EXTREME VALUES OF SYMMETRIC POWER L-FUNCTIONS AT 1

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Abstract. We obtain an almost all result on the size of the symmetric mth power L-functions ($m = 1, 2, 3, 4$) for the normalized Hecke eigencuspforms at $s = 1$, which extends results of Elliott and Montgomery & Vaughan on Dirichlet L-functions to higher degree L-functions.

§ 1. Introduction

The study on the extreme values of Dirichlet L-functions at the point 1 has a long and rich history. The research in this topic was originated with a paper of Littlewood [15] in 1928 and was pursued by many authors (cf. [1], [2], [6], [7], [8], [23], [17] and [9]). A very good historical account can be found in [9], where Granville & Soundararajan made a very important new progress on the distribution of the extreme values of $L(1, \chi_d)$ for a real primitive character χ_d of modulus $|d|$.

Among the family of L-functions attached to the automorphic cuspidal representations for $GL_n(\mathbb{Q})$ where $n \geq 1$, the Dirichlet L-functions constitute only a small part corresponding to $n = 1$. The GL_2 class consists of those L-functions associated to holomorphic cusp forms or Maass forms. The symmetric mth power of a GL_2 L-function yields, under Langlands functoriality conjecture if $m \geq 5$, an automorphic GL_{m+1} L-function which is defined as a Euler product of degree $m + 1$ (and thus called a L-function of degree $m + 1$). The properties of these L-functions are of great current interests and their values at 1 are recently delved. Luo [16] investigated the case of symmetric square L-functions for Maass forms with large eigenvalue. Royer [18, 19], Habsieger & Royer [10], Royer & Wu [20] considered the first two symmetric power L-functions attached to holomorphic cusp forms with large squarefree level † while Cogdell & Michel [3] and Royer & Wu [21] considered all the symmetric power L-functions. Besides Lau & Wu [14] studied similar problems in the weight aspect. In this paper we shall further study the extreme values of symmetric power L-functions at 1.

Let us introduce our notation. For a positive even integer k, we denote by $\mathrm{H}_k^*(1)$ the set of all normalized Hecke primitive eigencuspforms of weight

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†An integer n is called squarefree if $p^2 \nmid n$ for all prime numbers p.

k for the modular group $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$. It is a finite set with cardinality

$$|\mathrm{H}_k^*(1)| = \frac{k}{12} + O(1). \tag{1.1}$$

Here the normalization is taken in the way that the Fourier series expansion at the cusp ∞,

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} e^{2\pi i n z} \qquad (\Im m z > 0), \tag{1.2}$$

has its first coefficient equal to one (i.e. $\lambda_f(1) = 1$). Inherited from the Hecke operators, the Fourier coefficient $\lambda_f(n)$ satisfies the following relation

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \lambda_f\left(\frac{mn}{d^2}\right) \tag{1.3}$$

for all integers $m \geq 1$ and $n \geq 1$. According to Deligne [4], for any prime number p there is a (complex) number $\alpha_f(p)$ such that

$$|\alpha_f(p)| = 1 \tag{1.4}$$

and

$$\lambda_f(p^\nu) = \alpha_f(p)^\nu + \alpha_f(p)^{\nu-2} + \cdots + \alpha_f(p)^{-\nu} \tag{1.5}$$

for all integers $\nu \geq 1$. Hence $\lambda_f(n)$ is a real multiplicative function of n.

Associated to each $f \in \mathrm{H}_k^*(1)$, the symmetric mth power L-function ($m \in \mathbb{N}$) is defined as

$$L(s, \mathrm{sym}^m f) := \prod_p \prod_{0 \leq j \leq m} \left(1 - \alpha_f(p)^{m-2j} p^{-s}\right)^{-1} \tag{1.6}$$

for $\sigma > 1$, where and in the sequel σ and τ mean tacitly the real and imaginary part of s, i.e. $s = \sigma + i\tau$. Multiplying out the Euler product, we see that it admits a Dirichlet series representation:

$$L(s, \mathrm{sym}^m f) = \sum_{n=1}^{\infty} \lambda_{\mathrm{sym}^m f}(n) n^{-s} \tag{1.7}$$

for $\sigma > 1$, where $\lambda_{\mathrm{sym}^m f}(n)$ is a multiplicative function. By (1.4) and (1.6), we have for $n \geq 1$,

$$|\lambda_{\mathrm{sym}^m f}(n)| \leq \tau_{m+1}(n). \tag{1.8}$$

As customary $\tau_{m+1}(n)$ denotes the number of solutions in positive integers $n_1, \ldots, n_{m+1}$ of the equation $n = n_1 \cdots n_{m+1}$. The case $m = 1$ in (1.8) is commonly known as Deligne's inequality. For $m = 1, 2, 3, 4$, the symmetric power function $L(s, \mathrm{sym}^m f)$ can be analytically prolonged to $\mathbb{C}$ and satisfies the functional equation

$$L_\infty(s, \mathrm{sym}^m f) L(s, \mathrm{sym}^m f) = \varepsilon(\mathrm{sym}^m f) L_\infty(1-s, \mathrm{sym}^m f) L(1-s, \mathrm{sym}^m f),$$

where $\varepsilon(\mathrm{sym}^m f) = \pm 1$ and $L_\infty(s, \mathrm{sym}^m f)$ is the corresponding gamma factor (cf. [3, Section 1.1]).

In [14], Lau & Wu proved the following results on the extreme values of $L(1,\mathrm{sym}^m f)$ in the weight aspect. Let $m = 1,2,3,4$ and $2 \mid k$. For any $f \in \mathrm{H}_k^*(1)$, under GRH for $L(s,\mathrm{sym}^m f)$, we have

$$\{1+o(1)\}(2B_m^- \log_2 k)^{-A_m^-} \le L(1,\mathrm{sym}^m f) \le \{1+o(1)\}(2B_m^+ \log_2 k)^{A_m^+} \tag{1.9}$$

as $k \to \infty$. In the opposite direction, it was shown unconditionally that there are $f_m^\pm \in \mathrm{H}_k^*(1)$ such that for $k \to \infty$,

$$L(1,\mathrm{sym}^m f_m^+) \ge \{1+o(1)\}(B_m^+ \log_2 k)^{A_m^+}, \tag{1.10}$$

$$L(1,\mathrm{sym}^m f_m^-) \le \{1+o(1)\}(B_m^- \log_2 k)^{-A_m^-}. \tag{1.11}$$

Here (and in the sequel) $\log_j$ denotes the j-fold iterated logarithm. The constants $A_m^\pm$ and $B_m^\pm$ are explicitly evaluated,

$$\begin{cases} A_m^+ = m+1, & B_m^+ = e^\gamma \qquad (m=1,2,3,4), \\ A_m^- = m+1, & B_m^- = e^\gamma \zeta(2)^{-1} \qquad (m=1,3), \\ A_2^- = 1, & B_2^- = e^\gamma \zeta(2)^{-2}, \\ A_4^- = \frac{5}{4}, & B_4^- = e^\gamma B_4'^-, \end{cases} \tag{1.12}$$

where $\zeta(s)$ is the Riemann zeta-function, γ denotes the Euler constant and $B_4'^-$ is a positive constant given by a rather complicated Euler product (cf. [14], (1.16)).

The results in (1.9), (1.10) and (1.11) determine completely, at least under GRH, the order of magnitude of $L(1,\mathrm{sym}^m f)$. Then it is interesting and natural to try removing the assumption of GRH and closing up the gap coming from the factor 2. We shall prove an almost all result towards this delicate problem, which can be regarded as analogues, in the higher degree L-function case, of results of Elliott ([6], [7]) and Montgomery & Vaughan [17] on Dirichlet L-functions. It leads to a consequence that the forms f satisfying (1.10) or (1.11) are rather rare in the sense of being density zero.

In what follows we shall assume k to be any sufficiently large even integer (but the parity will be repeatedly emphasized).

Theorem 1.1. *Let $m \in \{1,2,3,4\}$, $\theta_1 > 0$ and $\theta_2 > 0$ such that $1 - 2\theta_1 - \theta_2 > 0$ and $\theta_3 \in (0,\ \min\{1/2\theta_1 - 1,\ 1\}]$ be fixed. Then for $2 \mid k$ and $z \ge (\log_2 k)^{1/\theta_1}$, we have*

$$L(1,\mathrm{sym}^m f) = \bigg\{1 + O\bigg(\frac{1}{z^{\theta_2}} + \frac{1}{(\log k)^{\theta_3}}\bigg)\bigg\} \prod_{p \le z} \prod_{0 \le j \le m} \bigg(1 - \frac{\alpha_f(p)^{m-2j}}{p}\bigg)^{-1}$$

for all but $O\big(k e^{-z_0^{\theta_1}}\big)$ forms $f \in \mathrm{H}_k^*(1)$, where $z_0 := \min\big\{z, (\log k)^2\big\}$ and the implied constants depend on θ_1, θ_2 and θ_3 only.

Corollary 1.2. *Let $\varepsilon > 0$ be an arbitrarily small positive number, $m \in \{1, 2, 3, 4\}$ and $2 \mid k$. Then there is a subset E_k^* of $\mathrm{H}_k^*(1)$ such that*

$$|\mathrm{E}_k^*| \ll k e^{-(\log k)^{1/2-\varepsilon}}$$

and for each $f \in \mathrm{H}_k^*(1) \smallsetminus \mathrm{E}_k^*$, we have

$$\begin{aligned}\big\{1 + O\big((\log k)^{-\varepsilon}\big)\big\}(B_m^- \log_2 k)^{-A_m^-} &\le L(1, \mathrm{sym}^m f) \\ &\le \big\{1 + O\big((\log k)^{-\varepsilon}\big)\big\}(B_m^+ \log_2 k)^{A_m^+}.\end{aligned}$$

The implied constants depend on ε only.

Remarks. (i) These results can be generalized (with a little extra effort) to $\mathrm{H}_k^*(N)$, where N is squarefree and $\mathrm{H}_k^*(N)$ denotes the set of all normalized Hecke primitive eigencuspforms of weight k for the congruence subgroup $\Gamma_0(N)$. Our method can also be applied to establish similar results in the level aspect for N squarefree and free of small prime factors.

(ii) We consider the case $1 \le m \le 4$ because the required properties of the high symmetric power L-functions are only known in these cases. Other higher degree case will follow along the same line of argument when the (expected) corresponding properties are established.

Our results above are analogues of Theorem 1 of [17] (see also [7]), where the case $L(1, \chi_d)$ was investigated. However their methods seem not to be directly generalized to the symmetric power L-functions. Following their approaches, one can see that correspondingly the key point of proof is to study the large sieve type inequality

$$\sum_{f \in \mathrm{H}_k^*(1)} \bigg| \sum_{P < p \le 2P} \frac{\lambda_{\mathrm{sym}^m f}(p)}{p} \bigg|^{2j}. \tag{1.15}$$

But then two difficulties come up. First, $\lambda_{\mathrm{sym}^m f}(n)$ is not completely multiplicative and second, the instantaneously available (almost) orthogonality property following from the large sieve result (developed in [5] for the level case and in [14] for the weight) is not adequate. As was indicated by Cogdell & Michel in [3, Section 1.3], the second difficulty seemed a bit problematic. In order to get around this difficulty, we shall appeal to Petersson's trace formula with the observation $\lambda_{\mathrm{sym}^m f}(n) = \lambda_f(n^m)$ for squarefree n. But then the harmonic weight (in the trace formula) needs further treatment as its trivial bound is not admissible for our purpose. To this end, we make use of (see (2.6) below)

$$1 = \frac{k-1}{12}\,\omega_f \sum_{n \le k^{7/2}} \frac{\lambda_f(n^2)}{n} + O_\varepsilon\big(k^{-1+\varepsilon}\big),$$

where ω_f is the harmonic weight (see (2.5) below). However, only a short initial section of the newly introduced sum is manageable by the Petersson trace formula. The remaining part will be handled with the idea in ([13], Lemma 6) by virtue of the large sieve result in [14]. Clearly our result for (1.15) (see the proposition below) is of independent interest and has other applications which will be presented elsewhere.

§ 2. A large sieve type inequality

This section is devoted to establish a large sieve type inequality, which will be our key tool for the proof of Theorem 1.1. For $2 \mid k$, $f \in \mathrm{H}_k^*(1)$, $m \in \mathbb{N}$ and $1 \le P < Q \le 2P$, we consider the sum

$$T_{\mathrm{sym}^m f}(P, Q) := \sum_{P < p \le Q} \frac{\lambda_{\mathrm{sym}^m f}(p)}{p}.$$

Our aim is to prove the following result, which reveals a good control over the tail part of the Dirichlet series representation of $\log L(1, \mathrm{sym}^m f)$ for most forms f.

Proposition 2.1. *Let $m \in \mathbb{N}$ be fixed. Then, we have*

$$\sum_{f \in \mathrm{H}_k^*(1)} |T_{\mathrm{sym}^m f}(P, Q)|^{2j} \ll_m k(\log k)^{\theta(m)} e^{2j \log j} P^{-j} + (j!)^2 k^{20/21} \tag{2.1}$$

uniformly for

$$2 \mid k, \qquad j \in \mathbb{N}, \qquad 1 \le P^j \le k^{7/(6m+24)} \qquad \text{and} \qquad P < Q \le 2P, \tag{2.2}$$

where $\theta(m) := (m+1)^4 + m + 7$ and the implied constant depends on m only.

To prove it, we need a couple of preliminary lemmas.

Although the function $\lambda_{\mathrm{sym}^m f}(n)$ is not completely multiplicative on $\mathbb{N}$, its restriction on the subset of squarefree integers recaptures this property and furthermore

$$\lambda_{\mathrm{sym}^m f}(n) = \prod_{p \mid n} \sum_{0 \le j \le m} \alpha_f(p)^{m-2j} = \lambda_f(n^m) \tag{2.3}$$

for n squarefree, which follows immediately from (1.5), (1.6) and (1.7). Thus we give an upper estimate to $|T_{\mathrm{sym}^m f}(P, Q)|^{2j}$ in terms of sums over squarefree integers.

Lemma 2.2. *Let $j \in \mathbb{N}$, $2 \mid k$, $m \in \mathbb{N}$ and $1 \le P < Q \le 2P$. For any $f \in \mathrm{H}_k^*(1)$, we have*

$$\begin{aligned} &\big|T_{\mathrm{sym}^m f}(P, Q)\big|^{2j} \\ &\ll_m (j \log Q)^{(m+1)^4} \sideset{}{^\natural}\sum_{n_2 \le Q^j} \frac{1}{n_2^{3/2}} \Bigg| \sideset{}{^\flat}\sum_{\substack{P^j/n_2 < n_1 \le Q^j/n_2 \\ (n_1, n_2) = 1}} \lambda_f(n_1^m) \frac{a_j(n_1 n_2)}{n_1} \Bigg|^2, \end{aligned}$$

where

$$a_j(n) = a_j(n; P, Q) := |\{(p_1, \dots, p_j) : p_1 \cdots p_j = n,\ P < p_i \le Q\}|. \tag{2.4}$$

The summations $\sum^\natural$ *and* $\sum^\flat$ *indicate run over squarefull* [‡] *and squarefree integers, respectively. The implied constant depends on* m *only.*

Proof. Multiplying out the product $T_{\mathrm{sym}^m f}(P,Q)^j$, we obtain a summation over integers in $(P^j, Q^j]$. As every integer n decomposes uniquely into a product of coprime integers $n = n_1 n_2$ with n_1 squarefree and n_2 squarefull, it then follows that

$$T_{\mathrm{sym}^m f}(P,Q)^j = \sideset{}{^\natural}\sum_{n_2 \le Q^j} \frac{1}{n_2} \prod_{p^\nu \| n_2} \lambda_{\mathrm{sym}^m f}(p)^\nu \sideset{}{^\flat}\sum_{\substack{P^j/n_2 < n_1 \le Q^j/n_2 \\ (n_1,n_2)=1}} \lambda_{\mathrm{sym}^m f}(n_1) \frac{a_j(n_1 n_2)}{n_1}.$$

Next we remove the products of $\lambda_{\mathrm{sym}^m f}(p)$ over squarefull integers by the Cauchy-Schwarz inequality and (1.8):

$$\begin{aligned} &|T_{\mathrm{sym}^m f}(P,Q)|^{2j} \\ &\le \sideset{}{^\natural}\sum_{n \le Q^j} \frac{(m+1)^{2\Omega(n)}}{n^{1/2}} \sideset{}{^\natural}\sum_{n_2 \le Q^j} \frac{1}{n_2^{3/2}} \bigg| \sideset{}{^\flat}\sum_{\substack{P^j/n_2 < n_1 \le Q^j/n_2 \\ (n_1,n_2)=1}} \lambda_{\mathrm{sym}^m f}(n_1) \frac{a_j(n_1 n_2)}{n_1} \bigg|^2. \end{aligned}$$

Here $\Omega(n)$ denotes the number of prime factors of n counted with multiplicity. Consequently, we get our result with (2.3) and the estimate below obtained by Rankin's trick

$$\sideset{}{^\natural}\sum_{n \le x} \frac{(m+1)^{2\Omega(n)}}{n^{1/2}} \le \prod_{p \le x} \bigg(1 + \frac{(m+1)^4}{p} + O_m\Big(\frac{1}{p^{3/2}}\Big)\bigg) \ll_m (\log x)^{(m+1)^4}$$

(see the proofs of Theorems II.1.2 & II.1.13 in [22] for paradigms). □

In view of Lemma 2.2, we invoke naturally the Petersson trace formula to prove our proposition. However the summation on the left-side of (2.1) runs over $f \in \mathrm{H}_k^*(1)$ without the harmonic weight

$$\omega_f := \frac{\Gamma(k-1)}{(4\pi)^{k-1}\|f\|} = \frac{12\zeta(2)}{(k-1)L(1, \mathrm{sym}^2 f)}. \tag{2.5}$$

(See [11, §2] for the last equality.) We borrow the technique in [13]. The underlying principle is built on approximating the factor $L(1, \mathrm{sym}^2 f)$ with a finite Dirichlet series.

Lemma 2.3. *Let* $2 \mid k$, $f \in \mathrm{H}_k^*(1)$ *and* $y \ge 1$. *For any fixed* $\varepsilon > 0$, *we have*

$$L(1, \mathrm{sym}^2 f) = \zeta(2) \sum_{n \le y} \lambda_f(n^2) n^{-1} + O_\varepsilon\big(k^\varepsilon (k^{3/4} y^{-1/2} + k^{-1})\big).$$

The implied constant depends on ε *only.*

[‡]An integer n is called squarefull if $p \mid n \Rightarrow p^2 \mid n$.

Proof. For $\sigma > 1$, we have

$$L(s, \mathrm{sym}^2 f) = \zeta(2s) \sum_{n \geq 1} \lambda_f(n^2) n^{-s}.$$

Applying the Perron formula ([22], Corollary II.2.1 with $B(x) = x^\varepsilon$ and $\alpha = 3$), we deduce that

$$\sum_{n \leq y} \frac{\lambda_f(n^2)}{n} = \frac{1}{2\pi i} \int_{1/\log y - ik}^{1/\log y + ik} \frac{L(1+s, \mathrm{sym}^2 f)}{\zeta(2+2s)} \frac{y^s}{s} \, \mathrm{d}s + O_\varepsilon\big((ky)^\varepsilon (k^{-1} + y^{-1})\big).$$

By moving the segment of integration to $\sigma = -\frac{1}{2} + \varepsilon$ and using the convexity bound for $L(s, \mathrm{sym}^2 f)$ (see [14], Proposition 3.1):

$$L(s, \mathrm{sym}^2 f) \ll_\varepsilon (k + |\tau|)^{\frac{3}{2} \max\{0, 1-\sigma\} + \varepsilon},$$

it follows that

$$\sum_{n \leq y} \frac{\lambda_f(n^2)}{n} = \frac{L(1, \mathrm{sym}^2 f)}{\zeta(2)} + O_\varepsilon\big((ky)^\varepsilon (k^{-1} + k^{3/4} y^{-1/2})\big),$$

which is equivalent to the required result. □

Taking $y = k^{7/2}$ and using the bound $\omega_f \ll (\log k)/k$ (cf. [11]), Lemma 2.3 with (2.5) gives

$$1 = \frac{k-1}{12} \omega_f \sum_{n \leq y} \frac{\lambda_f(n^2)}{n} + O_\varepsilon\big(k^{-1+\varepsilon}\big). \tag{2.6}$$

As mentioned in the introduction, the (short enough) initial section of the sum in (2.6) is under control of the Petersson trace formula. For the remaining part, we proceed with the idea in [13] to deduce that this part is small on average in virtue of the large sieve result developed in [14]. Define

$$\omega_f^*(x, y) := \sum_{x < n \leq y} \lambda_f(n^2) n^{-1}.$$

Then we give below the analogues of Lemmas 4 and 3 in [13], where the sum

$$\sum \lambda_{\mathrm{sym}^2 f}(n) n^{-1}$$

is used instead but it seems that our choice will lead to simpler manipulations.

Lemma 2.4. *Let $i \geq 1$, $2 \mid k$ and $f \in \mathrm{H}_k^*(1)$. Then we have*

$$\omega_f^*(x, y)^i = \sum_{x^i < d\ell \leq y^i} \lambda_f(\ell^2) \frac{c_i(d, \ell)}{d\ell}, \tag{2.7}$$

where $c_i(d, \ell) = 0$ unless $d = d^\flat d^*$ with $d^\flat$ squarefree and d^* squarefull such that $d^\flat \mid \ell$ and $(d^\flat, d^*) = 1$. Furthermore, we have

$$|c_i(d, \ell)| \leq \tau_i(d\ell) \tau_{i-1}(d) \tag{2.8}$$

where $\tau_i(\cdot)$ is the divisor function defined as in (1.8).

Proof. We proceed by induction on i. The case of $i = 1$ is trivial since we have $c_1(1, \ell) = 1$ and $c_1(d, \ell) = 0$ for $d \geq 2$. Assume that (2.7) holds for i as claimed. Thus by (1.3) we have

$$\begin{aligned}\omega_f^*(x,y)^{i+1} &= \sum_{x<n_{i+1}\le y} \frac{1}{n_{i+1}} \sum_{x^i<d\ell\le y^i} \frac{c_i(d,\ell)}{d\ell} \sum_{d_i\mid(\ell,n_{i+1})^2} \lambda_f\bigg(\Big(\frac{\ell n_{i+1}}{d_i}\Big)^2\bigg)\\ &= \sum_{x^{i+1}<d_0\ell_0\le y^{i+1}} \frac{\lambda_f(\ell_0^2)}{d_0\ell_0} c_{i+1}(d_0,\ell_0)\end{aligned}$$

with

$$c_{i+1}(d_0,\ell_0) = \sum_{x<n_{i+1}\le y} \sum_{\substack{x^i<d\ell\le y^i \\ \ell n_{i+1}=d_i\ell_0,\ d_0=dd_i}} \sum_{d_i\mid(\ell,n_{i+1})^2} c_i(d,\ell).$$

We write uniquely $d_0 = d_0^\flat d_0^*$ into a product of coprime integers with $d_0^\flat$ squarefree and d_0^* squarefull. We claim that

$$c_{i+1}(d_0,\ell_0) \neq 0 \quad \Rightarrow \quad d_0^\flat \mid \ell_0.$$

Let $d_0^\flat = d'd_i'$ with $d' \,\|\, d$ and $d_i' \,\|\, d_i$. [§] Then, $(d', d_i) = (d_i', d) = 1$ as $d_0^\flat \,\|\, dd_i$ and $d_0^\flat$ is squarefree. Since $d_i \mid (\ell, n_{i+1})^2$ and $\ell n_{i+1} = d_i\ell_0$, we have $d_i' \mid \ell_0$ (by noting $d_i' \,\|\, d_i$). On the other hand, by the induction hypothesis we see that $c_i(d,\ell) \neq 0$ implies $d' \mid \ell$, thus $d' \mid \ell_0$ for $(d', d_i) = 1$. This follows $d_0^\flat \mid \ell_0$ as $d_0^\flat = d'd_i'$ is squarefree.

It remains to verify (2.8), which is an immediate consequence of the formula:

$$c_i(d,\ell) := \sum_{\substack{x<n_1,\dots,n_i\le y \\ d\ell=n_1\cdots n_i}} \sum_{d_1\mid(n_1,n_2)^2} \sum_{d_2\mid(n_1n_2/d_1,n_3)^2} \cdots \sum_{\substack{d_{i-1}\mid(n_1\cdots n_{i-1}/d_1\cdots d_{i-2},n_i)^2 \\ d=d_1\cdots d_{i-1}}} 1.$$

This completes the proof of Lemma 2.4. □

Lemma 2.5. *For any $A > 0$, $\varepsilon > 0$ and integer $i \geq 1$, we have*

$$\sum_{f\in \mathrm{H}_k^*(1)} \omega_f^*(x,y)^{2i} \ll_{A,\varepsilon,i} k^\varepsilon \tag{2.9}$$

uniformly for $2 \mid k$ and $k^5 \leq x^i < y^i \leq k^A$.

Proof. The main ingredients of proof are Lemma 2.4 and the following large sieve type inequality: Suppose $a(n) \ll_\varepsilon n^{-1+\varepsilon}$ for any $\varepsilon > 0$. Then

$$\sum_{f\in \mathrm{H}_k^*(1)} \bigg|\sum_{L<\ell\le 2L} a(\ell)\lambda_f(\ell^2)\bigg|^2 \ll_\varepsilon (kL)^\varepsilon\big(1 + k^{5/2}L^{-1/2}\big) \tag{2.10}$$

holds uniformly for $2 \mid k$ and $L \geq 1$.

[§] The notation $d \,\|\, n$ means that $v_p(d) = v_p(n)$ for all $p \mid d$, where $v_p(n)$ is the exponent of p in the canonical factorization of n.

The inequality (2.10) is a consequence of the relation

$$\lambda_f(\ell^2) = \sum_{dn^2=\ell} \lambda_{\mathrm{sym}^2 f}(d)\mu(n) \tag{2.11}$$

where $\mu(n)$ is the Möbius function, and the large sieve inequality in ([14], Proposition 4.1 with $m=2$): For any $\varepsilon > 0$ we have

$$\sum_{f\in \mathrm{H}_k^*(1)} \Big|\sum_{\ell\le L} b_\ell \lambda_{\mathrm{sym}^2 f}(\ell)\Big|^2 \ll_\varepsilon k^\varepsilon\big(L + k^{5/2}L^{1/2+\varepsilon}\big) \sum_{\ell\le L} |b_\ell|^2$$

uniformly for $2 \mid k$, $L \ge 1$ and $\{b_\ell\}_{1\le\ell\le L} \subset \mathbb{C}$.

From (2.11), we write the inner sum in (2.10) into

$$\sum_{L<\ell\le 2L} a(\ell)\lambda_f(\ell^2) = \sum_{d\le 2L} \lambda_{\mathrm{sym}^2 f}(d) \sum_{\sqrt{L/d}<n\le\sqrt{2L/d}} \mu(n)a(dn^2)$$

and apply the large sieve inequality to the right-side. Then (2.10) follows because the condition $a(n) \ll n^{-1+\varepsilon}$ yields

$$\sum_{d\le 2L} \Big| \sum_{\sqrt{L/d}<n\le\sqrt{2L/d}} \big|a(dn^2)\big| \Big|^2 \ll L^{-1+\varepsilon}.$$

Now we prove (2.9). Firstly, we divide the sum in (2.7) dyadically

$$\omega_f^*(x,y)^i = \sum_{j\le(\log y^i)/\log 2} \ \sum_{x^i/2^{j+1}<\ell\le y^i/2^j} \lambda_f(\ell^2)\frac{c_j(\ell)}{\ell},$$

where

$$c_j(\ell) := \sum_{\substack{2^j<d\le 2^{j+1}\\ x^i/\ell<d\le y^i/\ell}} \frac{c_i(d,\ell)}{d}.$$

Then, by the Cauchy-Schwarz inequality, we obtain

$$\sum_{f\in\mathrm{H}_k^*(1)} \omega_f^*(x,y)^{2i} \tag{2.12}$$
$$\ll_A (\log k) \sum_{j\le(\log y^i)/\log 2} \ \sum_{f\in\mathrm{H}_k^*(1)} \Big| \sum_{x^i/2^{j+1}<\ell\le y^i/2^j} \lambda_f(\ell^2)\frac{c_j(\ell)}{\ell}\Big|^2.$$

From (2.8) and $\tau_i(d\ell) \le \tau_i(d)\tau_i(\ell)$, we have

$$c_j(\ell) \le \tau_i(\ell)^3 \sum_{d\mid\ell} \frac{1}{d} \sideset{}{^\natural}\sum_{2^j/d<d^*\le 2^{j+1}/d} \frac{\tau_i(d^*)^2}{d^*}.$$

By the Rankin trick again, it is easy to see that

$$\sideset{}{^\natural}\sum_{d\le t} \frac{\tau_i(d)^2}{\sqrt{d}} \ll (\log t)^{\theta_0(i)}$$

with $\theta_0(i) := ((i+1)i/2)^2$, and hence

$$c_j(\ell) \ll \tau_i(\ell)^3\tau(\ell)2^{-j/2}(\log 2^j)^{\theta_0(i)}.$$

From (2.12) and (2.10) with $a(\ell) = 2^{j/2}(\log 2^j)^{-\theta_0(i)} c_j(\ell)/\ell$, we infer that

$$\begin{aligned}\sum_{f\in \mathrm{H}_k^*(1)} &\omega_f^*(x,y)^{2i} \\ &\ll_{A,\varepsilon,i} k^\varepsilon \sum_{j\ll \log k} \frac{1}{2^j} \sum_{f\in \mathrm{H}_k^*(1)} \bigg| \sum_{x^i/2^{j+1}<\ell\le y^i/2^j} \frac{2^{j/2}(\log 2^j)^{-\theta_0(i)}c_j(\ell)}{\ell} \lambda_f(\ell^2)\bigg|^2 \\ &\ll_{A,\varepsilon,i} k^\varepsilon \sum_{j\ll\log k} 2^{-j}\big\{1 + k^{5/2}(x^i 2^{-j-1})^{-1/2}\big\} \\ &\ll_{A,\varepsilon,i} k^\varepsilon\end{aligned}$$

for $k^5 \le x^i \le y^i \le k^A$. □

Now we are ready to prove Proposition 2.1.

Proof of Proposition 2.1. By Lemma 2.2 and (2.6), we deduce that

$$\begin{aligned}\sum_{f\in \mathrm{H}_k^*(1)} &|T_{\mathrm{sym}^m f}(P,Q)|^{2j} \\ &\ll_m (\log k)^{(m+1)^4}\bigg(k \sideset{}{^\natural}\sum_{n_2\le Q^j} \frac{|M(n_2)|}{n_2^{3/2}} + O(R)\bigg),\end{aligned} \tag{2.13}$$

where

$$M(n_2) := \sum_{f\in \mathrm{H}_k^*(1)} \omega_f \sum_{n\le y} \frac{\lambda_f(n^2)}{n} \bigg| \sideset{}{^\flat}\sum_{\substack{P^j/n_2<n_1\le Q^j/n_2\\ (n_1,n_2)=1}} \lambda_f(n_1^m)\frac{a_j(n_1 n_2)}{n_1}\bigg|^2$$

with $y = k^{7/2}$, and

$$\begin{aligned}R &:= k^{-1+\varepsilon} \sideset{}{^\natural}\sum_{n_2\le Q^j} \frac{1}{n_2^{3/2}} \sum_{f\in \mathrm{H}_k^*(1)} \bigg| \sideset{}{^\flat}\sum_{\substack{P^j/n_2<n_1\le Q^j/n_2\\ (n_1,n_2)=1}} \lambda_f(n_1^m)\frac{a_j(n_1n_2)}{n_1}\bigg|^2 \\ &\ll j!^2 k^\varepsilon \sideset{}{^\natural}\sum_{n_2\le Q^j}\frac{1}{n_2^{3/2}}\bigg(\sum_{P^j/n_2<n\le Q^j/n_2} \frac{\tau(n^m)}{n}\bigg)^2 \\ &\ll j!^2 k^\varepsilon\end{aligned} \tag{2.14}$$

by the Deligne inequality, (1.1) and the trivial estimate for (2.4)

$$a_j(n) \le j!. \tag{2.15}$$

The remaining task is to estimate $M(n_2)$. We square out the innermost sum in $M(n_2)$ and explore the cancellation through the Petersson trace formula. But this approach is only effective for small n, hence we split $M(n_2)$ into two parts

$$M(n_2) = S_x + S_{x,y} \tag{2.16}$$

according to $n \le x$ and $x < n \le y$ respectively where $x = k^{1/2}$. The second term $S_{x,y}$ is handled by Lemma 2.5, as follows.

From the estimate $\omega_f \ll (\log k)/k$, the Deligne inequality and (2.15), we have

$$\begin{aligned}S_{x,y} &\ll k^{-1}(\log k) \sum_{f\in \mathrm{H}_k^*(1)} |\omega_f^*(x,y)| \Big(j! \sum_{n\le Q^j} \tau(n^m)n^{-1}\Big)^2 \\ &\ll (j!)^2 k^{-1}(\log k)^{2m+3} \sum_{f\in \mathrm{H}_k^*(1)} |\omega_f^*(x,y)|.\end{aligned}$$

Applying Hölder's inequality and Lemma 2.5 with $i=10$, we deduce that

$$\begin{aligned}S_{x,y} &\ll (j!)^2 k^{-1+\varepsilon}\Big(\sum_{f\in \mathrm{H}_k^*(1)} 1\Big)^{19/20}\Big(\sum_{f\in \mathrm{H}_k^*(1)} \omega_f^*(x,y)^{20}\Big)^{1/20} \\ &\ll (j!)^2 k^{-1/20+\varepsilon}.\end{aligned}$$

Now we treat S_x by the Petersson trace formula (see [12], Corollary 2.10)

$$\sum_{f\in \mathrm{H}_k^*(1)} \omega_f \lambda_f(a)\lambda_f(b) = \delta(a,b) + O\big(k^{-5/6}(ab)^{1/4}\tau_3((a,b))\log(2ab)\big)$$

where $\delta(a,b)$ is the Kronecker delta and the implied constant is absolute. Squaring out and using (1.3) and (2.15), we obtain

$$\begin{aligned}S_x \le (j!)^2 \sum_{n\le x}\frac{1}{n} &\sideset{}{^\flat}\sum_{P^j/n_2<n_1,n_1'\le Q^j/n_2}\sideset{}{^\flat}\sum \frac{1}{n_1n_1'} \\ &\times \sum_{d\mid (n_1,n_1')^m}\Big|\sum_{f\in \mathrm{H}_k^*(1)} \omega_f\lambda_f(n^2)\lambda_f\Big(\frac{(n_1n_1')^m}{d^2}\Big)\Big|.\end{aligned}$$

Let us write $n_1=d\ell$ and $n_1'=d\ell'$ where $d=(n_1,n_1')$. Then d, ℓ and ℓ' are squarefree and pairwisely coprime for squarefree n_1 and n_1'. Therefore,

$$\begin{aligned}S_x \le (j!)^2 &\sideset{}{^\flat}\sum\sideset{}{^\flat}\sum\sideset{}{^\flat}\sum_{\substack{P^j/n_2<d\ell,d\ell'\le Q^j/n_2\\ (\ell,\ell')=1}} \frac{1}{d^2\ell\ell'}\times \\ &\times \sum_{d_1\mid d^m}\sum_{n\le x}\frac{1}{n}\Big|\sum_{f\in \mathrm{H}_k^*(1)} \omega_f\lambda_f(n^2)\lambda_f\Big(\frac{(d^2\ell\ell')^m}{d_1^2}\Big)\Big|.\end{aligned}\tag{2.17}$$

The Petersson trace formula yields that the sum over $f\in \mathrm{H}_k^*(1)$ equals

$$\delta\big(n^2,(\ell\ell')^m(d^m/d_1)^2\big) + O\bigg(\frac{(d^2\ell\ell')^{m/4}n^{1/2}}{d_1^{1/2}k^{5/6}}\tau_3(n^2)\log k\bigg).$$

Clearly for $d_1\mid d^m$ and squarefree integers ℓ and ℓ' with $(\ell,\ell')=1$, we have

$$n^2=(\ell\ell')^m(d^m/d_1)^2 \;\Rightarrow\; \ell\ell'(d^m/d_1)\mid n.$$

Thus after summing over n, the δ-symbol contributes

$$\begin{aligned}\sum_{n\le x}\frac{1}{n}\delta\big(n^2,(\ell\ell')^m(d^m/d_1)^2\big) &\ll \frac{1}{\ell\ell'}\frac{d_1}{d^m}\sum_{n\le x/\ell\ell'(d^m/d_1)}\frac{1}{n} \\ &\ll \frac{\log k}{\ell\ell'},\end{aligned}$$

while the O-term produces a term trivially bounded by

$$\frac{\log k}{\ell\ell'}\frac{Q^{j(m/2+2)}}{k^{5/6}}\sum_{n\le x}\frac{\tau_3(n^2)}{\sqrt{n}}\ll(\log k)^6\ell\ell'$$

in view of our choices of x, j and Q.

Inserting these estimates into (2.17), it follows that

$$\begin{aligned}S_x&\ll(j!)^2(\log k)^6\sideset{}{^\flat}\sum\sideset{}{^\flat}\sum\sideset{}{^\flat}\sum_{P^j/n_2<d\ell,d\ell'\le Q^j/n_2}\frac{\tau(d^m)}{(d\ell\ell')^2}\\&\ll(j!)^2(\log k)^6\sideset{}{^\flat}\sum_{d\le Q^j/n_2}\frac{\tau(d^m)}{d^2}\bigg(\sideset{}{^\flat}\sum_{P^j/dn_2<\ell\le Q^j/dn_2}\frac{1}{\ell^2}\bigg)^2\\&\ll(j!)^2(\log k)^6\frac{n_2^2}{P^{2j}}\sideset{}{^\flat}\sum_{d\le Q^j/n_2}\tau(d^m).\end{aligned}$$

Together with the estimates of $S_{x,y}$ and (2.17), we get an upper bound for $M(n_2)$:

$$M(n_2)\ll(j!)^2(\log k)^6\frac{n_2^2}{P^{2j}}\sideset{}{^\flat}\sum_{d\le Q^j/n_2}\tau(d^m)+(j!)^2k^{-1/20+\varepsilon}.$$

In view of (2.13), we need to evaluate the following sum over squarefull integers.

$$\begin{aligned}\sideset{}{^\natural}\sum_{n\le Q^j}n^{1/2}\sideset{}{^\flat}\sum_{d\le Q^j/n}\tau(d^m)&\ll Q^j(\log k)^m\sideset{}{^\natural}\sum_{n\le Q^j}n^{-1/2}\\&\ll Q^j(\log k)^{m+1}\end{aligned}$$

as there are $O(\sqrt{t})$ squarefull integers less than t and

$$\sum_{d\le t}\tau(d^m)\ll t(\log t)^m.$$

Together with (2.13) and (2.14), we conclude that

$$\sum_{f\in\mathrm{H}_k^*(1)}|T_{\mathrm{sym}^m f}(P,Q)|^{2j}\ll k(\log k)^{(m+1)^4+m+7}(j!)^2Q^jP^{-2j}+(j!)^2k^{19/20+\varepsilon}$$

which gives our desired result, by Stirling's formula and $Q\le 2P$. $\square$

§ 3. Proof of Theorem 1.1

Let $m\in\mathbb{N}$, $2\mid k$ and $f\in\mathrm{H}_k^*(1)$. We have

$$\log L(s,\mathrm{sym}^m f)=\sum_{n=1}^{\infty}\frac{\Lambda_{\mathrm{sym}^m f}(n)}{n^s\log n}\qquad(\sigma>1),\tag{3.1}$$

where

$$\Lambda_{\mathrm{sym}^m f}(n) = \begin{cases} \left[\alpha_f(p)^{m\nu} + \alpha_f(p)^{(m-2)\nu} + \cdots + \alpha_f(p)^{-m\nu}\right] \log p & \text{if } n = p^\nu, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

Apparently $|\Lambda_{\mathrm{sym}^m f}(n)| \le (m+1)\log n$ for $n \ge 1$. To prove our theorem, we shall show that for almost all f, $\log L(1, \mathrm{sym}^m f)$ is well approximated by a short partial sum over primes. Actually, $\log L(1, \mathrm{sym}^m f)$ has a good approximation by a partial sum of moderate length when $L(s, \mathrm{sym}^m f)$ has a bigger zero-free region, which is available for most $f \in \mathrm{H}_k^*(1)$.

As in [14], for each $\eta \in (0, \frac{1}{100}]$, we define

$$\mathrm{H}^+_{k,\mathrm{sym}^m}(1;\eta) := \big\{ f \in \mathrm{H}_k^*(1) : L(s, \mathrm{sym}^m f) \ne 0 \text{ for } s \in \mathcal{S} \big\}, \tag{3.3}$$

where $\mathcal{S} := \{s : \sigma \ge 1-\eta, \, |\tau| \le 100k^\eta\} \cup \{s : \sigma \ge 1\}$, and

$$\mathrm{H}^-_{k,\mathrm{sym}^m}(1;\eta) := \mathrm{H}_k^*(1) \smallsetminus \mathrm{H}^+_{k,\mathrm{sym}^m}(1;\eta). \tag{3.4}$$

According to (1.11) of [14], we have

$$|\mathrm{H}^-_{k,\mathrm{sym}^m}(1;\eta)| \ll_\eta k^{31\eta}. \tag{3.5}$$

For $f \in \mathrm{H}^+_{k,\mathrm{sym}^m}(1;\eta)$, we have the following result.

Lemma 3.1. *Let $\eta \in (0, \frac{1}{100}]$ and $\delta_0 \in (0,1]$ be fixed and $m \in \{1,2,3,4\}$. Let $2 \mid k$ and $x = \exp\big\{[(\log k)/7(m+4)]^{\delta_0}\big\}$. Then for any $f \in \mathrm{H}^+_{k,\mathrm{sym}^m}(1;\eta)$, we have*

$$\log L(1, \mathrm{sym}^m f) = \sum_{p \le x} \sum_{0 \le j \le m} \log \left(1 - \frac{\alpha_f(p)^{m-2j}}{p}\right)^{-1} + O\bigg(\frac{1}{(\log k)^{\delta_0}}\bigg).$$

The implied constant depends on δ, η and m only.

Proof. Let $f \in \mathrm{H}_k^*(1)$, $T \ge 1$ and $x \ge 1$. By the Perron formula ([22], Corollary II.2.1 with $B(x) = 1$ and $\alpha = 1$), we have

$$\begin{aligned} \sum_{2 \le n \le x} \frac{\Lambda_{\mathrm{sym}^m f}(n)}{n \log n} &= \frac{1}{2\pi i} \int_{1/\log x - iT}^{1/\log x - iT} \log L(s+1, \mathrm{sym}^m f) \frac{x^s}{s} \, \mathrm{d}s \\ &\quad + O\bigg(\frac{\log(Tx)}{T} + \frac{1}{x}\bigg). \end{aligned}$$

Once $f \in \mathrm{H}^+_{k,\mathrm{sym}^m}(1;\eta)$, we have the upper estimate

$$\log L(s, \mathrm{sym}^m f) \ll_\eta \log k \tag{3.6}$$

uniformly for $\sigma \ge 1 - \frac{1}{4}\eta$ and $|\tau| \le (\log k)^{4/\eta}$. This is a particular case of Proposition 3.5 of [14] (with $\alpha = \frac{1}{4}\eta$).

Now for $f \in \mathrm{H}^+_{k,\mathrm{sym}^m}(1;\eta)$, we move the line of integration to $\sigma = -\frac{1}{4}\eta$ and estimate $\log L(s+1, \mathrm{sym}^m f)$ by (3.6) over the contour. We see that

(3.7)
$$\begin{aligned}\sum_{2\le n\le x} \frac{\Lambda_{\mathrm{sym}^m f}(n)}{n\log n} &= \log L(1, \mathrm{sym}^m f) + O\bigg(\frac{\log(kTx)}{T} + \frac{(\log k)(\log T)}{x^{\eta/4}}\bigg) \\ &= \log L(1, \mathrm{sym}^m f) + O\bigg(\frac{1}{(\log k)^{4/\eta - 1}}\bigg)\end{aligned}$$

by taking the parameters $T = (\log k)^{4/\eta}$ and $x = \exp\big\{[(\log k)/7(m+4)]^{\delta_0}\big\}$.

On the other hand, we have

$$\begin{aligned}\sum_{2\le n\le x} \frac{\Lambda_{\mathrm{sym}^m f}(n)}{n\log n} &= \sum_{p\le x}\sum_{\nu\le(\log x)/\log p} \frac{\Lambda_{\mathrm{sym}^m f}(p^\nu)}{p^\nu \log p^\nu} \\ &= \sum_{p\le x}\sum_{0\le j\le m}\sum_{\nu\le(\log x)/\log p} \frac{\alpha_f(p)^{(m-2j)\nu}}{\nu p^\nu} \\ &= \sum_{p\le x}\sum_{0\le j\le m}\bigg\{\log\bigg(1-\frac{\alpha_f(p)^{m-2j}}{p}\bigg)^{-1} + O\bigg(\frac{1}{x}\bigg)\bigg\}.\end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8), we get the required result. □

The size of x given in Lemma 3.1, even though being quite small, is still insufficient for our purpose. Making use of the proposition to remove the "exceptional forms", we are able to further reduce its size in the next two lemmas.

Lemma 3.2. *Let $m \in \mathbb{N}$, $\delta_1 > 0$ and $\delta_2 > 0$ such that $\delta_1 - \delta_2 - 2 > 0$ be fixed. Suppose that*

$$2 \mid k \qquad \text{and} \qquad (\log k)^{\delta_1} \le P \le Q \le 2P \le k^{14/15(m+4)}. \tag{3.9}$$

Then we have

$$\big|T_{\mathrm{sym}^m f}(P,Q)\big| \le \frac{1}{(\log k)^{\delta_2}} \tag{3.10}$$

for all but $O_{\delta_1,\delta_2,m}\big(k^{1-\theta_0}\big)$ forms $f \in \mathrm{H}^*_k(1)$, where

$$\theta_0 := (\delta_1 - \delta_2 - 2)/10(m+4)\delta_1 > 0.$$

Proof. Define

$$\mathrm{E}^1_m(P,Q) := \{f \in \mathrm{H}^*_k(1) : (3.10) \text{ fails}\}. \tag{3.11}$$

We shall use the proposition in Section 2 with the choices

$$j = \bigg[c'\frac{\log k}{\log P}\bigg] + 1, \qquad c' := \frac{1}{5(m+4)}$$

to count $|\mathrm{E}^1_m(P,Q)|$. Plainly we have

$$k^{1/(3m+12)} \le P^j < (2P)^j \le k^{7/(6m+24)}$$

by (3.9), whence the proposition is applicable. It follows that

$$|\mathrm{E}^1_m(P,Q)| \ll k\big((\log k)^{\theta(m)} e^{2j\log j} P^{-j} + e^{2j\log j} k^{-1/21}\big)(\log k)^{2\delta_2 j}. \tag{3.12}$$

On the other hand, the lower bound for P in (3.9) yields that

$$\begin{aligned}
&-j\log P + j(2\log j + \delta_2 \log_2 k) + \theta(m)\log_2 k \\
&\quad\le -c'\log k + \big(c'(\log k)/\log P + 1\big)(2+\delta_2)\log_2 k + \theta(m)\log_2 k \\
&\quad\le -c'\{(\delta_1-\delta_2-2)/\delta_1\}\log k + (\theta(m)+2+\delta_2)\log_2 k \\
&\quad\le -\tfrac12 c'\{(\delta_1-\delta_2-2)/\delta_1\}\log k
\end{aligned}$$

and

$$\begin{aligned}
&-\tfrac{1}{21}\log k + j(2\log j + \delta_2\log_2 k) \\
&\quad\le -\tfrac{1}{21}\log k + \big(c'(\log k)/\log P + 1\big)(2+\delta_2)\log_2 k \\
&\quad\le -\big(\tfrac{1}{21} - c'(2+\delta_2)/\delta_1\big)\log k + (2+\delta_2)\log_2 k \\
&\quad\le -\tfrac12\big(\tfrac{1}{21} - c'(2+\delta_2)/\delta_1\big)\log k.
\end{aligned}$$

Inserting these two estimates into (3.12) and noticing $\frac{1}{21} - c'(2+\delta_2)/\delta_1 \ge c'(\delta_1-\delta_2-2)/\delta_1$, we get the desired result. This completes the proof. □

Lemma 3.3. *Let $m \in \mathbb{N}$, $\delta_3 > 0$ and $\delta_4 > 0$ such that $1 - 2\delta_3 - \delta_4 > 0$ be fixed. Suppose that*

$$2 \mid k \qquad \textit{and} \qquad (\log_2 k)^{1/\delta_3} \le P \le Q \le 2P \le (c\log k)^{1/\delta_3}, \tag{3.13}$$

where $c = (1-2\delta_3-\delta_4)/24(m+4)(\theta(m)+2) > 0$. Then we have

$$\big|T_{\mathrm{sym}^m f}(P,Q)\big| \le P^{-\delta_4} \tag{3.14}$$

for all but $O_{\delta_3,\delta_4,m}\big(ke^{-(\theta(m)+2)P^{\delta_3}}\big)$ forms $f \in \mathrm{H}^*_k(1)$.

Proof. The argument is similar to Lemma 3.2. Define

$$\mathrm{E}^2_m(P,Q) := \{f \in \mathrm{H}^*_k(1) : (3.14) \text{ fails}\}. \tag{3.15}$$

But this time we apply the proposition with another choice of parameters

$$j = \left[c'\frac{P^{\delta_3}}{\log P}\right] + 1, \qquad c' := \frac{2\theta(m)+4}{1-2\delta_3-\delta_4}.$$

By (3.13), it is easy to verify that $e^{c'P^{\delta_3}} \le P^j \le k^{2cc'} = k^{1/6(m+4)} < k^{7/(6m+24)}$. Thus we deduce by the proposition that

$$|\mathrm{E}^2_m(P,Q)| \ll k\big((\log k)^{\theta(m)} e^{2j\log j} P^{-j} + e^{2j\log j} k^{-1/21}\big)P^{2\delta_4 j}. \tag{3.16}$$

Now, in view of our choices of c' and c, we have

$$\begin{aligned}
&-(1-\delta_4)j\log P + 2j\log j + \theta(m)\log_2 k \\
&\quad\le -(1-\delta_4)c'P^{\delta_3} + 2\big(c'P^{\delta_3}/\log P + 1\big)\delta_3\log P + \theta(m)\log_2 k \\
&\quad\le -c'(1-2\delta_3-\delta_4)P^{\delta_3} + (\theta(m)+2)\log_2 k \\
&\quad\le -\tfrac12 c'(1-2\delta_3-\delta_4)P^{\delta_3}
\end{aligned}$$

by the lower bound for P in (3.13), and

$$\begin{aligned}
&-\tfrac{1}{21}\log k + j(2\log j + \delta_4 \log P)\\
&\quad\le -\tfrac{1}{21}\log k + \big(c'P^{\delta_3}/\log P + 1\big)(2\delta_3+\delta_4)\log P\\
&\quad\le -\tfrac{1}{21}\log k + 2c'(2\delta_3+\delta_4)P^{\delta_3}\\
&\quad\le -\big(\tfrac{1}{21c} - 2c'(2\delta_3+\delta_4)\big)P^{\delta_3}\\
&\quad\le -\tfrac{1}{2}c'(1-2\delta_3-\delta_4)P^{\delta_3}
\end{aligned}$$

by the upper bound in (3.13). We get the required result by these two estimates with (3.16). This completes the proof. □

Now we finish the proof of Theorem 1.1.

Let $\eta \in (0, \frac{1}{100}]$ and $\delta_0 \in (0, 1]$ be fixed and $m \in \{1, 2, 3, 4\}$. Take δ_i $(1 \le i \le 4)$ such that

$$1/\theta_1 > \delta_1 > 2/(1-\theta_2), \qquad \delta_2 = 2\delta_0 = 2\theta_3, \qquad \delta_3 = \theta_1, \qquad \delta_4 = \theta_2.$$

It is easy to verify that δ_1 and δ_3 fulfill the conditions in Lemmas 2.4 and 2.5 respectively, and $1/\delta_3 > \delta_1$. Define

$$x = \exp\big\{[(\log k)/7(m+4)]^{\delta_0}\big\}, \qquad y_1 := (\log k)^{\delta_1}, \qquad y_2 := (\log_2 k)^{1/\delta_3}.$$

Then we consider the following three cases according to the size of z.

1° *The case $z \ge x$*

The required formula follows immediately from Lemma 3.1 with a better upper bound $O(k^{31\eta})$ for the exceptional set in view of (3.5).

2° *The case $y_1 \le z < x$*

Using Lemma 3.1 with $x = \exp\{[(\log k)/7(m+4)]^{\delta_0}\}$, we can write

$$\begin{aligned}
\log L(1, \mathrm{sym}^m f) \;&=\; \sum_{p\le z}\sum_{0\le j\le m}\log\bigg(1-\frac{\alpha_f(p)^{m-2j}}{p}\bigg)^{-1} + O\bigg(\frac{1}{(\log k)^{\delta_0}}\bigg)\\
&\quad + R_1(\mathrm{sym}^m f)
\end{aligned}\tag{3.17}$$

for any $f \in \mathrm{H}^+_{k,\mathrm{sym}^m}(1;\eta)$, where

$$R_1(\mathrm{sym}^m f) := -\sum_{z<p\le x}\sum_{0\le j\le m}\log\bigg(1-\frac{\alpha_f(p)^{m-2j}}{p}\bigg).$$

This case will be done if we show that $R_1(\mathrm{sym}^m f)$ is negligible apart from a small exceptional set of f. Clearly,

$$\begin{aligned}
R_1(\mathrm{sym}^m f) &= \sum_{z<p\le x}\bigg\{\frac{\lambda_{\mathrm{sym}^m f}(p)}{p} + O_m\bigg(\frac{1}{p^2}\bigg)\bigg\}\\
&= \sum_{z<p\le x}\frac{\lambda_{\mathrm{sym}^m f}(p)}{p} + O\bigg(\frac{1}{z}\bigg).
\end{aligned}$$

Define

$$P_i := 2^{i-1}z, \quad Q_i := \min\{2^i z, x\}, \quad \mathrm{E}_m^1 := \mathrm{H}^-_{k,\mathrm{sym}^m}(1;\eta) \cup \bigcup_{i \ll \log x} \mathrm{E}_m^1(P_i, Q_i),$$

where $\mathrm{E}_m^1(P_i, Q_i)$ is defined as in (3.11). According to Lemma 3.2, we have

$$|\mathrm{E}_m^1| \ll k^{31\eta} + \sum_{i \ll \log x} |\mathrm{E}_m^1(P_i, Q_i)| \ll (\log k)^{\delta_0} k^{1-\theta_0}$$

and for $f \notin \mathrm{E}_m^1$,

$$\begin{aligned} R_1(\mathrm{sym}^m f) &\ll \sum_{i \ll \log x} \big|T_{\mathrm{sym}^m f}(P_i, Q_i)\big| + \frac{1}{z} \\ &\ll \frac{1}{(\log k)^{\delta_2 - \delta_0}} + \frac{1}{z}. \end{aligned}$$

Inserting it into (3.17), we find that for $f \notin \mathrm{E}_m^1$,

$$\begin{aligned} \log L(1, \mathrm{sym}^m f) \ &= \ \sum_{p \le z} \sum_{0 \le j \le m} \log \left(1 - \frac{\alpha_f(p)^{m-2j}}{p}\right)^{-1} \\ &\quad + O\left(\frac{1}{(\log k)^{\min\{\delta_0,\, \delta_2 - \delta_0\}}} + \frac{1}{z}\right), \end{aligned} \tag{3.18}$$

which will give the required result.

3° *The case* $y_2 \le z < y_1$

We truncate the tail as in (3.17), and use the estimate in the second case. Thus it remains to evaluate

$$R_2(\mathrm{sym}^m f) := -\sum_{z < p \le y_1} \sum_{0 \le j \le m} \log \left(1 - \frac{\alpha_f(p)^{m-2j}}{p}\right).$$

Let us take

$$P_i := 2^{i-1}z, \quad Q_i := \min\{2^i z, y_1\}, \quad \mathrm{E}_m^2 := \mathrm{H}^-_{k,\mathrm{sym}^m}(1;\eta) \cup \bigcup_{i \ll \log_2 k} \mathrm{E}_m^2(P_i, Q_i).$$

By Lemma 3.3, we have

$$\begin{aligned} |\mathrm{E}_m^2| &\ll k^{31\eta} + k e^{-(\theta(m)+2) z^{\delta_3}} \log_2 k \\ &\ll k e^{-\theta(m) z^{\delta_3}} \end{aligned}$$

and

$$\begin{aligned} R_2(\mathrm{sym}^m f) &\ll \sum_{i \ll \log_2 k} \big|T_{\mathrm{sym}^m f}(P_i, Q_i)\big| + \frac{1}{z} \\ &\ll \sum_{i \ll \log_2 k} \frac{1}{(2^{i-1}z)^{\delta_4}} + \frac{1}{z} \\ &\ll \frac{1}{z^{\delta_4}} \end{aligned} \tag{3.19}$$

for all $f \notin \mathrm{E}_m^2$.

Finally define $\mathrm{E}_k^* := \mathrm{E}_m^1 \cup \mathrm{E}_m^2$, then we have

$$|\mathrm{E}_k^*| \ll k e^{-\theta(m) z^{\delta_3}}.$$

In view of (3.19) and (3.18), we derive that

$$\log L(1,\mathrm{sym}^m f) = \sum_{p\le z}\sum_{0\le j\le m}\log\left(1-\frac{\alpha_f(p)^{m-2j}}{p}\right)^{-1} + O\left(\frac{1}{(\log k)^{\delta_5}}+\frac{1}{z^{\delta_4}}\right) \tag{3.20}$$

for $f\in\mathrm{H}_k^*(1)\smallsetminus\mathrm{E}_k^*$, where $\delta_5 := \min\{\delta_0,\,\delta_1,\,\delta_2-\delta_0\}$. Obviously this is equivalent to our required result. The proof of Theorem 1.1 is thus complete. □

§ 4. Proof of Corollary 1.2

By Theorem 1.1 with the choice of

$$z=\log k,\qquad \theta_1=\tfrac12-\varepsilon,\qquad \theta_2=\theta_3=\varepsilon,$$

there is a subset E_k^* of $\mathrm{H}_k^*(1)$ such that $|\mathrm{E}_k^*|\ll ke^{-(\log k)^{1/2-\varepsilon}}$ and

$$L(1,\mathrm{sym}^m f)=\left\{1+O\left(\frac{1}{(\log k)^{\varepsilon}}\right)\right\}\prod_{p\le z}\prod_{0\le j\le m}\left(1-\frac{\alpha_f(p)^{m-2j}}{p}\right)^{-1}$$

for each $f\in\mathrm{H}_k^*(1)\smallsetminus\mathrm{E}_k^*$. In view of (1.4) and the prime number theorem, it follows that

$$\begin{aligned} L(1,\mathrm{sym}^m f) &\le \left\{1+O\left(\frac{1}{(\log k)^{\varepsilon}}\right)\right\}\prod_{p\le z}\left(1-\frac1p\right)^{-(m+1)} \\ &= \left\{1+O\left(\frac{1}{(\log k)^{\varepsilon}}\right)\right\}(e^{\gamma}\log_2 k)^{m+1} \end{aligned}$$

for all $f\in\mathrm{H}_k^*(1)\smallsetminus\mathrm{E}_k^*$. This proves the upper bound result in Corollary 1.2 and one can treat the lower bound in the same way. □

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