Mean square estimate for twisted automorphic $L$-functions on weight aspect

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Abstract We study the mean square estimate for the twisted automorphic $L$-functions averaged over Hecke eigencuspforms at large weight. The upper bound obtained is sharp, and a direct application yields an unconditional version for a result of Kohnen and Sengupta.

1. Introduction. Let $k$ be an even positive integer. The space of all holomorphic cusp forms of weight $k$ with respect to the full modular group has a basis $B_k$ of normalized Hecke eigencuspforms. More explicitly, for $f \in B_k$, $T_n f = \lambda_f(n)n^{(k-1)/2}f$ where $T_n$ ($n = 1, 2, \cdots$) are the Hecke operators, and $f$ has the Fourier series

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n)n^{(k-1)/2}e(nz)$$

where $e(\alpha) = e^{2\pi i \alpha}$. The eigenvalues $\lambda_f(n)$ are all real, and furthermore, $\lambda_f(1) = 1$ and

$$|\lambda_f(n)| \leq d(n) \quad (1.1)$$

(Deligne’s bound) where $d(n) = \sum_{d|n} 1$ is the divisor function.

Let $\chi$ be a primitive Dirichlet character of conductor $D \geq 1$. We associate each $f$ with a twisted $L$-function

$$L(f \otimes \chi, s) = \sum_{n=1}^{\infty} \chi(n)\lambda_f(n)n^{-s} \quad (\Re s > 1). \quad (1.2)$$

Analogously to the classical $L$-functions, the twisted automorphic $L$-function factors into an Euler product. Moreover, let us define

$$\Lambda(f \otimes \chi, s) = (\frac{D}{2\pi})^s \Gamma(s + (k-1)/2)L(f \otimes \chi, s).$$

The completed function $\Lambda(f \otimes \chi, s)$ can be holomorphically continued to the whole $\mathbb{C}$. It is bounded on any vertical strip, and satisfies the functional equation

$$\Lambda(f \otimes \chi, s) = \epsilon_k(\chi)\Lambda(f \otimes \overline{\chi}, 1-s) \quad (1.3)$$
where $\epsilon_k(\chi) = i^k \tau(\chi)^2 / D$ and $\tau(\chi)$ is the Gaussian sum (see [3, Theorem 7.6]). In addition to this similarity, it is conjectured that the Grand Riemann Hypothesis holds, and

$$L(f \otimes \chi, 1/2 + it) \ll_{\epsilon} (k(1 + |t|))^{\epsilon}$$  \hspace{1cm} (1.4)$$

for any $\epsilon > 0$. These (twisted) automorphic $L$-functions play an important role in $GL_2$ theory and are interesting; for example, their values at the central point reveal arithmetical information.

When $\chi$ is quadratic, Kohnen and Sengupta [7] proved that

$$\sum_{f \in B_k} L(f \otimes \chi, 1/2) \ll_{\epsilon, D} k^{1+\epsilon} \hspace{1cm} (k \to \infty, i^k D > 0),$$  \hspace{1cm} (1.5)$$

and deduced that under the assumption $L(f \otimes \chi, 1/2) \ll_{\delta, D} k^{\delta}$ where $\delta \geq 0$,

$$\# \{ f \in B_k : L(f \otimes \chi, 1/2) \neq 0 \} \gg_{\delta, D} k^{1-\delta} / \log k.$$  \hspace{1cm} (1.6)$$

Our main objective is to establish a mean square estimate for $L(f \otimes \chi, 1/2 + it)$ averaging over the basis $B_k$ on the weight aspect. We obtain an estimate (see Theorem 2 below) which supports the validity of (1.4) on the weight $k$. As an application it yields an unconditional lower bound for (1.6).

Now let us fix our notation: denote by $D(k)$ a positive function such that $\log D(k) = o(\log k)$, and define for $f \in B_k$,

$$w_f = \frac{\Gamma(k-1)}{(4\pi)^{k-1} \|f\|^2}$$

where $\|f\|^2 = \int_F y^{k-2} |f(z)|^2 \, dx \, dy$. ($F$ is the fundamental domain for the full modular group.) The first theorem is a weighted form for (1.5) in the critical strip, and the next theorem on mean square estimate is our main result.

**Theorem 1** Let $k$ be any sufficiently large even integer and $1/2 \leq \Re s \leq 1$. Suppose that $\chi$ is a primitive character of conductor $D$ with $1 \leq D \leq D(k)$. For any $\lambda \geq -\Re s$ and any arbitrarily small $\epsilon > 0$,

$$\sum_{f \in B_k} w_f L(f \otimes \chi, s) = 1 + \epsilon_k(\chi) \left( \frac{D}{2\pi} \right)^{1-2s} \frac{\Gamma(1-s+(k-1)/2)}{\Gamma(s+(k-1)/2)} + O(|s|^{2\Re s + \lambda + \epsilon} k^{-\lambda})$$

where the implied constant in the $O$-term depends on $\epsilon$ and $\lambda$ but is uniform for $1 \leq D \leq D(k)$. (Recall $B_k$ is the basis containing all normalized Hecke eigencuspforms.)

Remark. Due to (1.3), there is no loss of generality to assume $1/2 \leq \Re s \leq 1$. 


**Theorem 2** Under the same assumptions as in Theorem 1, we have for any \(\epsilon > 0\),

\[
\sum_{f \in \mathcal{B}_k} w_f |L(f \otimes \chi, 1/2 + it)|^2 \ll \epsilon k^\epsilon (|t| + 1)^{2+\epsilon} \quad (t \in \mathbb{R})
\]

where the implied constant depends only on \(\epsilon\) and \(t\).

The case \(t = 0\) yields (1.5) immediately by Cauchy-Schwarz's inequality and \(\sum_{f \in \mathcal{B}_k} w_f \ll 1\) from (2.1) below with the choice \(m = n = 1\). Another consequence is about the non-vanishing of the central values.

**Corollary 3** Under the assumptions in Theorem 1, it holds that for any \(\epsilon > 0\)

\[
\# \{ f \in \mathcal{B}_k : L(f \otimes \chi, 1/2) \neq 0 \} \gg \epsilon |1 + \epsilon_k(\chi)|^{2k^{1-\epsilon}}.
\]

**Proof.** Taking \(s = 1/2\) in Theorem 1 and using the Cauchy-Schwarz inequality, we obtain that

\[
|1 + \epsilon_k(\chi)|^2 - O(k^{-1}) \leq \left( \sum_{f \in \mathcal{B}_k} w_f L(f \otimes \chi, 1/2) \right)^2
\]

\[
\leq \left( \sum_{f \in \mathcal{B}_k} w_f \right) \left( \sum_{f \in \mathcal{B}_k} w_f |L(f \otimes \chi, 1/2)|^2 \right).
\]

Together with the bound \(w_f \ll \log k/k\), it follows that for any \(\epsilon > 0\),

\[
\sum_{f \in \mathcal{B}_k \atop L(f \otimes \chi, 1/2) \neq 0} 1 \gg \epsilon |1 + \epsilon_k(\chi)|^{2k^{1-\epsilon}}.
\]

This completes the proof.

Remark. Note that \(\epsilon_k(\chi) = i^k D/|D|\) for real \(\chi\). This gives (1.6) an unconditional lower bound with essentially the same quality.

Finally we outline the proof of the main result Theorem 2. We represent the value of \(L(f \otimes \chi, 1/2 + it)\) by a fast convergent series. Averaging over \(\mathcal{B}_k\) enables us to apply the Petersson trace formula. It turns out a sum that involves the Kloosterman sum and the Bessel function. This process is quite common, such as in [1] and [5]. These articles focus on the level aspect but not the weight. In [4] there is a tool to treat the average over weight but not the individual. Hence we need other auxiliary tools for our situation. Using the periodicity of the Kloosterman sum, we pass to an exponential sum over an arithmetic progression. Then the resulting exponential integral will be handled by a ‘saddle-point’ theorem, see Lemma 2.1. (A smooth version of the saddle-point theorem is studied by Jutila [6].)
2. Some Preparation. Our key tool is the Petersson trace formula:

\[ \sum_{f \in B_k} w_f \lambda_f(m) \lambda_f(n) = \delta_{m,n} + 2\pi i^{-k} \sum_{c \geq 1} c^{-1} S(m, n, c) J_{k-1}(\frac{4\pi \sqrt{mn}}{c}) \]  

(2.1)

where \( \delta_{m,n} \) is the Kronecker delta, \( S(m, n, c) \) is the Kloosterman sum and \( J_{k-1} \) is the Bessel function of order \( k - 1 \). Note that

\[ |S(m, n, c)| \leq (m, n, c)^{1/2} c^{1/2} d(c). \]  

(2.2)

\((d(\cdot) \text{ is the divisor function.})\) Concerning the Bessel function, we have the integral representations:

(i) \( J_{k-1}(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(k-1)\theta + ix \sin \theta} d\theta \) for \( x > 0 \),

(ii) \( J_{k-1}(x) = \frac{1}{\sqrt{\pi} \Gamma(k - 1/2)} \left( \frac{x}{2} \right)^{k-1} \int_{-1}^{1} (1-t^2)^{k-3/2} e^{ixt} dt. \)

Writing \( f(\theta) = (k-1) \theta + x \sin \theta \), we have \( f'(\theta) \gg k \) for \( x > 0 \) and \( \theta \in [0, \pi/2] \). This yields

\[ \int_{0}^{\pi/2} e^{\pm if(\theta)} d\theta \ll k^{-1} \]

by the first derivative test ([8, Lemma 4.3]). In addition, it follows from the same argument that

\[ \int_{\pi/2-k^{-\delta}}^{\pi/2} e^{ix \sin \theta - i(k-1)\theta} d\theta \ll k^{-1} \]

for \( x \ll k^{1+\delta} \) where \( 0 < \delta < 1 \). Dividing the \( \theta \)-integral in (i) into suitable ranges, we see that for any \( 0 < \delta < 1 \) and \( 0 < x \ll k^{1+\delta} \),

\[ J_{k-1}(x) = \Re \left\{ \frac{1}{\pi} \int_{0}^{\pi/2-k^{-\delta}} e^{-i(k-1)\theta + ix \sin \theta} d\theta + O(k^{-1}). \right\} \]  

(2.3)

Trivial estimate of (ii) gives

\[ J_{k-1}(x) \ll \frac{1}{\Gamma(k - 1/2)} \left( \frac{x}{2} \right)^{k-1}. \]  

(2.4)

The next lemma is another main tool.

**Lemma 2.1** Let \( p, q \in \mathbb{C} \) with \( \Re p, \Re q \ll 1 \) and \( 0 < \theta < 1 \) be any fixed number. Suppose \( 1 \leq c^2 \leq Q \leq K^\theta \). Then, for any \( \epsilon > 0 \),

\[ \int_{0}^{1} \left| \sum_{\substack{K < m, n \leq 2K \\text{and} \ n \equiv \overline{a} \pmod{Q}}} m^{-1/2-p} n^{-q \epsilon(\frac{2\lambda}{c} \sqrt{mn})} \right| d\lambda \ll (|p| + 1)(|q| + 1) \frac{Q}{c} K^{1/2+\epsilon-\Re(p+q)} \]

where the implied constant depends on \( \epsilon \) only.
Proof. We only need to consider the case for large $K$. Integration by parts gives

$$\sum_{\substack{A<h\leq B \\ h \equiv \gamma(\Delta)}} h^{-(u+v)}e(\sqrt{h}\phi)$$

$$= B^{-v} \sum_{\substack{A<h\leq B \\ h \equiv \gamma(\Delta)}} h^{-u}e(\sqrt{h}\phi) + v \int_A^B y^{-v-1} \sum_{\substack{A<h\leq y \\ h \equiv \gamma(\Delta)}} h^{-u}e(\sqrt{h}\phi) dy.$$

Applying it with $u = 1/2$, $v = p$ for the sum over $m$, and $u = 0$, $v = q$ for the sum over $n$, the integrand is expressed as

$$\sum_{K<m,n \leq 2K \\ m \equiv a, \ n \equiv b} m^{-1/2-p}n^{-q}e(\frac{2\lambda}{c} \sqrt{mn})$$

$$= (2K)^{-(p+q)} \sum_{K<m \leq 2K \ \ m \equiv a} m^{-1/2} \sum_{K<n \leq 2K \ \ n \equiv b} e(\frac{2\lambda}{c} \sqrt{mn})$$

$$+ p(2K)^{-p} \int_K^{2K} x^{-1-p} \sum_{K<m \leq x \ \ m \equiv a} m^{-1/2} \sum_{K<n \leq 2K \ \ n \equiv b} e(\frac{2\lambda}{c} \sqrt{mn}) dx$$

$$+ q(2K)^{-p} \int_K^{2K} y^{-1-q} \sum_{K<m \leq x \ \ m \equiv a} m^{-1/2} \sum_{K<n \leq y \ \ n \equiv b} e(\frac{2\lambda}{c} \sqrt{mn}) dy$$

$$+ pq \int_K^{2K} \int_K^{2K} x^{-1-p} y^{-1-q} \sum_{K<m \leq x \ \ m \equiv a} m^{-1/2} \sum_{K<n \leq y \ \ n \equiv b} e(\frac{2\lambda}{c} \sqrt{mn}) dx dy$$

Integrating with respect to $\lambda$, we obtain for some $M, N \in [K, 2K]$,

$$\int_0^1 \left| \sum_{K<m,n \leq 2K \ \ m \equiv a, \ n \equiv b} m^{-1/2-p}n^{-q}e(\frac{2\lambda}{c} \sqrt{mn}) \right| d\lambda$$

$$\ll (|p| + 1)(|q| + 1)K^{-\Re(p+q)} \int_0^1 \left| \sum_{K<m \leq M \ \ m \equiv a} m^{-1/2} \sum_{K<n \leq N \ \ n \equiv b} e(\frac{2\lambda}{c} \sqrt{mn}) \right| d\lambda. \ (2.5)$$

We proceed to transform the sum over $n$ in (2.5) by an extension of [8, Lemma 4.7], namely, for any $\alpha \in \mathbb{Z}$ and $\beta \in \mathbb{N}$,

$$\sum_{\substack{a<n<b \ \ n \equiv \alpha(Q)}} e(f(n)) = \sum_{Qf'(b)-1/2<v<Qf'(a)+1/2} \int_a^b e(f(x) - \frac{\nu x}{Q}) dx \cdot \frac{1}{Q} e\left(\frac{\alpha \nu}{Q}\right)$$

$$+ O(\log(Q(f'(a) - f'(b)) + 2))$$

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where $f$ is twice continuously differentiable with decreasing $f'$. This yields

$$
\sum_{\substack{K < n \leq N \\ n \equiv b(Q)}} e\left(\frac{2\lambda}{c} \sqrt{mn}\right) = Q^{-1} \sum_{\mathcal{U}(m) < \nu < \mathcal{V}(m)} e\left(\frac{\nu b}{Q}\right) \int_{K}^{N} e\left(\frac{2\lambda}{c} \sqrt{mt} - \frac{\nu t}{Q}\right) dt \ + \ O(\log K)
$$

(as $c^2 \leq Q \leq K^\theta$) where

$$
\mathcal{U}(m) = \frac{\lambda Q}{c} \sqrt{\frac{m}{N}} - \frac{1}{2} \quad \text{and} \quad \mathcal{V}(m) = \frac{\lambda Q}{c} \sqrt{\frac{m}{K}} + \frac{1}{2} \ll \frac{\lambda Q}{c} + 1. \quad (2.6)
$$

The term for $\nu = 0$ (if exists) can be handled by the first derivative test, as follows:

$$
\int_{K}^{N} e\left(\frac{2\lambda}{c} \sqrt{mt}\right) dt \ll \min\left(\frac{c}{|\lambda|} \sqrt{\frac{N}{m}}, N\right) \ll \min(K, c|\lambda|^{-1}).
$$

Thus,

$$
\sum_{\substack{K < n \leq N \\ n \equiv b(Q)}} e\left(\frac{2\lambda}{c} \sqrt{mn}\right) = Q^{-1} \sum_{\nu \geq 1} e\left(\frac{\nu b}{Q}\right) \int_{K}^{N} e\left(\frac{2\lambda}{c} \sqrt{mt} - \frac{\nu t}{Q}\right) dt
$$

$$
+ \ O(K^\epsilon \min(K, c|\lambda|^{-1})). \quad (2.7)
$$

To make use of the sum over $\nu$, we need to evaluate the exponential integral in a quite precise form. To this end, we apply [2, Theorem 2.2] with $F(x) = \lambda K/c$ and $\mu(x) = x/2$. It follows that

$$
\int_{K}^{N} e\left(\frac{2\lambda}{c} \sqrt{mt} - \frac{\nu t}{Q}\right) dt = \sqrt{2m\lambda} \left(\frac{Q}{\nu}\right)^{3/2} e\left(\frac{Q\lambda^2}{\nu c^2 m} - \frac{1}{8}\right)
$$

$$
+ \ O\left(\frac{Q^2}{\nu^2} \sqrt{\frac{\lambda}{cK}} + E_{\nu,m}(\lambda, K) + E_{\nu,m}(\lambda, N)\right) \quad (2.8)
$$

where

$$
E_{\nu,m}(\lambda, h) = c \left(\lambda - \frac{eh}{Q} \sqrt{\frac{h}{m}} + \sqrt{\frac{\lambda}{K}}\right)^{-1}. \quad (2.9)
$$

(Remark. The classical formula [8, Lemma 4.6] is not sufficient, its error term involving the third derivative is too big.)

Inserting (2.8) into (2.7), we have

$$
\int_{0}^{1} \left| \sum_{\substack{K < m \leq M \\ m \equiv a(Q)}} m^{-1/2} \right| \sum_{\substack{K < n \leq N \\ n \equiv b(Q)}} e\left(\frac{2\lambda}{c} \sqrt{mn}\right) |d\lambda| \ll \frac{\sqrt{Q}}{c} I_1 + K^\epsilon I_2 + Q^{-1} I_3 \quad (2.10)
$$
where

\[ I_1 = \int_0^1 \lambda \left| \sum_{K < m \leq M \atop m \equiv a \pmod{Q}} \sum_{\nu \geq 1} \nu^{-3/2} e\left( \frac{Q\lambda^2}{\nu c^2} m + \frac{\nu b}{Q} \right) \right| d\lambda \tag{2.11} \]

\[ I_2 = \sum_{K < m \leq 2K} m^{-1/2} \int_0^1 \left( \min(K, \frac{c}{|\lambda|}) + \frac{Q\sqrt{\lambda}}{\sqrt{cK}} \right) d\lambda \]

\[ I_3 = \sum_{K < m \leq 2K} m^{-1/2} \int_0^1 \sum_{\nu \geq 1 \atop \nu \equiv m \pmod{\nu \in \nu(V)} (m)} (E_{\nu,m}(\lambda, K) + E_{\nu,m}(\lambda, N)) d\lambda \tag{2.12} \]

Appearing \( I_2 \ll c^{K^{1/2+\epsilon}} + Q/\sqrt{c} \). Observing that for any real number \( H \),

\[ \int_0^1 \frac{d\lambda}{|\lambda - H| + \sqrt{\lambda/K}} \ll K^{1/2} \int_{0<\lambda<1} \frac{d\lambda}{\sqrt{\lambda}} + \int_{0<\lambda<1} \frac{d\lambda}{|\lambda - H|} \ll \log K, \]

we see that by (2.6), (2.9) and (2.12), \( I_3 \ll QK^{1/2+\epsilon} \) and hence with \( c^2 \leq Q < K \),

\[ K^\epsilon I_2 + Q^{-1} I_3 \ll cK^{1/2+\epsilon} + Q/\sqrt{c} \ll K^{1/2+\epsilon} Q/c. \tag{2.13} \]

Finally we estimate \( I_1 \). Interchanging the summations, the double sum in (2.11) equals

\[ \sum_{\nu \geq 1 \atop \nu \equiv m \pmod{\nu \in \nu(V)} (m)} \nu^{-3/2} e\left( \frac{\nu b}{Q} \right) \sum_{m \in \mathcal{I}} e\left( \frac{Q\lambda^2}{\nu c^2} m \right) \]

where \( m \) runs over the set

\[ \mathcal{I} = \{ m \equiv a \pmod{Q} : K < m \leq M, \ (\frac{c(\nu-1/2)}{Q\lambda})^2 K < m \leq (\frac{c(\nu+1/2)}{Q\lambda})^2 N \}. \]

Using that for any \( \alpha \in \mathbb{Z} \) and \( \phi \in \mathbb{R} \),

\[ \sum_{n \equiv \alpha \pmod{Q}} e(n\phi) \ll \min(1 + |x - \alpha|/Q, |\sin(\pi Q\phi)|^{-1}), \]

the double sum (in 2.11) is thus

\[ \ll \sum_{\nu \geq 1 \atop \nu \equiv m \pmod{\nu \in \nu(V)} (m)} \nu^{-3/2} \min(K, |\sin(\pi \frac{Q^2\lambda^2}{\nu c^2})|^{-1}). \]

We conclude that

\[ I_1 \ll \int_0^1 \sum_{\nu \geq 1 \atop \nu \equiv m \pmod{\nu \in \nu(V)} (m)} \nu^{-3/2} \lambda \min(K, |\sin(\pi \frac{Q^2\lambda^2}{\nu c^2})|^{-1}) d\lambda \]

\[ \ll \log K. \tag{2.14} \]
The last line follows from the estimate
\[ \int_0^1 \lambda \min(K, |\sin(\pi \alpha \lambda^2)|^{-1}) d\lambda \ll \frac{\alpha + 1}{\alpha} \int_0^{\pi/2} \min(K, |u|^{-1}) du \ll \log K \]
for any \( \alpha \gg 1 \). This completes the proof in view of (2.5), (2.10), (2.13) and (2.14).

**Lemma 2.2** Let \( B, B', C > 0 \) be fixed numbers and \( H \geq 2(B + B' + C) \) be any large number. Suppose \( \alpha, \beta \in \mathbb{C} \) satisfy \(-B' \leq \Re\alpha < B\) and \( |\Re\beta| \leq C \). Then,
\[ \frac{\Gamma(H + \beta + \alpha)}{\Gamma(H + \beta)} \ll (|\beta| + 1)^B H^{\Re\alpha} \frac{\Gamma(B - \Re\alpha)}{|\Gamma(B - \alpha)|} \]
where the implied constant depends on \( B, B' \) and \( C \).

**Proof.** Using Stirling’s formula, one can prove that for any \( \sigma, t \in \mathbb{R} \) with \(|\sigma| \leq C\),
\[ \frac{\Gamma(H + \sigma + it)}{\Gamma(H + it)} \ll (H + |t|)^\sigma. \]
(2.15)

It is known that \( \Gamma(x)\Gamma(y)/\Gamma(x + y) = \int_0^1 t^{x-1}(1 - t)^{y-1} dt \) for \( \Re x \) and \( \Re y > 0 \). Hence,
\[ \frac{\Gamma(H + \beta + \alpha)\Gamma(B - \alpha)}{\Gamma(H + \beta + B)} = \int_0^1 t^{B-\alpha-1}(1 - t)^{H+\beta+\alpha-1} dt \ll \int_0^1 t^{B-\Re\alpha-1}(1 - t)^{H+\Re(\beta+\alpha)-1} dt \]
\[ = \frac{\Gamma(H + \Re(\beta + \alpha))}{\Gamma(H + \Re \beta + B)} \Gamma(B - \Re\alpha) \ll H^{\Re\alpha-B} \Gamma(B - \Re\alpha) \]
by Stirling’s formula. The implied constants depend on \( B, B' \) and \( C \). Our result follows, after applying (2.15) to \( \Gamma(H + \beta + B)/\Gamma(H + \beta) \).

**3. Proof of Theorems.** We assume \( 1/2 \leq \Re s \leq 1 \). Throughout we use \( c_i \) \( (i = 1, 2, \ldots) \) to denote unspecified positive constants, and let \( 0 < \delta < 1, A_0 > (2 + \delta + |\lambda|)/\delta \). (Later we take \( \delta = 1/(1 + \epsilon) \).) We apply the residue theorem to the integral
\[ \frac{1}{2\pi i} \int_{\mathcal{R}} \Lambda(f \otimes \chi, 1/2 + w) G(w) dw \]
where
\[ G(w) = \frac{1}{w} \frac{\Gamma(A_0 - w)\Gamma(A_0 + w)}{\Gamma(A_0)^2}. \]
The contour $\mathcal{R}$ is given by the positively oriented rectangle with vertices at $\pm 2 \pm iT$. After taking $T \to \infty$, we have from the functional equation (1.3) that

$$
(D/(2\pi))^{s} \Gamma(s + (k - 1)/2) L(f \otimes \chi, s)
= \frac{1}{2\pi i} \int_{(2)} \Lambda(f \otimes \chi, s + w) G(w) \, dw + \epsilon_{k}(\chi) \frac{1}{2\pi i} \int_{(2)} \Lambda(f \otimes \chi, 1 - s + w) G(w) \, dw.
$$

This leads to

$$
L(f \otimes \chi, s) = \sum_{n=1}^{\infty} \frac{\lambda_{f}(n)\chi(n)}{n^{s}} V_{s}(n) + \epsilon_{k}(\chi) \sum_{n=1}^{\infty} \frac{\lambda_{f}(n)\chi(n)}{n^{1-s}} V_{1-s}(n) \quad (3.1)
$$

where

$$
V_{s}(y) = \frac{1}{2\pi i} \int_{(2)} \left( \frac{D}{2\pi} \right)^{w+z-s} \frac{\Gamma(z+w+(k-1)/2)}{\Gamma(s+(k-1)/2)} y^{-w} G(w) \, dw. \quad (3.2)
$$

Moving the line of integration to the line $\Re w = A$ with $|A| < A_{0}$, we see that

$$
V_{z}(y) \ll_{A} |s|^{A} y^{-A} D^{A}(k|s|)^{A+\Re(z-s)}. \quad (3.3)
$$

Note that the $\Gamma$-factors in the right side of (3.2) are treated by Lemma 2.2 with $B = A + \epsilon; \text{indeed for } z = s \text{ or } 1 - s,$

$$
\frac{\Gamma(z+w+(k-1)/2)}{\Gamma(s+(k-1)/2)} \ll |s|^{\epsilon}(k|s|)^{A+\Re(z-s)}(|\Im w| + 1)^{1/2-\epsilon} e^{\pi|\Im w|/2}. \quad (3.4)
$$

Shifting the integral path to $\Re w = -1,$ we obtain

$$
V_{z}(y) = \left( \frac{D}{2\pi} \right)^{z-s} \frac{\Gamma(z+(k-1)/2)}{\Gamma(s+(k-1)/2)} + O(yk^{\Re(z-s)-1}). \quad (3.5)
$$

Truncating the series in (3.1) at $N = k^{1+\delta},$ the contribution of the tail is, by using (3.3), (1.1) \text{and } \Re s \geq 1/2,

$$
\ll |s|^{A+\epsilon} D^{A} k^{A} \sum_{n \geq N} d(n) (n^{-(\Re s+A)} + n^{-(1-\Re s+A)}) \quad (3.6)
$$

$$
\ll |s|^{A+\epsilon} D^{A} k^{A} N^{\Re s-A} \log N \ll_{\delta} |s|^{A+\epsilon} k^{-\lambda},
$$

where $A$ is taken to satisfy $A + (1 + \delta)(\Re s - A) = -\lambda - \delta \epsilon.$ Note that as $\lambda \geq -\Re s$, $A \geq \Re s + \epsilon$ which is needed in (3.6). (Also we have used $D = o(\log k)$.). Hence, it follows from (3.1) that

$$
L(f \otimes \chi, s) = \sum_{n \leq k^{1+\delta}} \frac{\lambda_{f}(n)\chi(n)}{n^{s}} V_{s}(n)
+ \epsilon_{k}(\chi) \sum_{n \leq k^{1+\delta}} \frac{\lambda_{f}(n)\chi(n)}{n^{1-s}} V_{1-s}(n) + O(|s|^{A+\epsilon} k^{-\lambda}). \quad (3.7)
$$
Summing over \( f \in B_k \), we obtain by (2.1)
\[
\sum_{f \in B_k} w_f L(f \otimes \chi, s) = V_s(1) + \epsilon_k(\chi)V_{1-s}(1) \\
+ 2\pi i^{-k} \sum_{n \leq k^{1+\delta}} \left( \frac{\chi(n)}{n^s} V_s(n) + \epsilon_k(\chi) \frac{\psi(n)}{n^{1-s}} V_{1-s}(n) \right) \\
\times \sum_{c \geq 1} c^{-1} S(1, n, c) J_{k-1}(\frac{4\pi \sqrt{n}}{c}) + O(|s|^{A+\epsilon} k^{-\lambda}).
\]

From (2.4), we see that for \( n \leq k^{1+\delta}, J_{k-1}(4\pi \sqrt{n}/c) \ll c^k k^{-(1-\delta)k/2} c^{-1} \). Hence, with (3.3), the double sum above is \( \ll c^k k^{-(1-\delta)k/2} \sum_{n \leq k^{1+\delta}} n^{-1-\epsilon} \sum_{c \geq 1} d(c) c^{-3/2} \ll k^{-1} \).

By (3.5), we obtain
\[
\sum_{f \in B_k} w_f L(f \otimes \chi, s) = V_s(1) + \epsilon_k(\chi)V_{1-s}(1) + O(|s|^{A+\epsilon} k^{-\lambda})
\]
\[
= 1 + \epsilon_k(\chi) \left( \frac{D}{2\pi} \right)^{1-2s} \frac{\Gamma(1-s+ (k-1)/2)}{\Gamma(s+(k-1)/2)} + O(|s|^{A+\epsilon} k^{-\lambda}).
\]

This completes the proof of Theorem 1, by taking \( \delta = 1/(1+\epsilon) \). (Recall \( A = ((1+\delta) \Re(s + \lambda)/\delta + \epsilon) \).

Now we move to prove Theorem 2, and thus take \( s = 1/2 + it \). Using (3.3) and (3.7) with \( \delta = 1/(1+\epsilon), \lambda = -1/2 \) (and thus \( A = 1/2 + 3\epsilon \)), it follows that \( L(f \otimes \chi, 1/2+it) \ll (k(|t|+1))^{1/2+\epsilon} \). For simplicity, we write
\[
W_A(m, n) = \frac{1}{(2\pi i)^2} \int_{(A)} \int_{(A)} \frac{\Gamma(k/2 + it + w_1) \Gamma(k/2 - it + w_2)}{\Gamma(k/2 + it)} \left( \frac{D}{2\pi} \right)^{w_1+w_2} \\
\times G(w_1)G(w_2)m^{-w_1-it}n^{-w_2+it} \, dw_1 dw_2
\]
for \( A > 0 \) and
\[
|S_K|^2 = \sum_{K<m,n \leq 2K} \frac{\chi(m)\chi(n)}{\sqrt{mn}} W_A(m, n) \sum_{f \in B_k} w_f \lambda_f(m) \lambda_f(n).
\]
Shifting the line of integration, we have \( W_A = W_{A'} \) for any \( A, A' > 0 \). We take \( \lambda = -1/2 \) in (3.7) and correspondingly \( A = 1/2 + \epsilon \). Cauchy-Schwarz inequality yields that
\[
\sum_{f \in B_k} w_f |L(f \otimes \chi, 1/2+it)|^2 \ll \sum_{K=2^i < k^{1+\delta}} |S_K|^2 \log k + |s|^{1+i} k^{-1}
\]
where \( s = 1/2 + it \) and \( K \) runs over all powers of two greater than zero but less than
\[ k^{1+\delta}. \text{ In view of (3.9), it follows by (2.1) that} \]
\[
|S_K|^2 = \sum_{K<n\leq 2K} \frac{|\chi(n)|^2}{n} W_A(n,n) \\
+ 2\pi i^{-k} \sum_{K<m,n\leq 2K} \chi(m)\overline{\chi(n)} \frac{W_A(m,n)}{\sqrt{mn}} \\
\times \sum_{c\geq 1} S(m,n,c) J_{k-1}(\frac{4\pi \sqrt{mn}}{c}) \tag{3.11}
\]

With the choice \( A = \epsilon \), the first sum is \( \ll (|s|k)^{\epsilon} \) as \( W_\epsilon(n,n) \ll (|s|kD)^{\epsilon} \) from (3.3) and (3.8). When \( K \leq k^{1-2\delta} \), we take \( A = 1 \) in (3.3), and this yields by (2.4),
\[
\sum_{K<m,n\leq 2K} \sum_{c\geq 1} \ll (|t| + 1)^{2+\epsilon} (Dk)^2 \frac{(2\pi k^{1-\delta})^{k-1}}{\Gamma(k-1/2)} \sum_{m,n\leq k^{1-2\delta}} (mn)^{-3/2} \sum_{c\geq 1} d(c)c^{1-k} \\
\ll k^3 k^{-\delta k} (|t| + 1)^{2+\epsilon} \ll k^{-1}(|t| + 1)^{2+\epsilon}. \tag{3.12}
\]

It remains to consider the range \( k^{1-2\delta} < K < k^{1+\delta} \). Similarly to (3.12), one can see (with \( A = 1 \)) that
\[
\sum_{K<m,n\leq 2K} \sum_{c\geq k^{3\delta}} \ll (|t| + 1)^{2+\epsilon} (Dk)^2 \frac{(2\pi k^{1-\delta})^{k-1}}{\Gamma(k-1/2)} \sum_{m,n\leq k^{1+\delta}} (mn)^{-3/2} \sum_{c\geq k^{3\delta}} d(c)c^{(1-k)/3} \\
\ll k^3 k^{-\delta k} (|t| + 1)^{2+\epsilon} \ll k^{-1}(|t| + 1)^{2+\epsilon}. \tag{3.13}
\]

We set \( A = \epsilon \) for the other case. Using (2.3), the remnant of the double sum in (3.11) is
\[
\sum_{K<m,n\leq 2K} \sum_{c<k^{3\delta}} \ll |J| + k^{-1} \sum_{K<m,n\leq 2K} \sum_{c<k^{3\delta}} \frac{|W_\epsilon(m,n)| |S(m,n,c)|}{\sqrt{mn}} \tag{3.14}
\]

where we write \( f_{m,n}(\theta) = 4\pi c^{-1} \sqrt{mn} \sin \theta - (k-1)\theta \) and
\[
J = \sum_{K<m,n\leq 2K} \chi(m)\overline{\chi(n)} \frac{W_\epsilon(m,n)}{\sqrt{mn}} \\
\times \sum_{c\geq k^{3\delta}} S(m,n,c) c \int_0^{\pi/2-k^{-4\delta}} \exp(if_{m,n}(\theta)) d\theta. \tag{3.15}
\]

By (3.3) with \( A = \epsilon \), the sum in the right side of (3.14) is
\[
\ll (|t| + 1)^{2\epsilon k^{2\epsilon-1}} \sum_{K<m,n\leq 2K} (mn)^{-1/2} \sum_{c<k^{3\delta}} d(c) \\
\ll (|t| + 1)^{2\epsilon K k^{3\delta+2\epsilon-1} \log k} \ll (|t| + 1)^{2\epsilon k^{3\delta+\epsilon}}. \tag{3.16}
\]
Using the periodicity of $S(\cdot, \cdot, c)$ and $\chi(\cdot)$, we express (3.15) as

$$J = \sum_{c < k^{3\delta}} \sum_{\alpha_1, \alpha_2 (c)} c^{-1} S(\alpha_1, \alpha_2, c) \sum_{\beta_1, \beta_2 (D)} \chi(\beta_1) \chi(\beta_2)$$

$$\times \sum_{K < m, n < 2K} W_c(m, n) \frac{\sqrt{mn}}{\sqrt{mn}} \int_{0}^{\pi/2-k^{-4\delta}} e\left(\frac{2\sqrt{mn}}{c} \sin \theta\right) e^{-i(k-1)\theta} d\theta$$

where the condition (*) denotes $m \equiv \alpha_1 (c)$, $m \equiv \beta_1 (D)$ and $n \equiv \alpha_2 (c)$, $n \equiv \beta_1 (D)$. The congruence system $m \equiv \alpha_1 (c)$ and $m \equiv \beta_1 (D)$ is solvable if and only if $(c, D)|(\alpha_1 - \beta_1)$. Assume $(c, D)|(\alpha_1 - \beta_1)$. The solution is given by $m \equiv \gamma_1 (cd)$ for some $\gamma_1$ where $d = D/(c, D)$. Similarly, the system $n \equiv \alpha_2 (c)$ and $n \equiv \beta_2 (D)$ is given by $n \equiv \gamma_2 (cd)$ for some $\gamma_2$. Hence, with (3.8) we can write (3.17) as

$$J = \sum_{c < k^{3\delta}} \sum_{\alpha_1, \alpha_2 (c)} c^{-1} S(\alpha_1, \alpha_2, c) \sum_{\beta_1, \beta_2 (D)} \chi(\beta_1) \chi(\beta_2)$$

$$\times \frac{1}{(2\pi i)^2} \int_{(\epsilon)} \int_{(\epsilon)} \frac{\Gamma(k/2 + it + w_1) \Gamma(k/2 - it + w_2)}{|\Gamma(k/2 + it)|^2}$$

$$\times \left(\frac{D}{2\pi}\right)^{w_1+w_2} H(w_1, w_2) G(w_1) G(w_2) dw_1 dw_2$$

where $H(w_1, w_2)$ denotes the integral

$$\int_{0}^{\pi/2-k^{-4\delta}} \sum_{K < m, n < 2K} m^{-1/2-it-w_1} n^{-1/2-it-w_2} e\left(\frac{2\sqrt{mn}}{c} \sin \theta\right) e^{-i(k-1)\theta} d\theta$$

with $d = D/(c, D)$. Substituting $\lambda = \sin \theta$, we have $d\theta = (\cos \theta)^{-1} d\lambda = O(k^{c\delta}) d\lambda$ as $0 < \theta < \pi/2 - k^{-4\delta}$, and hence

$$H(w_1, w_2)$$

$$\ll (|t| + 1)^{2+2\epsilon} k^{c\delta}$$

$$\int_{0}^{1} \left| \sum_{K < m, n < 2K} m^{-1/2-it-w_1} n^{-1/2-it-w_2} e\left(\frac{2\lambda}{c} \sqrt{mn}\right) \right| d\lambda$$

$$\ll (|w_1| + |t|)(|w_2| + |t|) k^{c\delta} K^{\epsilon - \text{Re}(w_1+w_2)}$$

by Lemma 2.1. This gives $J \ll D^7 k^{c\delta} (|t| + 1)^{2+\epsilon}$ by (3.4) and (3.18); together with (3.14) and (3.16), $\sum_{K < m, n < 2K} \sum_{c < k^{3\delta}}$ has an upper bound in this fashion. In view of (3.11) and (3.12), $|S_K|^2 \ll D^7 k^{c\delta} (|t| + 1)^{2+\epsilon}$ for all $K \leq k^{1+\delta}$. Our result follows from (3.10) and $\log D = o(\log k)$ with $\delta = c_{11}\epsilon$. 

12
References


