critical strip are interesting. Particularly for $s = 1$ and all large prime $q$, we have the asymptotic formula

$$\sum_{f \in B_2(q)} L(1, \text{sym}^2 f) = \frac{\pi^4}{432} q + O\left(q^{9/10} \log^\beta q\right)$$

for some constants $0 < \alpha < 1$ and $\beta > 0$. Here, we are concerned with the size of the error term. In [1], Akbary proved that $\alpha = 45/46$ is admissible and recently Wu gives an improvement to $\alpha = 27/28$ (see [5]). Our purpose is to show the refinement below.

**Theorem**  Let $q$ be a prime. There is an absolute constant $c > 0$ such that

$$\sum_{f \in B_2(q)} L(1, \text{sym}^2 f) = \frac{\zeta(2)^3}{2\pi^2} q + O\left(q^{9/10} \log^\beta q\right).$$

*(Note that $\zeta(2)^3/(2\pi^2) = \pi^4/432$.)

Remark. In decimal form we have $\frac{45}{46} \approx 0.978$, $\frac{27}{28} \approx 0.964$ and $\frac{9}{10} = 0.9$.

2. Some Preparation.

**Lemma 1** Let $A > 1$ be any fixed constant and $q \ll y \ll q^A$ but $y \notin \mathbb{Z}$. We have

$$L(1, \text{sym}^2 f) = \zeta_q(2) \sum_{n \leq y} \lambda_f(n^2) \frac{n}{y} + O\left(q(y^{-1} + (\frac{2}{y})^{2/7})\right)$$

where $\epsilon > 0$ is an arbitrarily small constant and the implied constant in the $O$-term depends on $\epsilon$. 

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Proof. This follows from the truncated Perron’s formula. Using the estimate
\[ \Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s+2}{2}\right) \asymp |t|^{(3\sigma+1)/2}e^{-3\sigma|t|/4} \]
for \( s = \sigma + it \) where \( \sigma \ll 1 \) and \( |t| \gg 1 \), we can derive from the functional equation the convexity bound: for \( 0 \leq \sigma \leq 1 \),
\[ (4) \quad L(\sigma + it, \sym^2 f) \ll (q|t|^{3/2})^{1-\sigma+\epsilon}. \]
By [2, Lemma 12.1], we see that for any \( T \gg 1 \),
\[ (5) \quad \zeta_q(2) \left( \sum_{n \leq y} \frac{\lambda_f(n^2)}{n} \right) = \frac{\zeta_q(2)}{2\pi i} \int_{y-iT}^{y+iT} \sum_{n=1}^{\infty} \frac{\lambda_f(n^2) y^n}{s} \, ds + O(y^\epsilon \sum_{n=1}^{\infty} \frac{\tau(n)^2}{n^{1+\epsilon}} \min(1, (T|\log y/n|)^{-1})). \]
To evaluate the \( O \)-term, we split the summation over \( n \) into three pieces: \( n \leq y/2, n \geq 3y/2 \) and \( y/2 < n < 3y/2 \). As \( |\log(y/n)| \gg 1 \) in the first two pieces, these two sums are \( O(T^{-1}y') \). The third one is
\[ \ll y'T^{-1} \sum_{y/2 < n < 3y/2 \atop |n-y| \geq 1} |y-n|^{-1} + y^{-1+\epsilon} \ll y'(T^{-1} + y^{-1}). \]
Thus, the overall contribution is absorbed in the \( O \)-term in our lemma.

From (2), we can replace \( \sum_{n=1}^{\infty} \lambda_f(n^2) n^{-(1+s)} \) in (5) by \( \zeta_q(2+2s)^{-1} L(1+s, \sym^2 f) \). Then we apply the residue theorem to the rectangular contour with vertices at \( \epsilon \pm iT \) and \(-1/2 + \epsilon \pm iT\). The integral in (5) equals a sum of two terms: the main term \( L(1, \sym^2 f) \) from the pole at \( s = 0 \), and the remainder term which is
\[ \ll \int_{-1/2+\epsilon}^{\epsilon} \left| \frac{L(1+\alpha + iT, \sym^2 f)}{\zeta_q(2+2\alpha + 2iT)} \right| \frac{y^{\alpha}}{T} \, d\alpha + y^{-1/2+\epsilon} \int_{-iT}^{iT} \left| \frac{L(1/2 + \epsilon + it, \sym^2 f)}{\zeta_q(1+2\epsilon + i2t)} \right| \, dt / (1+|t|). \]
Using the bound \( \zeta(\sigma + it)^{-1} \ll \log(1+|t|) \) for \( \sigma \geq 1 \) and \( |t| \gg 1 \), the two \( O \)-terms are
\[ \ll (qT)^\epsilon (y^{-1/2}q^{1/2}T^{3/4} + T^{-1}). \]
The proof is complete after setting \( T = (y/q)^{2/7} \).

Our next task is to extend the admissible range in [5, Lemma 2]. To this end, we modify the mean square estimate result in [4, Corollary 1]. Suppose \( M \leq q^9 \) and \( \{a_n\}_{1 \leq n \leq M} \) is a sequence of complex numbers. Then by taking \( a_n = 0 \) for \( M < n \leq q^9 \), [4, Proposition 1] with \( N = q^9 \) gives
\[ (6) \quad \sum_{f \in B_2(q)} \left| \sum_{n \leq M} a_n \rho_f(n) \right|^2 \ll q^{9} (\log q)^{15} \sum_{n \leq M} |a_n|^2 \]
where \( \rho_f(n) = \sum_{n^2 = n} \zeta_q(m) \lambda_f(l^2) \). (Note that \( B_2(q) = S_2(q)^* \) in [4] for prime \( q \).)
Lemma 2 Let $M \gg 1$ and suppose that $\{a(n)\}_{M<n \leq 2M}$ satisfies

$$a(n) \ll \frac{(\tau(n) \log n)^A}{n}$$

for some constant $A > 0$. There exists a constant $B = B(A) \geq 0$ such that

$$\sum_{f \in B_2(q)} \left| \sum_{M<n \leq 2M} a(n)\lambda_f(n^2) \right|^2 \ll \max(1, q^9 M^{-1}) \log^B(qM).$$

The implied constant depends on $A$.

Proof. When $M \geq q^9$, it follows immediately from [4, Corollary 1] (by taking $N = M$). Consider the case $M < q^9$. From [4, (16)], we have

$$S := \sum_{f \in B_2(q)} \left| \sum_{M<n \leq 2M} a(n)\lambda_f(n^2) \right|^2 = \sum_{f \in B_2(q)} \left| \sum_{l<M} a_l \rho_f(l) \right|^2$$

where

$$a_l = \sum_{\sqrt{M/l} < m \leq \sqrt{2M/l}} \mu(m) \epsilon_q(m) a(lm^2) \ll \frac{(\tau(l) \log 2l)^A}{l} \sum_{\sqrt{M/l} < m \leq \sqrt{2M/l}} \frac{(\tau(m) \log 2m)^{2A}}{m^2} \ll (Ml)^{-1/2} (\log Ml)^B$$

(see the proof of [4, Corollary 1] as well). $B$ denotes an unspecified positive constant depending on $A$ and its value may differ at each occurrence in the proof. By (6),

$$S \ll q^9 (\log q)^{15} \sum_{l<M} (Ml)^{-1} (\log Ml)^B \ll q^9 M^{-1} \log^B(qM).$$

Define for $1 \leq x < y$,

$$\omega_f(x, y) = \sum_{x \leq n < y} \frac{\lambda_f(n^2)}{n}.$$

Lemma 3 Let $x > 0$ and $x < y \ll q^A$ for some constant $A > 0$. Suppose $r \geq 1$ is a fixed integer satisfying $x^r \geq q^9$. Then there exists a constant $D = D(r) > 0$ such that

$$\sum_{f \in B_2(q)} \omega_f(x, y)^{2r} \ll (\log q)^D$$

where the implied constant depends on $A$ and $r$.
Proof. Following the argument in the proof of [4, Lemma 4], one can show that
\[ \omega_f(x, y)^r = \sum_{x^r \leq mn < y^r} \lambda_f(m^2) \frac{c(m, n)}{mn} \]
where \( c(m, n) \) is independent of \( f \) and \( c(m, n) = 0 \) if \( n \) is not of the form \( n = dn_1 \) where \( d|m \) and \( n_1 \) is squarefull. Moreover, \( |c(m, n)| \leq \tau(mn)^\gamma \) for some integer \( \gamma = \gamma(r) > 0 \) depending on \( r \). Then, we write
\[ \omega_f(x, y)^r = \sum_{H=2^k} \sum_{x^r \leq mn < y^r} \lambda_f(m^2) \frac{c(m, n)}{mn} \]
where the first summation runs over all nonnegative integers \( k \). Define
\[ c_H(m) = \sum_{H \leq n < 2H} \frac{c(m, n)}{n} \]
Then, using \( \sum_{n \leq z} \tau(n) \ll z^{1/2} (\log z)^{2\gamma} \), we have
\[ c_H(m) \ll \tau(m) \sum_{d|m} \frac{1}{d} \sum_{H \leq n < 2H} \frac{\tau(n)^\gamma}{n} \]
\[ \ll \tau(m) \gamma \left( \sum_{d|m} d^{-1} \sum_{H \leq n < 2H} \frac{\tau(n)^\gamma}{n} \right) \]
\[ \ll H^{-1/2} (\tau(m)(\log m)(\log H))^D. \]
(7)
Here we use \( D \) to denote a positive constant (depending on \( r \)) which may assume different values at other places. Making use of (7) for \( H \geq q \),
\[ \omega_f(x, y)^r = \sum_{H=2^k} \sum_{H \leq m < y^r/H} \lambda_f(m^2) \frac{c_H(m)}{m} + O(q^{-1/2} \log D q). \]
Squaring both sides and averaging over all \( f \in B_2(q) \) yields
\[ \sum_{f \in B_2(q)} \omega_f(x, y)^{2r} \ll \left( \sum_{H=2^k} H^{-1} \sum_{H \leq m < y^r/H} \lambda_f(m^2) \frac{c_H(m)\sqrt{H}}{m} \right)^2 + 1 \right) \log D q \]
(8)
as \( (\sum_{i \in I} a_i)^2 \ll |I| \sum_{i \in I} a_i^2 \) and \( |B_2(q)| \ll q \). For each \( H \), we split the range of the summation over \( m \) into dyadic intervals \( M < m \leq 2M \) where \( M \geq x^r/(2H) \). It follows from Lemma 2 and (7) that
\[ \sum_f \left( \sum_{H=2^k} \right. \frac{\lambda_f(m^2) c_H(m)\sqrt{H}}{m} \right)^2 \ll \max(1, q^9 x^{-r} H) \log D q. \]
Inserting it into (8), we conclude that

\[ \sum_{f \in B_2(q)} \omega_f(x, y) 2^r \ll \log q \sum_{H=2^k < q} \max(H^{-1}, q^9 x^{-r}), \]

and our result follows in view of the condition \( x^r \geq q^9 \).

3. **Proof of the Theorem.** Define for \( f \in B_2(q) \), \( w_f = 4\pi(f, f) \), which is a positive real number. We have from [3, Lemma 2.5] that \( w_f = (2\pi^2)^{-1}qL(1, \text{sym}^2 f) \) and from [3, Corollary 2.2] (with \( \tau_3((m, n)) \leq \tau((m, n))^2 \leq \tau(m)\tau(n) \)),

\[ (9) \sum_{f \in B_2(q)} w_f^{-1}\lambda_f(m^2)\lambda_f(n^2) = \delta(m, n) + O(q^{-1}(mn)^{1/2}(\tau(m)\tau(n))^2 \log 2mn) \]

for \( \min(m, n) < q \), where \( \delta(\cdot, \cdot) \) is the Kronecker delta. (Note that \( w_f^{-1} = \omega_f \) in [4].) In particular, \( \sum_f w_f^{-1} \ll 1 \) as \( \lambda_f(1) = 1 \).

We split the sum over \( n \) in Lemma 1 into two subsums \( \sum_{n \leq x} + \sum_{x < n \leq y} \) where \( 1 < x < q < y \). (Our choice will be \( x = q^{9/10} \) and \( y = q^{173/110} \).) Squaring the formula in Lemma 1 together with the bound \( L(1, \text{sym}^2 f) \ll \log q \) (from [4, (18)]), we deduce that

\[ (10) \sum_{f \in B_2(q)} L(1, \text{sym}^2 f) = \frac{q}{2\pi^2} \sum_{f \in B_2(q)} w_f^{-1}L(1, \text{sym}^2 f)^2 \]

where

\[ S_1 = \sum_f w_f^{-1} \left( \sum_{n \leq x} \frac{\lambda_f(n^2)}{n} \right)^2, \]

\[ S_2 = \sum_f w_f^{-1} \left( \sum_{n \leq x} \frac{\lambda_f(n^2)}{n} \right) \left( \sum_{x < n \leq y} \frac{\lambda_f(n^2)}{n} \right), \]

\[ S_3 = \sum_f w_f^{-1} \left( \sum_{x < n \leq y} \frac{\lambda_f(n^2)}{n} \right)^2. \]

It follows from the bound \( w_f^{-1} \ll q^{-1} \log q \) (see [4, (20)]) and Lemma 3 that if \( x^r \geq q^9 \),

\[ S_3 \ll \frac{\log q}{q} \sum_f \omega_f(x, y)^2 \ll (\sum_f \omega_f(x, y)^2)^{1/r} |B_2(q)|^{1-1/r} q^{-1} \log q \]

\[ \ll q^{-1/r} \log c_1 q. \]

Throughout \( c_i, i = 1, 2, \cdots \), denote unspecified positive constants. Using (9), we obtain that for \( x < q \),

\[ S_1 = \sum_{n \leq x} n^{-2} + O(q^{-1} \sum_{m,n \leq x} (mn)^{-1/2} \tau(m)^2 \tau(n)^2 \log 2mn) \]

\[ = \zeta(2) + O(x^{-1} + q^{-1} x \log c_2 q). \]
To treat $S_2$, we split it into two parts: let $z = qx^{-1}$,

$$
S_2 = \sum_f w_f^{-1} \sum_{n \leq z} \frac{\lambda_f(n^2)}{n} \sum_{x < n \leq y} \frac{\lambda_f(n^2)}{n} + \sum_f w_f^{-1} \sum_{z < n \leq x} \frac{\lambda_f(n^2)}{n} \sum_{x < n \leq y} \frac{\lambda_f(n^2)}{n}
$$

(13) \quad = S_{21} + S_{22}, \text{ say.}

By (9), we have, provided that $z \leq x$ (or equivalently $x \geq q^{1/2}$),

$$
S_{21} \ll q^{-1} (\log^c q) \sum_{m \leq z} \sum_{n \leq y} \tau(m)^2 \tau(n)^2 (mn)^{-1/2} \ll \sqrt{y/q} \log^c q.
$$

Applying the argument in (12), we get that

$$
\sum_f w_f^{-1} \left( \sum_{z < n \leq x} \frac{\lambda_f(n^2)}{n} \right)^2 \ll z^{-1} + q^{-1} x \log^c q \ll q^{-1} x \log^c q.
$$

By $ab \ll |a|^2 + |b|^2$ and (11), we have $S_{22} \ll (q^{-1/r} + q^{-1} x) \log^c q$. Hence, by (13),

$$
S_2 \ll (q^{-1/r} + q^{-1} x + \left(\frac{y}{q^x}\right)^{1/2}) \log^c q.
$$

Putting this estimate, (11) and (12) into (10), we infer that as $\zeta_q(2) = \zeta(2) + O(q^{-2})$,

$$
\sum_{f \in B_2(q)} L(1, \text{sym}^2 f) = \frac{\zeta(2)}{2\pi^2} q + qO\left( (q^{-1/r} + q^{-1} x) \log^c q + q^r (x^{-1} + \left(\frac{y}{q^x}\right)^{1/2} + \left(\frac{y}{q^x}\right)^{2/7}) \right).
$$

Subject to the condition $x^r \geq q^9$, we take $x = q^{9/r}$ and select $r = 10$, $x = q^{9/10}$ and $y = q^{173/110}$ by equating $q^{-1/r} = q^{-1} x$. This ends the proof.

References


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